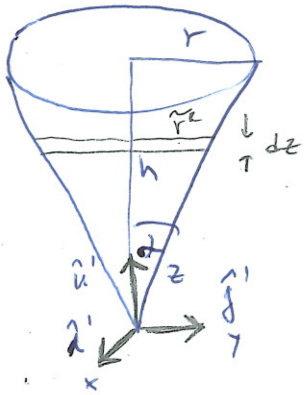
STOŽEC NA NAGNJENI PODLAGI - 2. Kolokvij 2016

Na ~~ne~~ nagnjeno podlago položimo stožec in ga upnemo v vrhu (okrog njega se lahko prosto vrti). Stožec ima višino h, radij osnovne ploskve r in maso M. Zapiši kinetično energijo stožca z Eulerjevimi koti. Stožec se kotali brez zdrsavanja. Izračunaj vztrajnostne momente, zapiši Lagrangeovo funkcijo in izračunaj frekvenco nihanja stožca za majhne odmike iz ravnovesne lege!

- Vztrajnostni momenti:



Koordinatni sistem $\hat{i}', \hat{j}', \hat{k}'$ je lastni sistem stožca, tako da je v njem $\underline{\underline{J}}$ diagonalen:

$$\underline{\underline{J}} = \begin{pmatrix} J & & \\ & J & \\ & & J' \end{pmatrix}$$

$J_x = J_y = J$ zaradi osne simetrije stožca okrog $\hat{k}'$.

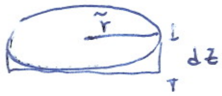
Volumen stožca:

$$V = \int dV = \pi \left(\frac{r}{h}\right)^2 \int_0^h z^2 dz = \frac{1}{3}\pi r^2 h$$

$$dV = \pi \tilde{r}^2 dz = \pi \left(\frac{r}{h}\right)^2 z^2 dz, \qquad \tilde{r} = \frac{r}{h} z$$

$$J' = \int dJ'(z) = \frac{1}{2}\left(\frac{r}{h}\right)^4 \rho \pi \int_0^h z^4 dz = \frac{1}{10} r^4 h \pi \overbrace{\frac{M}{\pi r^2 h \frac{1}{3}}}^{\rho} = \frac{3}{10} M r^2$$

$dJ'(z)$ - vztrajnostni moment diska pri z:



$$dJ'(z) = \frac{1}{2} dm \tilde{r}^2 = \frac{1}{2}\left(\frac{r}{h}\right)^2 z^2 \rho dV = \frac{1}{2}\rho \pi \left(\frac{r}{h}\right)^4 z^4 dz$$

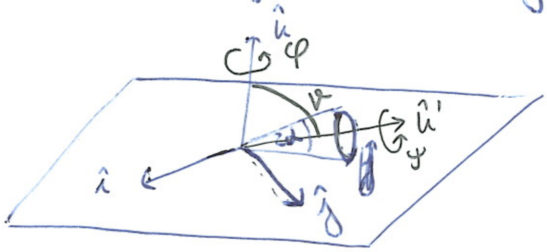
$$2J = J_x + J_y = \int \rho dV (x^2 + z^2 + y^2 + z^2) = \underbrace{\int \rho dV (x^2 + y^2)}_{J'} + 2\int \rho dV z^2$$

$$2\int \rho dV z^2 = 2\rho \pi \left(\frac{r}{h}\right)^2 \int_0^h z^4 dz = \frac{2}{5}\rho \pi r^2 h^3 = \frac{2}{5}\pi r^2 h^3 \frac{M}{\pi r^2 h/3} = \frac{6}{5} M h^2$$

$$\Rightarrow \boxed{J = \frac{3}{20} M r^2 + \frac{3}{5} M h^2 \quad ; \quad J' = \frac{3}{10} M r^2}$$

Kinetična energija:

Izrazimo jo z Eulerjevimi koti:



Lastna os vrtavke je za kot α, $\tan\alpha = \frac{r}{h}$ nad podlago.

Torej je:

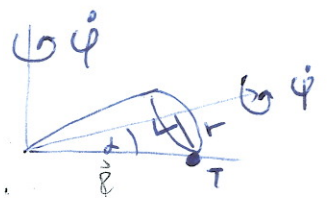
- ϑ: kot nutacije: $\pi/2 - \alpha$

  $\Rightarrow$ $\dot\vartheta = 0$ saj $\vartheta = konst$
- $\dot\psi$: vrtenje okrog osi vrtavke $\hat u'$
- $\dot\varphi$: vrtenje okrog osi $\hat u$.

Že izpeljan izraz za T:

$$T = \frac{1}{2} J \dot\varphi^2 \sin^2\vartheta + \frac{1}{2} J' (\dot\varphi \cos\vartheta + \dot\psi)^2$$

Ker se vrtavka kotali brez vrtenja, je njena hitrost na stiku s površino enaka 0:



V točki T ima hitrost: $\vec v = \vec{\dot\varphi} \times \vec R + \vec{\dot\psi} \times \vec r \equiv 0$

$v = \dot\varphi r + \dot\varphi \sqrt{r^2+h^2} \equiv 0$!

Torej velja zveza:

$$\dot\psi = -\frac{\sqrt{r^2+h^2}}{r}\dot\varphi$$

To in $\sin\vartheta = \cos\alpha = \frac{h}{\sqrt{r^2+h^2}}$ in $\cos\vartheta = \sin\alpha = \frac{r}{\sqrt{r^2+h^2}}$ vstavimo v T in dobimo:

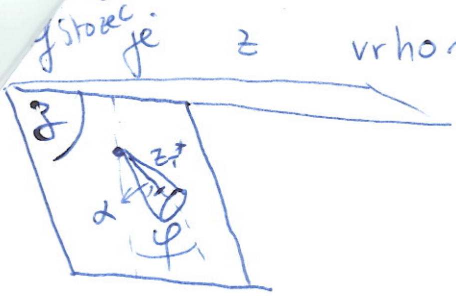
$$T = \frac{1}{2} J \dot\varphi^2 \frac{h^2}{r^2+h^2} + \frac{1}{2} J' \dot\varphi^2 \Big(\underbrace{\frac{r}{\sqrt{r^2+h^2}} - \frac{\sqrt{r^2+h^2}}{r}}_{\frac{1}{r\sqrt{r^2+h^2}}(r^2 - r^2 - h^2) = \frac{-h^2}{r\sqrt{r^2+h^2}}} \Big)^2$$

$$= \frac{1}{2}\dot\varphi^2 \left(J \frac{h^2}{r^2+h^2} + J' \frac{h^4}{r^2(r^2+h^2)} \right) = \frac{1}{2}\dot\varphi^2 \frac{h^2}{r^2+h^2}\left(J + J' \frac{h^2}{r^2} \right)$$

$$\boxed{\begin{aligned} T &= \frac{1}{2}\dot\varphi^2 J_{ef} \\ J_{ef} &= \frac{h^2}{r^2+h^2}\left(J + J'\frac{h^2}{r^2}\right) = \frac{h^2}{r^2+h^2}\left(\frac{3}{20}Mr^2 + \frac{3}{5}Mh^2 + \frac{3}{10}Mh^2\right) = \frac{h^2}{r^2+h^2} M \left(\frac{3}{20}r^2 + \frac{9}{10}h^2\right) \end{aligned}}$$

Potencialna energija:

stožec je z vrhom upet na nagnjeni plošči:

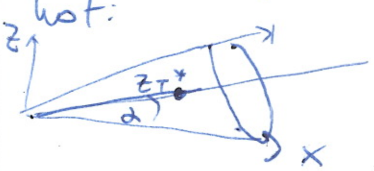


$$V = mg z_T$$

z_T izračunamo tako, da:

1. Izračun z_T^* na osi stožca
2. Zapis $\vec{r}_T'$, če leži v smeri x
3. Zapis $\vec{r}_T''$, ko ga zasukamo za φ okrog $\hat{h}$
4. Zapis $\vec{r}_T$, ko zasukamo ploščo za ζ okrog $\hat{j}$.

1.: z_T^*: oddaljenost težišča, ki leži na osi od vrha, izračunamo kot:

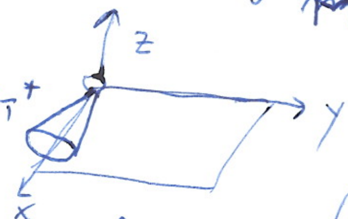


$$x_T^* = y_T^* = 0$$

$$z_T^* = \frac{\int dm\, z}{m} = \frac{\rho \int_0^h z\, \pi \left(\frac{r}{h}\right)^2 z^2 dz}{M} = \frac{\rho}{M} \pi \frac{r^2}{h^2} \frac{1}{4} h^4 = \frac{3}{4} h$$

$$V = \frac{1}{3}\pi r^2 h$$

2. $$\vec{r}_T' = \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} z_T^*$$



3. Zasuk za φ okrog $\hat{h}$:

$$\vec{r}_T'' = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ +\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ 0 \\ \sin\alpha \end{pmatrix} z_T^* = z_T^* \begin{pmatrix} \cos\varphi\cos\alpha \\ +\sin\varphi\cos\alpha \\ \sin\alpha \end{pmatrix}$$

4. Zasuk za ζ okrog $\hat{j}$:

$$\vec{r}_T = \begin{pmatrix} \cos\zeta & 0 & \sin\zeta \\ 0 & 1 & 0 \\ -\sin\zeta & 0 & \cos\zeta \end{pmatrix} \begin{pmatrix} \cos\varphi\cos\alpha \\ +\sin\varphi\cos\alpha \\ \sin\alpha \end{pmatrix} z_T^* = z_T^* \begin{pmatrix} * \\ * \\ -\sin\zeta\cos\alpha\cos\varphi + \cos\zeta\sin\alpha \end{pmatrix}$$

$$\Rightarrow z_T = z_T^* \left[\cos\zeta\sin\alpha - \sin\zeta\cos\alpha\cos\varphi\right]$$

Potencialna energija je torej: $V = V_0 - z_T^* \sin\zeta\cos\alpha\cos\varphi$

- $L = T - V = \frac{1}{2} J_{ef}\dot{\varphi}^2 + z_T^* \sin\zeta\cos\alpha\cos\varphi - V_0$
- Euler-Lagrangeove enačbe: $J_{ef}\ddot{\varphi} + z_T^* \sin\zeta\cos\alpha\sin\varphi = 0$

Pri majhnih odmikih je $\sin\varphi \approx \varphi$ in dobimo harmonsko nihanje s frekvenco:

$$\Omega^2 = \frac{3}{4} \frac{mgh \sin\zeta\cos\alpha}{J_{ef}}$$