KLM - Togo telo 1

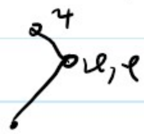
sreda, 29. april 2020 21:24

1. Izpelji tenzor vztrajnostnega momenta. (ponovitev predavanj)
2. Izpelji odvisnost komponent kotne frekvence od časa v lastnem sistemu togega telesa (t.i. Eulerjeve enačbe). (ponovitev predavanj)
3. Tanka deska dimenzij a, b s površinsko gostoto dm/dS = sigma je vpeta v os po diagonali. Z uporabo Eulerjevih enacb izračunaj navor na desko, če se deska vrti okrog osi s konstantno kotno frekvenco omega. Rešitev: $M_3 = \frac{a^2-b^2}{a^2+b^2}\frac{mab}{12}$; $m = \sigma a b$

4. a) Obravnavaj gibanje osnosimetričnega prostega togega telesa (tenzor J v lastnem sistemu diagonalne oblike z elementi J,J, J',vsota zunanjih sil 0). Zapiši Eulerjeve enačbe. Izpelji d/dt w3 = 0. (w1,w2) = const (sin($\tilde{w}$ t), cos ($\tilde{w}$ t)); $\tilde{w}$ = w3 (J'-J)/J. (ponovitev predavanj)
   b) Obravnavaj gibanje simetrijske osi (označimo jo z e3) telesa iz a). Pokaži, da oklepa konstantni kot theta z vrtilno količino L! Pokaži, da e3 okrog L precedira s kotno hitrostjo omega_pr = L/J= J' omega3/ (J cos (theta))!
   c) Zapiši vektor kotne hitrosti z Eulerjevimi koti v sistemu pripetem na togo telo. Poveži opis kotne hitrosti z rezultati iz a) in b)Upoštevaj d/dt theta=0 in v primerjavi z rezultati iz a) poveži psi z w! Rešitev psi= $\tilde{w}$ t.

Rešitve

Težišče miruje, položaj telesa
(množica masnih točk na fiksnih razdaljah)
določajo tri prostostne stopnje



kinematika : $\vec{\omega}$

vrtilna količina $\vec{L} = \underline{\underline{J}}\vec{\omega}$

$$\underline{\underline{J}} = \int dm \left(r^2 \delta_{ij} - r_i r_j\right)$$

$\int dm \to \int \rho\, dV \ldots$

$r_i = x, y, z$

$$\vec{L} = \sum_i m_i \vec{r}_i \times \vec{v}_i = \sum_i m_i \left(\vec{r}_i \times (\vec{\omega} \times \vec{r}_i)\right) =$$

$$\sum_i m_i \left(r_i^2 \vec{\omega} - (\vec{r}_i \cdot \vec{\omega})\vec{r}_i\right) = \underline{\underline{J}}\,\omega$$

$$\dot{\vec{L}} = \vec{M}$$

① Eulerjeve enačbe

gremo v sistem kjer $J = \begin{bmatrix} J_1 & & \\ & J_2 & \\ & & J_3 \end{bmatrix}$

ω_i .... komponente $\vec{\omega}$ v sistemu pripetem na telo

$$\vec{L} = \sum_i J_i \omega_i \hat{e}_i$$

$$\dot{\vec{L}} = \sum_i J_i \dot{\omega}_i \hat{e}_i + \sum_i J_i \omega_i (\vec{\omega} \times \hat{e}_i)$$

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ & 1 & \end{vmatrix} = -\omega_3 \hat{e}_1$$

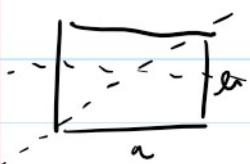
$$\dot{\vec{L}} \cdot \hat{e}_1 = J_1 \dot{\omega}_1 + J_2 \omega_2 (-\omega_3) + J_3 \omega_3 \omega_2$$

$$\begin{vmatrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ \omega_1 & \omega_2 & \omega_3 \\ & & 1 \end{vmatrix} = e_1 \omega_2$$

$$\boxed{\begin{aligned} M_1 &= J_1 \dot{\omega}_1 + \omega_2 \omega_3 (J_3 - J_2) \\ M_2 &= J_2 \dot{\omega}_2 + \omega_3 \omega_1 (J_1 - J_3) \\ M_3 &= J_3 \dot{\omega}_3 + \omega_1 \omega_2 (J_2 - J_1) \end{aligned}}$$

Navor na tanko desko s površinsko gostoto $\sigma = \frac{dm}{dS}$

$z = 0$ $\quad \rho(x,y,z) = \sigma(x,y)\,\delta(z)$



$$J_{xx} = \int [(x^2+y^2+z^2) - x^2]\, dm$$

$= \int y^2 \rho\, dV$

$$= \int y^2 \sigma\, dx\, dy$$

$$= \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} \sigma\, y^2 = \sigma a \frac{b^3}{3 \cdot 8} \cdot 2$$

$$= \sigma a b^3/12 = m \frac{b^2}{12}$$

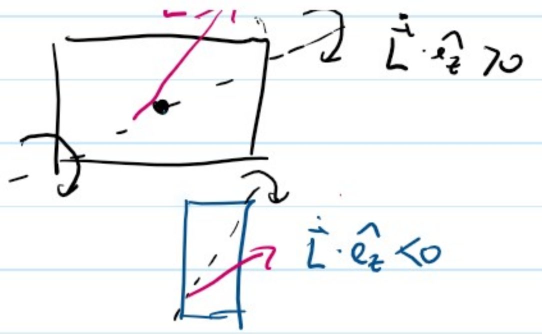
$$\underline{J} = \frac{m}{12} \begin{pmatrix} b^2 & & \\ & a^2 & \\ & & a^2+b^2 \end{pmatrix}$$

$$\vec{\omega} = \frac{(a, b, 0)}{\sqrt{a^2+b^2}}\, \omega$$

stalna, $\dot{\omega}_i = 0$

$$M_3 = (a^2 - b^2) \frac{m}{12} \cdot \frac{ab}{\sqrt{a^2+b^2}^2}$$

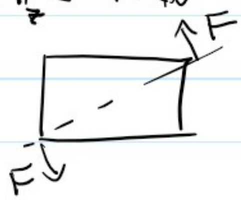
$\dot{\vec{L}} \cdot \hat{e}_z > 0$



navor ustvarjata sili v ležajih

$$\vec M = \vec r \times \vec F$$

$$(\vec M)_z = \sqrt{a^2+b^2}\cdot F \qquad F = \frac{(a^2-b^2)m}{12\,(\sqrt{a^2+b^2})}$$



Prosta, osno simetrična vrtavka

$$J_2 = J_1 = J \qquad J_3$$

$$0 = J_3\dot\omega_3 + 0 \qquad \dot\omega_3 = 0 \qquad \dot L_3 = 0$$

$$\boxed{\begin{aligned} M_1 &= J_1\dot\omega_1 + \omega_2\omega_3(J_3-J_2) \\ M_2 &= J_2\dot\omega_2 + \omega_3\omega_1(J_1-J_3) \\ M_3 &= J_3\dot\omega_3 + \omega_1\omega_2(J_2-J_1) \end{aligned}}$$

$$J\dot\omega_1 + \omega_3\Delta J\,\omega_2 = 0$$

$$J\dot\omega_2 + \omega_3\Delta J\,(-\omega_1) = 0 \quad \cdot i$$

$$\dot\xi + \underbrace{\frac{\omega_3\Delta J}{J}}_{\tilde\omega}\, i\,(-\xi) = 0$$

$$\xi = \xi_0\, e^{i\tilde\omega t}$$

$$\omega_1 = |\xi|\cos\tilde\omega t \ ; \ \omega_2 = |\xi|\sin\tilde\omega t$$

$\xi = \xi_0 e$

$\omega_1 = |\xi_0| \cos \tilde{\omega} t \; ; \; \omega_2 = |\xi_0| \sin \tilde{\omega} t$

$$\tilde{\omega} = \omega_3 \frac{\Delta J}{J}$$

primer: Zemlja

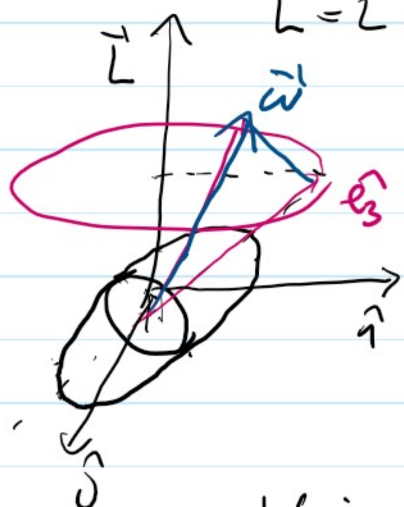
os okrog katere se vrti Zemlja malenkostno oploščena ; prosta precesija ; Chandlerjevo opletanje (Chandler - Wobble)

amplituda ~ 10 m ; 1/425 dni

Prosta sim. vrtavka – nadaljevanje (opis gibanja)

$\hat{i}, \hat{j}, \hat{k}$ zun. mirujoči k.s.

$\vec{L} = L \hat{k}$



$\dot{\vec{L}} = 0$

$\vec{L} = \sum J_i \omega_i \hat{e}_i$

$\vec{L} \cdot \hat{e}_3 = J_3 \omega_3 = \text{konst}$

$\vec{\omega} \cdot \vec{e}_3 = \text{konst}$

$\vec{L} = J \omega_1 \hat{e}_1 + J \omega_3 \hat{e}_3$ vektorji $\vec{L}, \omega, \hat{e}_3$ v isti ravnini

$\dot{\hat{e}}_3 = \vec{\omega} \times \hat{e}_3 = \omega_1 (-\hat{e}_2)$ (*)

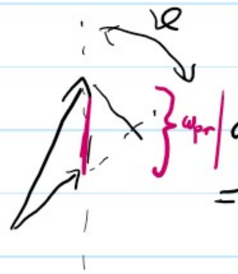
$$\omega_3 = \frac{L_3}{I_3} = \frac{\vec L \cdot \vec e_3}{I_3} = \frac{L\cos\vartheta}{I_3}$$

os vrtavke precesira okrog $\hat k$ s kotno hitrostjo ω_{pr}

Določimo ω_{pr}!

① geometrijsko

$$\vec\omega = \omega_{pr}\hat k + \omega'\hat e_3$$



$|\omega_{pr}\sin\vartheta =$

$= \omega_1 = \frac{L\sin\vartheta}{J}$

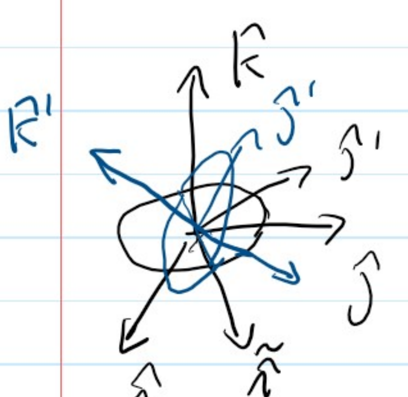
$$L_1 = J_1\omega_1$$
$$L\sin\vartheta = J\omega_1$$

$$\omega_{pr} = \frac{L}{J}$$

② s primerjavo z (*)

$$\dot{\hat e}_3 = \omega_{pr}\hat k \times \hat e_3 = \omega_{pr}\sin\vartheta$$
$$\omega_{pr}\sin\vartheta = \omega_1$$
$$\omega_{pr} = \omega_1/\sin\vartheta$$

Eulerjevi koti



$$\vec L = \frac{1}{2}J_1(\omega_1^2 + \omega_2^2)$$

$$\hat k = \hat k'\cos\vartheta + \sin\vartheta(\hat j'\cos\psi + \hat i'\sin\psi)$$

$$\omega = \dot\psi\hat k' + \dot\vartheta\hat\eta + \dot\phi\hat k$$

$$v = \dot{\psi}\,\hat{k}' + \dot{\vartheta}\left(\cos\psi\,\hat{\imath}' - \sin\psi\,\hat{\jmath}'\right)$$

$$\begin{aligned}\vec{\omega} = {} & \hat{\imath}'\left(\cos\psi\,\dot{\vartheta} + \sin\vartheta\,\sin\psi\,\dot{\phi}\right) \\ & + \hat{\jmath}'\left(-\sin\psi\,\dot{\vartheta} + \sin\vartheta\,\cos\psi\,\dot{\phi}\right) \\ & + \hat{k}'\left(\dot{\phi}\cos\vartheta + \dot{\psi}\right)\end{aligned}$$

Prosta vrtavka z Eulerjevimi koti

$\vartheta = \text{konst}$

$$\omega_3 = \dot{\phi}\cos\vartheta + \dot{\psi}$$

$$\begin{aligned}\vec{\omega} = {} & \hat{\imath}'\left(\sin\vartheta\,\sin\psi\,\dot{\phi}\right) \\ & + \hat{\jmath}'\left(\sin\vartheta\,\cos\psi\,\dot{\phi}\right)\end{aligned}$$

$$\frac{\Delta J}{J}\,\omega_3 = \dot{\psi}$$

$$\dot{\phi} = \frac{\omega_3 - \dot{\psi}}{\cos\vartheta} = \frac{\omega_3\,\frac{J - \Delta J}{J}}{\cos\vartheta} = J_3\,\omega_3 / J\cos\vartheta \qquad J_3\,\omega_3 = L\cos\vartheta$$

$$= L/J$$