THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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Appendix A
The dilatationally invariant system of units

That an electron here has the same mass as an electron there is also a triviality or a miracle. It is a triviality in quantum electrodynamics because it is assumed rather than derived. However, it is a miracle on any view that regards the universe as being from time to time "reprocessed".

—Charles W. Misner, Kip S. Thorne and John Archibald Wheeler[1]

We shall show how all the equations of physics can be cast in the system of units in which $\hbar = c = G = 4\pi\epsilon_0 = 1$. In spite of its usefulness for all sorts of calculations such a sytem of units is completely unknown.

Many authors of modern theoretical works use the system of units in which either $\hbar = c = 1$ or $c = G = 1$, etc.. This significantly simplifies equations and calculations, since various inessential $\hbar^3$, c^2, etc., are no longer present in formal expressions. But I have never seen the use of the next step, namely the units in which "all" fundamental constant are 1, that is $\hbar = c = G = 4\pi\epsilon_0 = 1$. Let us call such a system *the dilatationally invariant system of units*, briefly, the system D. It is introduced with the aid of the fine structure constant α, the Planck mass $M_{\rm P}$, the Planck time $T_{\rm P}$ and the Planck length $L_{\rm P}$ by setting $\hbar = c = G = 4\pi\epsilon_0 = 1$ in the usual MKSA expression for these quantities (Table A.1). That is, in the system D all quantities are expressed relative to the Planck units, which are dimensionless; the unit is 1. For practical reasons sometimes we will formally add the symbol D: so there holds 1 = 1D. With the aid of the formulas in Table A.1 we obtain the relation between the units MKSA and the units D (Table A.2).

[1]See ref. [136]

Table A.1. Physical constants in two systems of units

Description	*Symbol*	MKSA	D
Planck's constant/2π	$\hbar$	$1.0545887 \times 10^{-34}$ Js	1
Speed of light	c	$2.99792458 \times 10^{8}\mathrm{ms}^{-1}$	1
Gravitational constant	G	$6.6720 \times 10^{-11}\mathrm{kg}^{-1}\mathrm{m}^3\mathrm{s}^{-2}$	1
Dielectric constant of vacuum	ϵ_0	$8.8541876 \times 10^{-12}\mathrm{kg}^{-1}\mathrm{A}^2\mathrm{s}^4\mathrm{m}^{-3}$	$\frac{1}{4\pi}$
Induction constant of vacuum	μ_0	$1.2566371 \times 10^{-6}\mathrm{kg\,m\,s}^{-2}\mathrm{A}^{-2}$	4π
Electron's charge	e	$1.6021892 \times$ A s	$\alpha^{1/2}$
Electron's mass	m_e	9.109534×10^{-31} kg	$\kappa_0\alpha^{1/2}$
Boltzman constant	k_B	1.380622×10^{-23} J/^{o}K	1
Fine structure constant		$\alpha = 1/137.03604$	
'Fundamental scale'		$\kappa_0 = 0.489800 \times 10^{-21}$	
$e = \alpha^{1/2}(4\pi\epsilon_0\hbar c)^{1/2}$	$M_\mathrm{P} = (\hbar c/G)^{1/2}$	$T_\mathrm{P} = (\hbar G/c^5)^{1/2}$	$L_\mathrm{P} = (\hbar G/c^3)^{1/2}$

Let us now investigate more closely the system D. The Planck length is just the Compton wavelength of a particle with the Planck mass M_P

$$L_\mathrm{P} = \frac{\hbar}{M_\mathrm{P}c} \text{ (system MKSA)}, \qquad L_\mathrm{P} = \frac{1}{M_\mathrm{P}} = M_\mathrm{P} = 1 \text{ (system D)} \quad (A.1)$$

In addition to M_P and L_P we can introduce

$$M = \alpha^{1/2}M_\mathrm{P}\ , \qquad L = \alpha^{1/2}L_\mathrm{P}. \quad (A.2)$$

In the system D it is

$$e = M = L = \alpha^{1/2}. \quad (A.3)$$

The length L is the classical radius that a particle with mass M and charge e would have:

$$L = \frac{e^2}{4\pi\epsilon_0 Mc^2} \text{ (system MKSA)}, \qquad L = \frac{e^2}{M} \text{ (system D)}. \quad (A.4)$$

The classical radius of electron (a particle with the mass m_e and the charge e) is

$$r_c = \frac{e^2}{4\pi\epsilon_0 m_\mathrm{e}c^2} \text{ (system MKSA)}, \qquad r_c = \frac{e^2}{m_\mathrm{e}} \text{ (system D)}. \quad (A.5)$$

From (A.4) and (A.5) we have in consequence of (A.1) and Table A.1

$$\frac{r_\mathrm{c}}{L} = \frac{M}{m_\mathrm{e}} = \frac{e}{m_\mathrm{e}}(4\pi\epsilon_0 G)^{-1/2} \equiv \kappa_0^{-1} \quad (A.6)$$

Therefore

$$m_\mathrm{e} = \kappa_0 M = \kappa_0\alpha^{1/2} = \kappa_0 e \text{ (system D)}. \quad (A.7)$$

The ratio L/r_c represents the scale of the electron's classical radius relative to the length L. As a consequence of (A.4)–(A.7) we have $r_c = \kappa_0^{-1} e$.

Table A.2. Translation between units D and units MKSA

$$
\begin{aligned}
&1\mathrm{D} = (\hbar c/G)^{1/2} = 2.1768269 \times 10^{-8}\ \mathrm{kg}\\
&1\mathrm{D} = (\hbar G/c^5)^{1/2} = 5.3903605 \times 10^{-44}\ \mathrm{s}\\
&1\mathrm{D} = (\hbar G/c^3)^{1/2} = 1.6159894 \times 10^{-35}\ \mathrm{m}\\
&1\mathrm{D} = (4\pi\epsilon_0 \hbar c)^{1/2} = \alpha^{-1/2} e = 1.8755619 \times 10^{-18}\ \mathrm{As}\\
&1\mathrm{D} = c^3 (4\pi\epsilon_0/G)^{1/2} = 3.4794723 \times 10^{25}\ \mathrm{A}\\
&1\mathrm{D} = c^2 (4\pi\epsilon_0 G)^{-1/2} = 1.0431195 \times 10^{27}\ \mathrm{V}\\
&1\mathrm{D} = c^2 (\hbar c/G)^{1/2} = 1.9564344 \times 10^{9}\ \mathrm{J}\\
&1\mathrm{D} = 1.41702 \times 10^{32}\ {}^0\mathrm{K}
\end{aligned}
$$

At this point let us observe that the fundamental constants $\hbar$, c, G and ϵ as well as the quantities L_P, M_P, T_P, L, M, e, are by definition invariant under dilatations. The effect of a dilatation on various physical quantities, such as the spacetime coordinates x^μ, mass m, 4-momentum p_μ, 4-force f^μ, and 4-acceleration a^μ is [137]–[139]:

$$
\begin{aligned}
x^\mu &\rightarrow x'^\mu = \rho x^\mu,\\
p_\mu &\rightarrow p'_\mu = \rho^{-1} p_\mu,\\
m &\rightarrow m' = \rho^{-1} m,\\
f^\mu &\rightarrow f'^\mu = \rho^{-2} f^\mu,\\
a^\mu &\rightarrow a'^\mu = \rho^{-1} a^\mu.
\end{aligned} \tag{A.8}
$$

Instead of the *inhomogeneous coordinates* x^μ one can introduce [137]–[139] the *homogeneous coordinates* $\tilde{x}^\mu = \kappa x^\mu$ which are invariant under dilatations provided that the quantity κ transforms as

$$\kappa \rightarrow \kappa' = \rho^{-1} \kappa. \tag{A.9}$$

For instance, if initially $x^0 = 1$ sec, then after applying a dilatation, say by the factor $\rho = 3$, we have $x'^0 = 3$ sec, $\kappa = \frac{1}{3}$, $\tilde{x}'^0 = \tilde{x}^0 = 1$ sec. The quantity κ is the *scale* of the quantity x^μ relative to the corresponding invariant quantity $\tilde{x}^\mu$.

If we write a given equation we can check its consistency by comparing the dimension of its left hand and right hand side. In the MKSA system the dimensional control is in checking the powers of meters, kilograms, seconds and ampères on both sides of the equation. In the system D one has to verify that both sides transform under dilatations as the same power of

Table A.3. Some basic equations in the two systems of units

Description	*Symbol*	MKSA	D
Electric force between two electrons	F_e	$\dfrac{e^2}{4\pi\epsilon_0 r^2}$	$\dfrac{e^2}{r^2}$
Electric force between two electrons at the distance r_c	F_e	$\dfrac{e^2}{4\pi\epsilon_0 r_c^2}$	κ_0^2
Gravitational force between two electrons at the distance r_c	F_G	$\dfrac{\kappa^{-2} G m_e^2}{r_c^2}$	$\kappa^{-2}\kappa_0^4$
Ratio between electric and gravitational force	$\dfrac{F_e}{F_G}$	$\dfrac{e^2}{4\pi\epsilon_0 G m_e^2}$	$\kappa^2\kappa_0^{-2}$
Bohr radius	a_0	$\dfrac{4\pi\epsilon_0\hbar^2}{m_e e^2}$	$\kappa_0^{-1}e^{-3}$
Potential energy of electron at the distance a_0 from the centre	E_c	$\dfrac{e^2}{4\pi\epsilon_0 a_0}$	$\kappa_0 e^5$
Rydberg constant	Ry	$\dfrac{m_e e^4}{2(4\pi\epsilon_0\hbar)^2}$	$\dfrac{1}{2}\kappa_0 e^5$

ρ. For instance, eq. (A.7) is consistent, since $[m_\mathrm{e}] = \rho^{-1}$, $[\kappa_0] = \rho^{-1}$ and $[e] = 1$, where $[A]$ denotes the dimension of a generic quantity A.

In Table A.3 some well known equations are written in both systems of units. They are all covariant under dilatations. Taking G invariant, the equation $F = Gm^2/r^2$ is not dilatationally covariant, as one can directly check from (A.8). The same is true for the Einstein equations $G_{\mu\nu} = -8\pi G T^{\mu\nu}$ with $T^{\mu\nu} = (\rho + p)u^\mu u^\nu - p\, g^{\mu\nu}$, from which the Newtonian gravitation equation is derivable. Usually this non-covariance is interpreted as the fact that the gravitational coupling constant G is not dimensionless. One can avoid this difficulty by using the homogeneous coordinates $\tilde{x}^\mu$ and express the Einstein tensor $G^{\mu\nu}$, the rest mass density ρ and all other relevant quantities in terms of these homogeneous coordinates [139, 140]. Then the Einstein equations become $\widetilde{G}^{\mu\nu} = -8\pi G \widetilde{T}^{\mu\nu}$ with $\widetilde{T}^{\mu\nu} = (\tilde{\rho} + \tilde{p})\tilde{u}^\mu \tilde{u}^\nu - \tilde{p}\, \tilde{g}^{\mu\nu}$, where the quantities with tildes are invariant under dilatations. If the homogeneous Einstein equations are written back in terms of the inhomogeneous quantities, we have $G^{\mu\nu} = -8\pi G\kappa^{-2}T^{\mu\nu}$, which is covariant with respect to dilatations. If we choose $\kappa = 1$ then the equations for this particular choice correspond to the usual Einstein equations, and we may use either the MKSA system or the D system. Further discussion of this interesting and important subject would go beyond the scope of this book. More about the dilatationally and conformally covariant theories the reader will find in refs. [137]–[141].

Using the relations of Table A.2 all equations in the D system can be transformed back into the MKSA system. Suppose we have an equation in the D system:

$$a_0 = \frac{1}{m_\mathrm{e}e^2} = \kappa_0^{-1}e^{-3} = \kappa_0^{-1}\alpha^{-3/2} = \kappa_0^{-1}\alpha^{-3/2}\,\mathrm{D}. \tag{A.10}$$

We wish to know what form the latter equation assumes in the MKSA system. Using the expression for the fine structure constant $\alpha = e^2(4\pi\epsilon_0\hbar c)^{-1}$ and eq. (A.6) we have

$$a_0 = \frac{e}{m_\mathrm{e}(4\pi\epsilon_0 G)^{1/2}}\left(\frac{e^2}{4\pi\epsilon_0\hbar c}\right)^{-3/2}\mathrm{D}. \tag{A.11}$$

If we put 1D = 1 then the right hand side of the latter equation is a dimensionless quantity. If we wish to obtain a quantity of the dimension of length we have to insert $1\mathrm{D} = (\hbar G/c^3)^{1/2}$, which represents translation from meters to the D units. So we obtain

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{m_\mathrm{e}e^2}, \tag{A.12}$$

which is the expression for Bohr's radius.

Instead of rewriting equations from the D system in the MKSA system, we can retain equations in the D system and perform all the algebraic and numerical calculations in the D units. If we wish to know the numerical results in terms of the MKSA units, we can use the numbers of Table A.2. For example,

$$a_0 = \kappa_0^{-1}e^{-3} = 2.04136 \times 10^{21} \times 137.03604^{3/2}\mathrm{D}. \tag{A.13}$$

How much is this in meters? From Table A.2 we read $1\ \mathrm{D} = 1.615989^{-35}$ m. Inserting this into (A.13) we have $a_0 = 0.529177 \times 10^{-10}$m which is indeed the value of Bohr's radius.

Equations in the D system are very simple in comparison with those in the MKSA system. Algebraic calculations are much easier, since there are no inessential factors like $\hbar^2$, c^3, etc., which obsure legibility and clarity of equations. The transformation into the familiar MKSA units is quick with the aid of Table A.2 (and modern pocket calculators, unknown in the older times from which we inherit the major part of present day physics). However, I do not propose to replace the international MKSA system with the D system. I only wish to recall that most modern theoretical works do not use the MKSA system and that it is often very tedious to obtain the results in meters, seconds, kilograms and ampères. What I wish to point out here is that even when the authors are using the units in which, for example, $\hbar = c = 1$, or similar, we can easily transform their equations into the units in which $\hbar = c = G = 4\pi\epsilon_0 = 1$ and use Table A.2 to obtain the numerical results in the MKSA system.

To sum up, besides the Planck length, Planck time and Planck mass, which are composed of the fundamental constants $\hbar$, c and G, we have also introduced (see Table A.2) the corresponding electromagnetic quantity, namely the charge $E_{\rm P} = (4\pi\epsilon_0\hbar c)^{1/2}$ (or, equivalently, the current and the potential difference), by bringing into play the fundamental constant ϵ_0. We have then extended *the Planck system* of units [136] in which $c = \hbar = G = 1$ to the system of units in which $c = \hbar = G = 4\pi\epsilon_0 = 1$, in order to incorporate all known sorts of physical quantities.

Finally, let me quote the beautiful paper by Levy-Leblond [142] in which it is clearly stated that our progress in understanding the unity of nature follows the direction of eliminating from theories various (inessential) numerical constants with the improper name of "fundamental" constants. In fact, those constants are merely the constants which result from our unnatural choice of units, the choice due to our incomplete understanding of the unified theory behind.