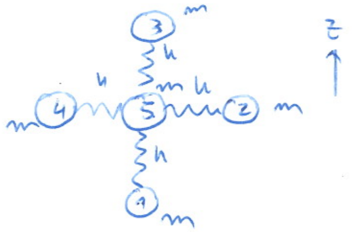
SISTEM PETIH UTEŽI, VEZANE V KRIŽ

Imamo sistem petih enakih uteži, povezane so z enakimi vzmetmi kot kaže slika. Utež 3 izmaknemo v z smeri, tako da bo sistem nihal le v z smeri. Kakšne so lastne frekvence in lastni načini?



- **Kinetična energija:**

$$T = \frac{1}{2} m \sum_{i=1}^{5} \dot{z}_i^2 = \frac{1}{2} m\, \dot{\underline{z}}^T \underline{\underline{T}}\, \dot{\underline{z}} \quad , \quad \underline{\underline{T}} = \begin{pmatrix} m &&&& \\ & m &&& \\ && m && \\ &&& m & \\ &&&& m \end{pmatrix} = m\,\underline{\underline{\mathbb{1}}} = m\,\tilde{\underline{\underline{T}}}$$

Kinetično energijo lahko predstavimo z diagonalno identično matriko.

- **Potencialna energija:**

$$V = \frac{1}{2} k \sum_{i=1}^{4} (z_i - z_5)^2 = \frac{1}{2} \underline{z}^T \underline{\underline{V}}\, \underline{z} \quad ; \quad \underline{\underline{V}} = k \begin{pmatrix} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} = k\,\tilde{\underline{\underline{V}}}$$

$$= \frac{1}{2} k \left(z_1^2 - 2 z_1 z_5 + z_5^2 + z_2^2 - 2 z_2 z_5 + z_5^2 + z_3^2 - 2 z_3 z_5 + z_5^2 + z_4^2 - 2 z_4 z_5 + z_5^2 \right)$$

- **Gibalna enačba:**

$$\underline{\underline{V}}\, \underline{a} = \omega^2 \underline{\underline{T}}\, \underline{a} \quad \rightarrow \quad k\, \tilde{\underline{\underline{V}}}\, \underline{a} = \omega^2 m\, \underline{\underline{\mathbb{1}}}\, \underline{a}$$

$$\left(\cancel{k}\, \tilde{\underline{\underline{V}}} - \underbrace{\omega^2/\omega_0^2}_{\lambda}\, \underline{\underline{\mathbb{1}}} \right) \underline{a} = 0 \quad ; \quad \omega_0^2 = \frac{k}{m}$$

Rešiti moramo problem lastnih vrednosti (saj $\underline{\underline{T}} = m\,\underline{\underline{\mathbb{1}}}$). Namesto, da se lotimo direktnega reševanja ~~reševanja~~ 5×5 determinante, bomo lastna nihanja uganili!
(To bo lahko narediti, saj so lastni vektorji $\underline{a}$ ortogonalni normalno $\underline{a}_k^T \underline{a}_l = \delta_{kl}$ in NE $\underline{a}_k^T \underline{\underline{T}}\, \underline{a}_l = \delta_{kl}$)

- **Ugibamo lastne vektorje:**

1) Tog premik - translacija: $\underline{a}_1^T = (1, 1, 1, 1, 1)$

Frekvenco dobimo iz:

$$\underline{\underline{V}}\, \underline{a}_1 = k \begin{pmatrix} 1 & & & & -1 \\ & 1 & & & -1 \\ & & 1 & & -1 \\ & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = k \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \omega_1^2 m \cdot \underline{\underline{T}}\, \underline{a}_1 = \omega^2 m\, \underline{a}_1 \Rightarrow \quad \omega_1^2 = 0$$

2) 1,3 gor, 2,4 dol: $\underline{a}_2^T = (1, -1, 1, -1, 0)$

$$\underline{\underline{V}}\,\underline{a}_2 = k\begin{pmatrix} 1 & & & & -1 \\ 0 & 1 & & & -1 \\ 0 & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = k\,\underline{a}_2 = \omega_2^2 m \underline{a}_2 \Rightarrow \omega_2^2 = \frac{k}{m} = \omega_0^2$$

3) 1,2 gor, 3,4 dol: $\underline{a}_3^T = (1, 1, -1, -1, 0)$

$$\underline{\underline{V}}\,\underline{a}_3 = k\begin{pmatrix} 1 & & & & -1 \\ 0 & 1 & & & -1 \\ 0 & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ -3 \end{pmatrix} = k\,\underline{a}_3 = \omega_3 m \underline{a}_3 \Rightarrow \omega_3^2 = \frac{k}{m} = \omega_0^2$$

5) 1,2,3,4 gor, 5 dol? → za koliko?

Nastavimo lastni vektor: $\underline{a}_5^T = (1, 1, 1, 1, \alpha)$

$$\underline{\underline{V}}\,\underline{a}_5 = k\begin{pmatrix} 1 & & & & -1 \\ 0 & 1 & & & -1 \\ 0 & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix} = k\begin{pmatrix} 1-4\alpha \\ 1-4\alpha \\ 1-\alpha \\ 1-\alpha \\ -4+4\alpha \end{pmatrix} = m\omega_5^2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \alpha \end{pmatrix}$$

pogoj za lastno nihanje

Koliko je α?

$(1-\alpha)(c) = 1$ neka konstanta $\Rightarrow c = \frac{1}{1-\alpha}$

$(-4+4\alpha)\,c = \alpha$

$-4 + 4\alpha = \frac{\alpha}{c} = \alpha(1-\alpha) = \alpha - \alpha^2$

$\alpha^2 + 3\alpha - 4 = 0 \Rightarrow (\alpha+4)(\alpha-1) = 0$

Če $\alpha = 1$, potem $\underline{a}^T = (1, 1, 1, 1, 1)$ translacija, že imamo.

Če $\alpha = -4$, $c = \frac{1}{5}$ in ~~imamo~~

$$\underline{\underline{V}}\,\underline{a}_5 = k\begin{pmatrix} 5 \\ 5 \\ 5 \\ 5 \\ -20 \end{pmatrix} = m\omega_5^2\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ -4 \end{pmatrix} \Rightarrow \omega_5^2 = 5\frac{k}{m} = 5\omega_0^2$$

4) 1,4 gor; 2,3 dol: $\underline{a}_4^T = (1, -1, -1, 1, 0)$

$$\underline{\underline{V}}\,\underline{a}_4 = k\begin{pmatrix} 1 & & & & -1 \\ 0 & 1 & & & -1 \\ 0 & & 1 & & -1 \\ 0 & & & 1 & -1 \\ -1 & -1 & -1 & -1 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = k\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} = m\omega_4^2\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \omega_4^2 = \frac{k}{m} = \omega_0^2$$