DELEC V POTENCIALU $V(r) = -\frac{k}{r}$

Poišči vse možne orbite delca v $V(r) = -\frac{k}{r}$ potencialu.

$r(\varphi)$ dobimo iz enačbe za energijo:

$$H = \frac{1}{2}m\dot{r}^2 + V_{ef}(r)$$

$$V_{ef}(r) = V(r) + \frac{p_\varphi^2}{2mr^2}$$

$$\dot{r}^2 = \frac{2H}{m} - \frac{p_\varphi^2}{m^2}\frac{1}{r^2} + \frac{2k}{m}\frac{1}{r}$$

$$p_\varphi = mr^2\dot{\varphi} = \text{konst}$$

Vpeljemo $u = \frac{1}{r}$, da lažje rešimo to dif. enačbo

$$\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt}$$

Zamenjamo odvod po t z odvodom po φ:

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \frac{p_\varphi}{mr^2}\frac{d}{d\varphi}$$

$$\dot{r} = -\frac{1}{u^2}\frac{p_\varphi}{m}\cdot u^2\frac{du}{d\varphi} = -\frac{p_\varphi}{m}\frac{du}{d\varphi} = -\frac{p_\varphi}{m}u'$$

Vstavimo v DE:

$$\left(-\frac{p_\varphi}{m}\right)^2 u'^2 = \frac{2H}{m} - \frac{p_\varphi^2}{m^2}u^2 + \frac{2k}{m}u \quad \Big/ \left(\frac{m}{p_\varphi}\right)^2$$

Odvajamo še enkrat po φ:

$$2u'u'' = -2uu' + \frac{2km}{p_\varphi^2}u'$$

$$u'\left(u'' + u - \frac{km}{p_\varphi^2}\right) = 0$$

- $u' = 0 \Rightarrow$ Rešitev: $u(\varphi) = A \Rightarrow r(\varphi) = \frac{1}{A} \Rightarrow$ kroženje
- $u'' + u - \frac{km}{p_\varphi^2} = 0 \xrightarrow{\text{HOM}} u_H(\varphi) = B\cos(\varphi - \varphi_0)$ iz $u'' + u = 0$

$$\xrightarrow{\text{PART}} u_p(\varphi) = \frac{km}{p_\varphi^2}$$

Rešitev: $u(\varphi) = \frac{km}{p_\varphi^2} + B\cos(\varphi - \varphi_0)$

Izrazimo konstanto B iz energijo H:

Recimo $\varphi = \varphi_0$: $u(\varphi_0) = \frac{km}{p_\varphi^2} + B$ in $u'(\varphi_0) = 0 \Rightarrow \dot{r}(\varphi_0) = 0$

$$H = -k\,u(\varphi_0) + \frac{p_\varphi^2}{2m}u(\varphi_0)^2$$

$$H = -\frac{K^2 m}{p_\varphi^2} - KB + \frac{p_\varphi^2}{2m}\left[\left(\frac{Km}{p_\varphi^2}\right)^2 + B^2 + \cancel{2B\frac{Km}{p_\varphi^2}}\right]$$

$$H = -\frac{K^2 m}{2p_\varphi^2} + \frac{p_\varphi^2}{2m}B^2 \Rightarrow B = \pm\sqrt{\frac{2mH}{p_\varphi^2} + \frac{K^2 m^2}{p_\varphi^4}}$$

Vstavimo v enačbo:

$$r(\varphi) = \frac{1}{u(\varphi)} = \frac{1}{\frac{Km}{p_\varphi^2} - \sqrt{\frac{2mH}{p_\varphi^2} + \frac{K^2 m^2}{p_\varphi^4}}\cos(\varphi - \varphi_0)}$$

$\pm$ predznak, vseeno, ga φ_0 določa

$$r(\varphi) = \frac{p_\varphi^2/Km}{1 - \sqrt{1 + \frac{2Hp_\varphi^2}{K^2 m}}\cos(\varphi - \varphi_0)}$$

$$\boxed{r(\varphi) = \frac{p}{1 - \varepsilon\cos(\varphi - \varphi_0)} \quad ; \quad p = p_\varphi^2/Km \quad \varepsilon = \sqrt{1 + \frac{2Hp_\varphi^2}{mK^2}}}$$

Enačba stožnice!

- $\varepsilon < 0$: omejeno gibanje po elipsi: ko $H < 0$
- $\varepsilon = 0$: kroženje: ko $H = -\frac{mK^2}{2p_\varphi^2}$
- $\varepsilon = 1$: neomejeno gibanje po paraboli: $H = 0$
- $\varepsilon > 1$: neomejeno gibanje po hiperboli: $H > 0$.