BEYOND SPACETIME: ON THE CLIFFORD ALGEBRA BASED GENERALIZATION OF RELATIVITY

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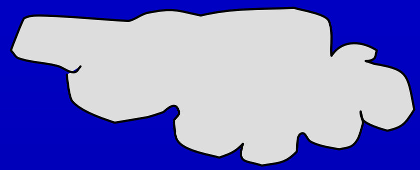
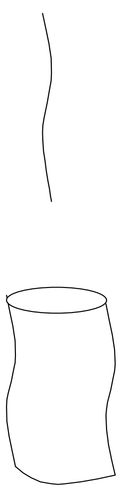
Contents

- Strings, branes

Theories of strings and higher dimensional extended objects, branes
- very promising in explaining the origin and interrelationship of the fundamental interactions, including gravity



But there is a cloud:
- what is a geometric principle behind string and brane theories and how to formulate them in a background independent way



$$I[g_{\mu\nu}] = \int \sqrt{-g}\, R \, \mathrm{d}^4 x$$

?

Configuration space for infinite dimensional objetcs - branes

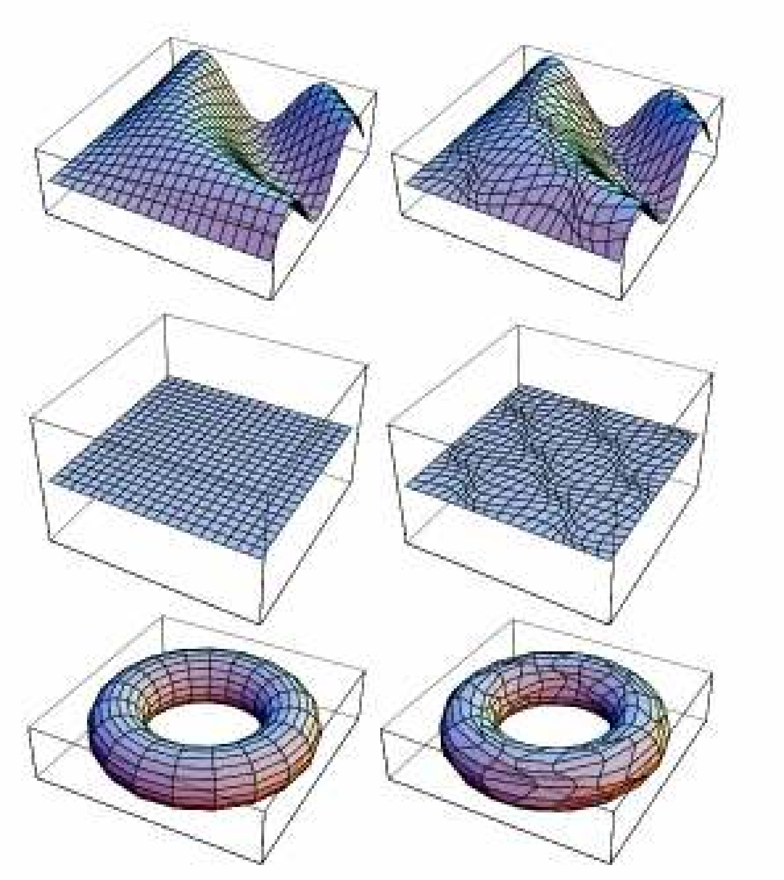
A brane can be considered as a point in infinite dimensional space with coordinates

$$X^{\mu}(\xi^{a}) \equiv X^{\mu(\xi)} \equiv X^{M}$$

This includes classes of tangentially deformed branes which we can interpret as physically different objects, not just reparametrizations.

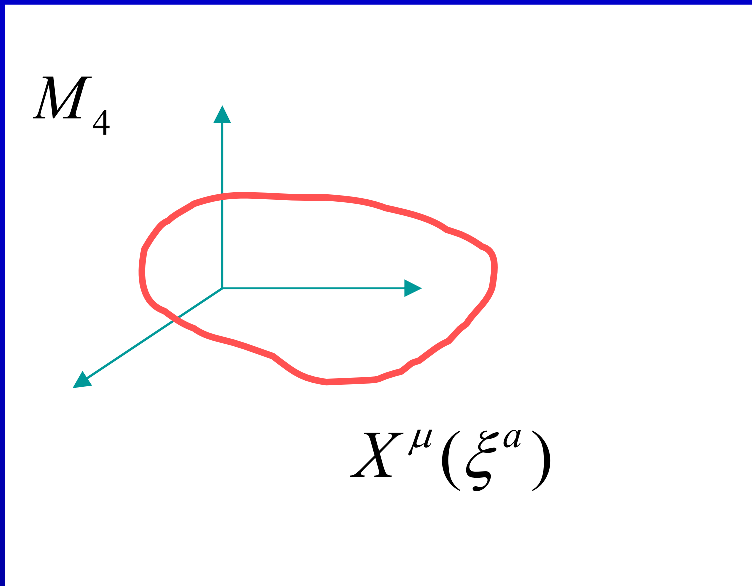
Mathematically the surfaces on the left and the right are the same.
Physically they are different.

They are represented by two different points in configuration space C

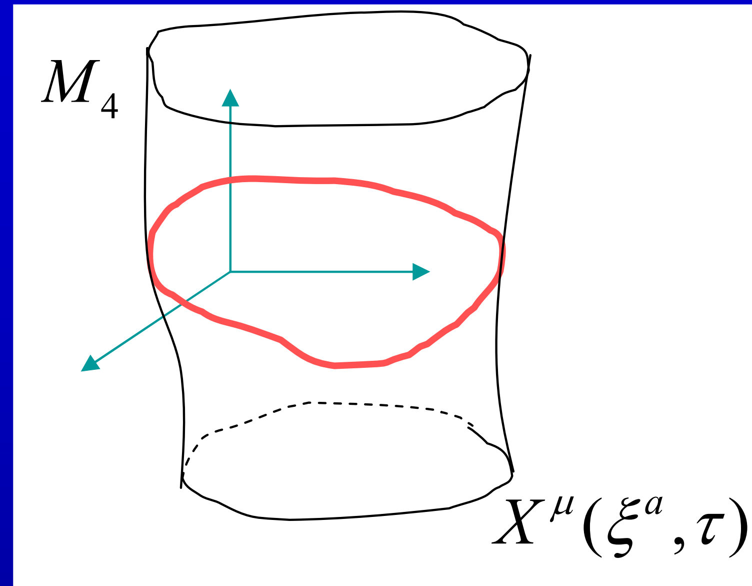


For the configuration space associated with a brane we will also use the name brane space $\mathcal{M}$

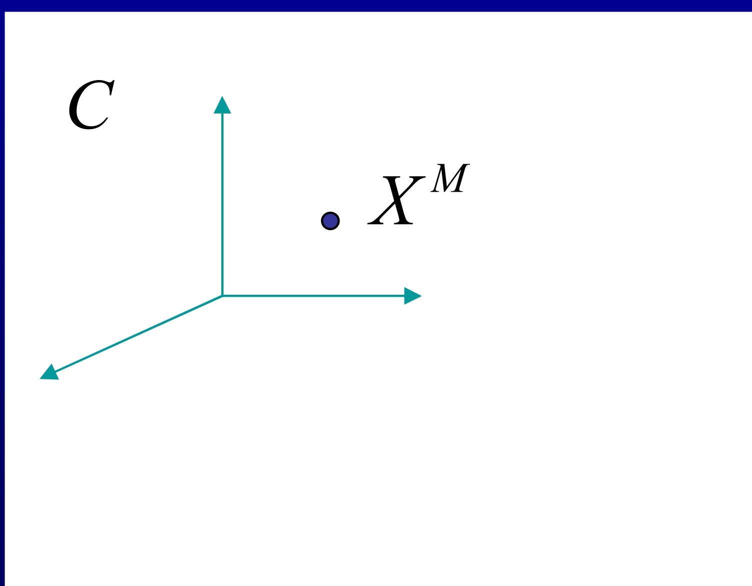
‘Instantaneous’ brane configuration in M_4

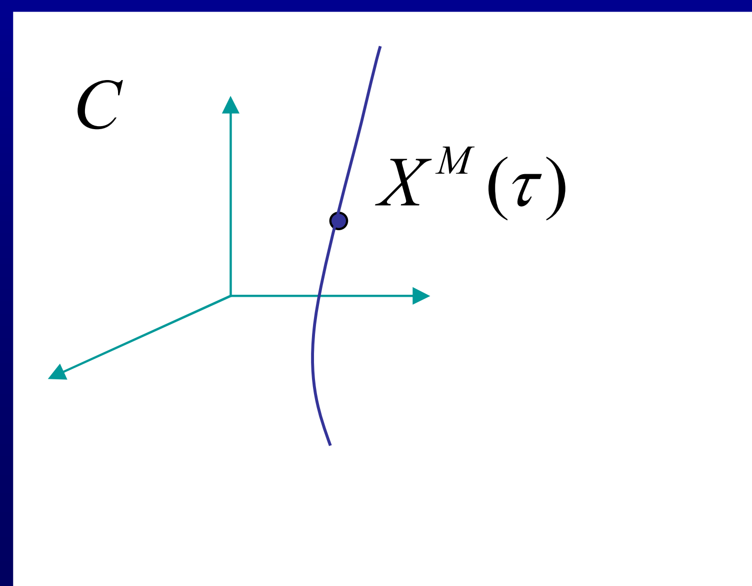


‘Evolution’ of a brane configuration in M_4



Representation in configuration space C





Action in the brane space $\mathcal{M}$

$$I[X^M] = \int \mathrm{d}\tau \, (\rho_{MN} \dot{X}^M \dot{X}^N)^{(1/2)}$$

Short hand notation

$$M \equiv \mu(\xi) \, , \qquad X^M \equiv X^{\mu(\xi)} \equiv X^\mu(\xi)$$

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} \right)^{1/2}$$

More explicit notation

If metric is given by

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \eta_{\alpha\beta}$$

$$f \equiv \det f_{ab} \, , \qquad f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

$$\dot{X}^2 \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

then the corresponding equations of motion are precisely those of a Dirac-Nambu-Goto brane!

In this theory we assume that the metric above is just one particular chose amongst many other possible metrics that are solution to the Einstein equations in the configuration space.

For more details see:

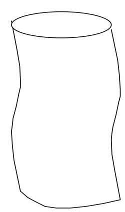
M. Pavšič: The Landscape of theoretical Physics (Kluwer, 2001), gr-qc/0610061 ; hep-th/0311060

We have taken the brane space $\mathcal{M}$ seriously as an arena for physics.

The arena itself is also a part of the dynamical system, it is not prescribed in advance.

The theory is thus background independent. It is based on the geometric principle which has its roots in the brane space $\mathcal{M}$

$$I[g_{\mu\nu}] = \int \mathrm{d}^4 x \sqrt{|g|}\, R$$



$$I[\rho_{\mu(\phi)\nu(\phi')}] = \int \mathcal{D}X \sqrt{|\rho|}\, \mathcal{R}$$

$$\phi \equiv \phi^A = (\tau, \xi^A)$$

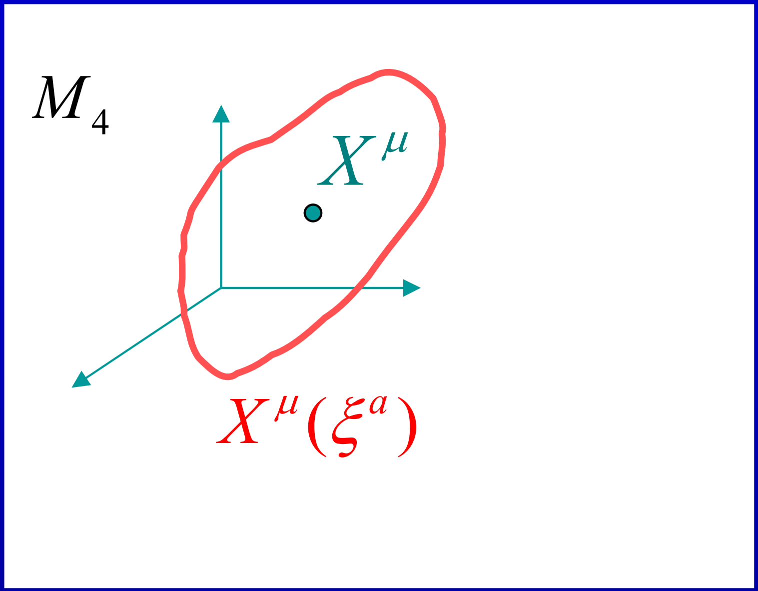
There is no pre-existing space and metric: they appear dynamically as solutions to the equations of motion.

Finite dimensional description of extended objects

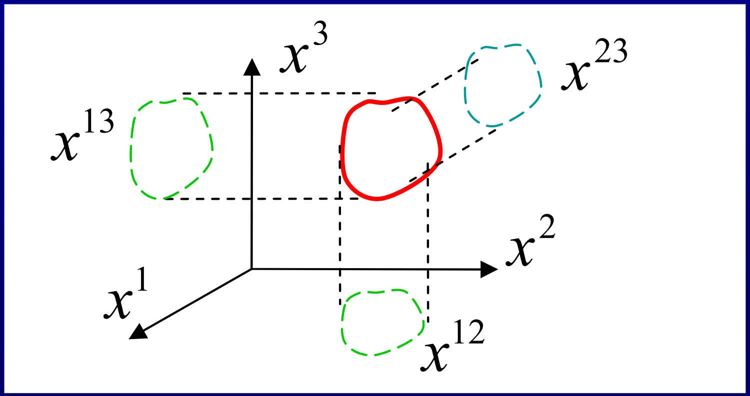
The Earth has a huge (practically infinite) number of degree of freedom. And yet, when describing the motion of the Earth around the Sun, we neglect them all, except for the coordinates of the centre of mass.

Instead of infinitely many degrees of freedom associated with an extended object, we may consider a finite number of degrees of freedom.

Strings and branes have infinitely many degrees of freedom.
But at first approximation we can consider just the centre of mass.

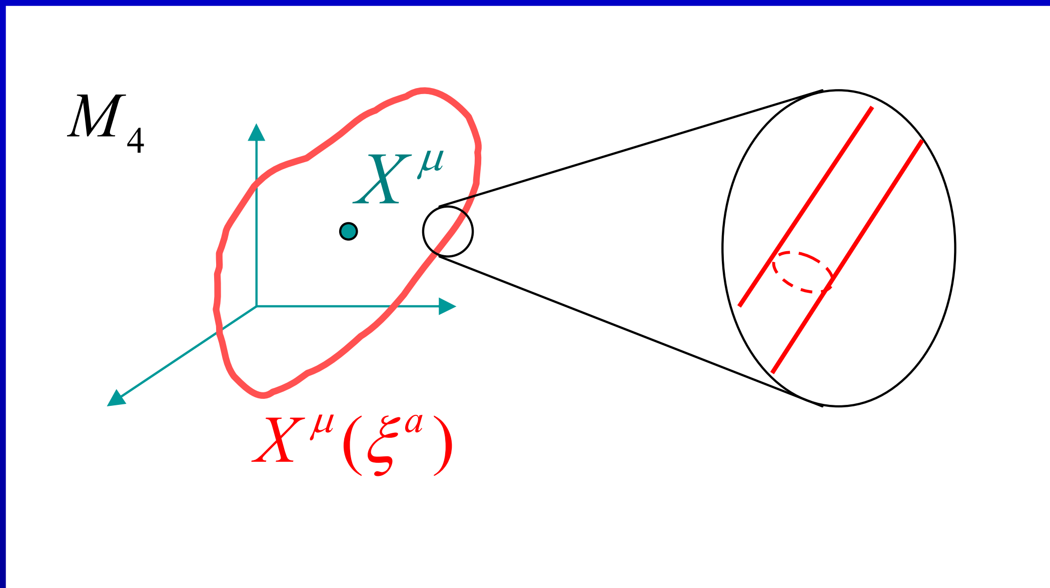


Next approximation is in considering the holographic coordinates of the oriented area enclosed by the string.



We may go further and search for eventual thickness of the object.

If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom.



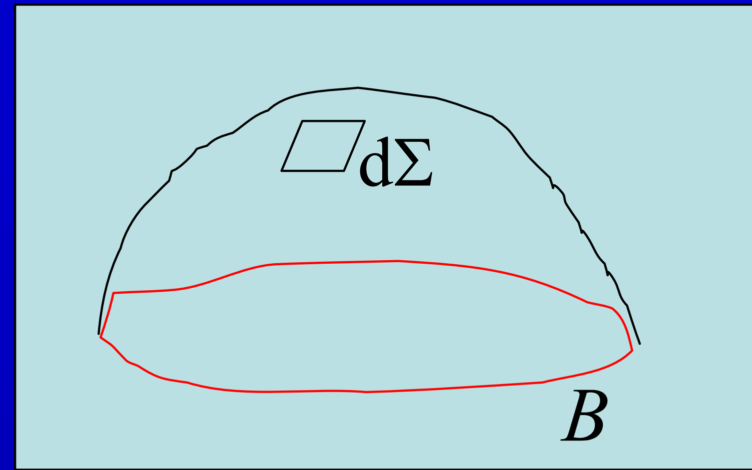
X^{123}

In general, for an extended object in M_4, we have 16 coordinates

$$x^M \equiv x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

They are the projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra.



$$\mathrm{d}\Sigma = \mathrm{d}\xi_1 \wedge \mathrm{d}\xi_2 = \mathrm{d}\xi_1^a\, \mathrm{d}\xi_2^b\, e_a \wedge e_b = \frac{1}{2}\mathrm{d}\xi^{ab}\, e_a \wedge e_b$$

$$\mathrm{d}\xi^{ab} = \mathrm{d}\xi_1^a\, \mathrm{d}\xi_2^b - \mathrm{d}\xi_2^a\, \mathrm{d}\xi_1^b$$

$$e_a = \partial_a X^\mu \gamma_\mu$$

$X^{\mu\nu}$

$$\int_{\Sigma_B} \mathrm{d}\Sigma \equiv \frac{1}{2} X^{\mu\nu}\, \gamma_\mu \wedge \gamma_\nu = \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab} \partial_a X^\mu \partial_b X^\nu\, \gamma_\mu \wedge \gamma_\nu$$

$$= \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab}\, \frac{1}{2}(\partial_a X^\mu \partial_b X^\nu - p_a X^\nu \partial_b X^\mu)\gamma_\mu \wedge \gamma_\nu$$

$$X^{\mu\nu}[B] = \frac{1}{2}\int_{\Sigma_B} \mathrm{d}\xi^{ab} (\partial_a X^\mu \partial_b X^\nu - \partial_a X^\nu \partial_b X^\mu)$$

$$X^{\mu\nu}[B] = \tfrac{1}{2}\oint_B \mathrm{d}s \left(X^\mu \frac{\partial X^\nu}{\partial s} - X^\nu \frac{\partial X^\mu}{\partial s} \right)$$

Mapping :

$$X^\mu\left(\xi^a\right) \longrightarrow X^{\mu\nu}$$

Instead of the usual relativity formulated in spacetime in which the interval is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$\mathrm{d}S^2 = G_{MN}\,\mathrm{d}x^M \mathrm{d}x^N \qquad \mathrm{d}x^M \equiv \mathrm{d}x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

Coordinates of Clifford space can be used to model extended objects. They are a generalization of the concept of center of mass.

Instead of describing extended objects in ``full detail'', we can describe them in terms of the center of mass, area and volume coordinates.

In particular, extended objects can be fundamental strings or branes.

Quadratic form in C-space

$$\mathrm{d}\, S^2 \equiv |\,\mathrm{d}X\,|^2 \equiv \mathrm{d}X^{\ddagger} * \mathrm{d}X = \mathrm{d}x^M \mathrm{d}x^N G_{MN} \equiv \mathrm{d}x^M \mathrm{d}x_M$$

where

$$\mathrm{d}X = \mathrm{d}x^M \gamma_M \equiv \mathrm{d}x^{\mu_1 \mu_2 \ldots \mu_r} \gamma_{\mu_1 \mu_2 \ldots \mu_r}\,, \qquad r = 0,1,2,3,4$$

Metric

$$G_{MN} = \gamma_M^{\ddagger} * \gamma_N \equiv \langle \gamma_M^{\ddagger}\, \gamma_N \rangle_0$$

Reversion

$$(\gamma_{\mu_1} \gamma_{\mu_2} \ldots \gamma_{\mu_r})^{\ddagger} = \gamma_{\mu_r} \ldots \gamma_{\mu_2} \gamma_{\mu_1}$$

Signature: $+ + + + + + + + - - - - - - - -$ (8,8)

In flat C-space:

$$\gamma_{\mu_1 \mu_2 \ldots \mu_r} = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \ldots \wedge \gamma_{\mu_r}$$

at every point $\mathcal{E} \in C$

Dynamics

Action:

$$I = \int d\tau \, (\eta_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Generalization of ordinary relativity

Equations of motion:

$$\ddot{X}^M \equiv \frac{\mathrm{d}^2 X^M}{\mathrm{d}\tau^2} = 0$$

These equations imply area (volume) motion

Metric:

$$\eta_{MN}$$

Diagonal metric

Signature:

$$+ + + + + + + + - - - - - - - -$$

(8,8)

The above dynamics holds for tensionless branes.
For the branes with tension one has to introduce
curved Clifford space.

Example: the Dirac membrane

$$X^\mu(\xi^a) = (X^0,\, r\sin\vartheta\cos\varphi,\, r\sin\vartheta\sin\varphi,\, r\cos\vartheta)$$

$$\xi^a = (\tau, \vartheta, \varphi)\,, \qquad X^0 = X_0$$

$$\gamma_{ab} = \begin{pmatrix} \dot{X}_0^2 - \dot{r}^2 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2\sin^2\vartheta \end{pmatrix}$$

X^0, r functions of τ

$$\sqrt{|\det\gamma|} \equiv \sqrt{|\gamma|} = \sqrt{\dot{X}_0^2 - \dot{r}^2}\; r^2 \sin\vartheta$$

$$I = \int d\tau\, d\vartheta\, d\varphi \sqrt{|\gamma|} = \int d\tau\, 4\pi r^2 \sqrt{\dot{X}_0^2 - \dot{r}^2}$$

Variation with respect to r and X^0

$$\frac{d}{d\tau}\left(\frac{\dot{r}}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) + \frac{2\dot{X}_0^2}{r\sqrt{\dot{X}_0^2 - \dot{r}^2}} = 0$$

$$\frac{d}{d\tau}\left(\frac{r^2\dot{X}_0}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) = 0$$

Example: the Dirac membrane

$$X^{\mu}(\xi^a) = (X^0,\ r\sin\vartheta\cos\varphi,\ r\sin\vartheta\sin\varphi,\ r\cos\vartheta)$$

$$\xi^a = (\tau, \vartheta, \varphi)\ , \qquad X^0 = X_0$$

$$\gamma_{ab} = \begin{pmatrix} \dot X_0^2 - \dot r^2 & 0 & 0 \\ 0 & -r^2 & 0 \\ 0 & 0 & -r^2\sin^2\vartheta \end{pmatrix}$$

$$\sqrt{|\det\gamma|} \equiv \sqrt{|\gamma|} = \sqrt{\dot X_0^2 - \dot r^2}\; r^2 \sin\vartheta$$

$$I = \int \mathrm{d}\tau\, \mathrm{d}\vartheta\, \mathrm{d}\varphi \sqrt{|\gamma|} = \int \mathrm{d}\tau\, 4\pi r^2 \sqrt{\dot X_0^2 \ldots}$$

Va

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot r}{\sqrt{\dot X_0^2 - \dot r^2}}\right) + \frac{2\dot X_0^2}{r\sqrt{\dot X_0^2 - \dot r^2}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r^2\,\dot X_0}{\sqrt{\dot X_0^2 - \dot r^2}}\right) = 0$$

$$I = \int \mathrm{d}S\,, \qquad \mathrm{d}S = \mathrm{d}\tau\, 4\pi r^2 \sqrt{\dot X_0^2 - \dot r^2}$$

$$X^{123} = \frac{1}{3!}\int \mathrm{d}r\, \mathrm{d}\vartheta\, \mathrm{d}\varphi\ \partial_{[a} X^1 \partial_b X^2 \partial_{c]} X^3 = \frac{4\pi r^3}{3}$$

$$\dot X^{123} = 4\pi r^2 \dot r$$

$$\frac{dX^{123}}{dS} = \frac{\dot X^{123}}{4\pi r^2 \sqrt{\dot X_0^2 - \dot r^2}} = \frac{\dot r}{\sqrt{\dot X_0^2 - \dot r^2}}$$

$$\frac{\mathrm{d}^2 X^{123}}{\mathrm{d}S^2} + \frac{2}{3X^{123}}\left(1 + \left(\frac{\mathrm{d}X^{123}}{\mathrm{d}S}\right)^2\right) = 0$$

Equation in new variables

Action in C-space

$$I[X^M] = \int \mathrm{d}S = \int d\tau \,(G_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

δX^M

$$\frac{1}{\sqrt{\dot{X}^2}} \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}^M}{\sqrt{\dot{X}^2}} \right) + \Gamma^M_{JK} \frac{\dot{X}^J \dot{X}^K}{\dot{X}^2} = 0$$

Let us consider a subspace $X^M = (X^0, X^{123})$

with the metric $G_{MN} = \begin{pmatrix} C\tilde{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix}$

$\tilde{X} \equiv X^{123}$

$$\frac{\mathrm{d}^2 X^{123}}{\mathrm{d}S^2} + \frac{2}{3X^{123}}\left(1 + \left(\frac{\mathrm{d}X^{123}}{\mathrm{d}S}\right)^2\right) = 0$$

The same equation as obtained directly for the Dirac membrane

Action in C-space

$$I[X^M] = \int dS = \int d\tau \, (G_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Let us consider a subspace $X^M = (X^0, X^{123})$

with the metric

$$G_{MN} = \begin{pmatrix} C\tilde{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{X} \equiv X^{123}$$

$$dS^2 = G_{00}(dX^0)^2 + G_{\tilde{X}\tilde{X}} d\tilde{X}^2$$

$$dS^2 = C\tilde{X}^{4/3}(dX^0)^2 - d\tilde{X}^2$$

$$\tilde{X} = \frac{4\pi r^3}{3}, \qquad d\tilde{X} = 4\pi r^3 dr$$

$$\tilde{X}^{4/3} = \left(\frac{4\pi}{3}\right)^{4/3} r^4$$

$$C\left(\frac{4\pi}{3}\right)^{4/3} = (4\pi)^2$$

$$dS^2 = (4\pi r^2)^2 \left(d(X^0)^2 - dr^2 \right)$$

$$I = \int d\tau \, (4\pi r^2)^2 \sqrt{(\dot{X}^0)^2 - \dot{r}^2}$$

The C-space action for this particular case is equivalent to the action for the Dirac membrane

C-space is a straightforward generalization of spacetime manifold M.

Choosing a point $\mathcal{P}$ of M,
the tangent space at $\mathcal{P}$ is the vector space $V_{1,3}$

$\bullet\mathcal{P}$

$$\gamma_\mu \in V_{1,3}$$

Generators of Clifford algebra

$$T_{\mathcal{P}}(M) = V_{1,3}$$

Choosing a point $\mathcal{P}_0$ as the origin, vectors ,

$\bullet\mathcal{P}_0$

$$x^\mu \gamma_\mu |_{\mathcal{P}_0} \in T_{\mathcal{P}_0}(M) = \mathbb{R}^{1,3}$$

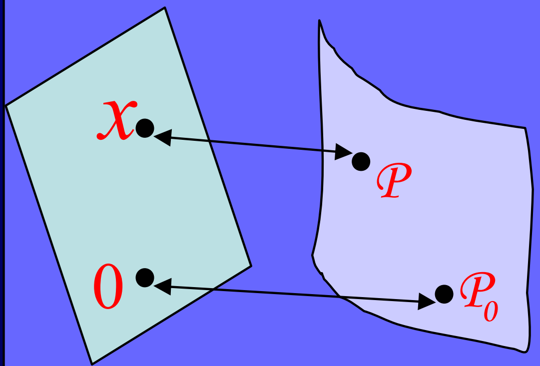
can be put into one-to one correspondence
with other point $\mathcal{P}$ of a region $B \subseteq M$

$$\mathbb{R}^{1,3} \leftrightarrow M$$

x^μ are then coordinates of $\mathcal{P}$

Position in M is described by vector

$$x \equiv x^\mu \gamma_\mu |_{\mathcal{P}_0}$$



$\mathcal{E}$ •

Choosing a point $\mathcal{E}$ of C ,
the tangent space at $\mathcal{E}$ is the Clifford algebra $Cl_{1,3}$

$$T_{\mathcal{E}}(C) = Cl_{1,3}$$

$$\gamma_{\mu_1 \mu_2 \ldots \mu_r} \equiv \gamma_M \in Cl_{1,3}$$

Basis elements of Clifford algebra

Isomorphic as a vector space

$\mathcal{E}_0$ •

Choosing a point $\mathcal{E}_0$ as the origin , polyvectors ,

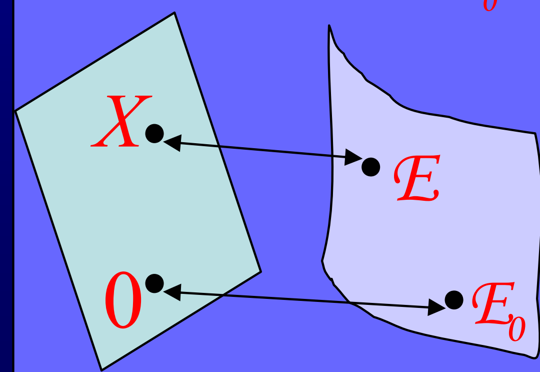
$$x^M \gamma_M \big|_{\mathcal{E}_0} \in T_{\mathcal{E}_0}(C) \sim \mathbb{R}^{8,8}$$

can be put in one-to one correspondence
with other point $\mathcal{E}$ of a region $\Omega \subseteq C$

$\mathbb{R}^{8,8} \leftrightarrow C$ $\qquad$ x^M are then coordinates of $\mathcal{E}$

Position in C is described by a polyvector

$$X \equiv x^M \gamma_M \big|_{\mathcal{E}_0}$$



Curved Clifford space

Coordinate basis

$$\gamma_M \equiv \gamma_{\mu_1 \dots \mu_n}$$

Depends on position $X = x^M \gamma_M |_{\mathcal{E}_0}$
No longer defined as wedge product

Orthonormal basis

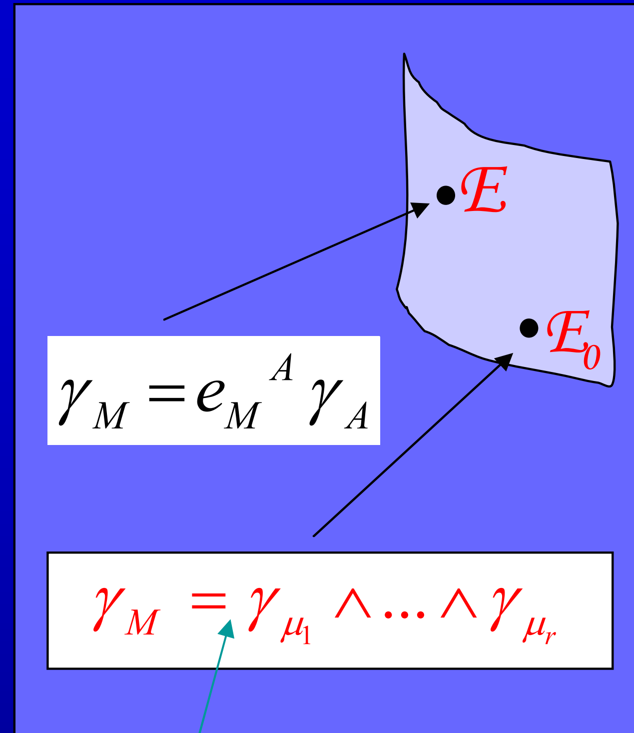
$$\gamma_A = \gamma_{a_1 a_2 \dots a_n} = \gamma_{a_1} \wedge \gamma_{a_2} \wedge \dots \wedge \gamma_{a_n}$$

C-space vielbein

$$\gamma_M = e_M{}^A \gamma_A$$

Indefinite grade

Definite grade



This may hold at point $\mathcal{E}_0$
but not at point $\mathcal{E}$

$$\gamma_A^{\ddagger} * \gamma_B = \eta_{AB}$$

Metric of the tangent space spanned by γ_A

$$\gamma_M^{\ddagger} * \gamma_M = g_{MN}$$

Metric of Clifford space

Derivative

$$\partial_M \phi = \frac{\partial \phi}{\partial x^M}$$

ϕ Scalar

$$\partial_M \gamma_N = \Gamma^J_{MN}\,\gamma_J$$

Connection for a coordinate frame field

$$\partial_M \gamma_A = -\Omega_A{}^B{}_M\,\gamma_B$$

Connection for orthonormal frame field

$$\partial_M \equiv \partial_{\gamma_M}$$

Derivative of a (poly)vector field

$$\partial_M (A^N \gamma_N) = (\partial_M A^N + \Gamma^N_{MK} A^K)\gamma_N \equiv \mathrm{D}_M A^N \gamma_N$$

Covariant derivative

$\partial_M A^N$ Partial derivative

$$\partial_M = \left(\frac{\partial}{\partial s}, \frac{\partial}{\partial x^{\mu_1}}, \frac{\partial}{\partial x^{\mu_1 \mu_2}}, \ldots, \frac{\partial}{\partial x^{\mu_1 \mu_2 \ldots \mu_n}}\right)$$

Other symbols used in the literature $\Box_M, \nabla_M, D_{\gamma_M}, \nabla_{\gamma_M}$

Reciprocal basis elements γ^M, γ^A

$$(\gamma^M)^{\ddagger} * \gamma_N = \delta^M{}_N, \quad (\gamma^A)^{\ddagger} * \gamma_B = \delta^A{}_B$$

Curvature of *C*-space

$$[\partial_M, \partial_N]\gamma_J = R_{MNJ}{}^K \gamma_K$$

$$R_{MNJ}{}^K = \partial_M \Gamma^K_{NJ} - \partial_N \Gamma^K_{MJ} + \Gamma^R_{NJ} \Gamma^K_{MR} - \Gamma^R_{MJ} \Gamma^K_{NR}$$

or:

$$[\partial_M, \partial_N]\gamma_A = R_{MNA}{}^B \gamma_B$$

$$R_{MNA}{}^B = -(\partial_M \Omega_A{}^B{}_N - \partial_N \Omega_A{}^B{}_M + \Omega_A{}^C{}_N \Omega_C{}^B{}_M - \Omega_A{}^C{}_N \Omega_C{}^B{}_M)$$

On the General Relativity in C-space

Concept of spacetime should be replaced by that of *C*-space.
Spacetime is just a start.
From its basis we can build a larger space – C-space.

Also physical!

It has 16 dimensions – therefore its can serve as a realization of Kaluza-Klein theory!

Kaluza-Klein theory without extra dimensions

$$I[X^M, G_{MN}] = M \int \mathrm{d}\tau \, (\dot{X}^M \dot{X}^N G_{MN})^{1/2} + \frac{1}{16\pi\kappa} \int \mathrm{d}x^{16} \, R$$

Action

$$\frac{1}{\sqrt{\dot{X}^2}} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{X}^M}{\sqrt{\dot{X}^2}} \right) + \Gamma^M_{JK} \frac{\dot{X}^J \dot{X}^K}{\dot{X}^2} = 0$$

Geodesic equation

$$R^{MN} - \tfrac{1}{2} G^{MN} R = 8\pi \kappa \int \mathrm{d}\tau \, \delta^{(C)}(x - X(\tau)) \dot{X}^M \dot{X}^N$$

Einstein's equation

Equations of motion for a point particle

Quadratic form in C

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \text{extra terms}$$

$$X^M = (X^\mu, X^{\bar{M}}), \qquad X^\mu \equiv X^{1\mu}$$

Ansatz for the metric

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu{}^{\bar{M}} A_\nu{}^{\bar{N}} \phi_{\bar{M}\bar{N}}, & A_\mu{}^{\bar{N}} \phi_{\bar{M}\bar{N}} \\ A_\nu{}^{\bar{N}} \phi_{\bar{M}\bar{N}}, & \phi_{\bar{M}\bar{N}} \end{pmatrix}$$

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$\dot{X}_{\bar{M}} = G_{\bar{M}N} \dot{X}^N = A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{N}} \dot{X}^{\bar{N}}$$

Split action

$$I = M \int d\tau \left[\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \phi^{\bar{M}\bar{N}} (A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{J}} \dot{X}^{\bar{J}})(A_{\bar{N}\nu} \dot{X}^\nu + \phi_{\bar{N}\bar{K}} \dot{X}^{\bar{K}}) \right]^{1/2}$$

Variation with respect to X^μ

$$\dot{X}^2 \equiv g_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma$$

$$\frac{1}{(\dot{X}^2)^{1/2}} \frac{d}{d\tau} \left(\frac{\dot{X}^\mu}{(\dot{X}^2)^{1/2}} \right) + \frac{1}{\dot{X}^2} \Gamma^\mu{}_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma + \text{extra terms} = 0$$

Phase space action

$$I\left[X^M, P_M, \Lambda\right] = \int d\tau \left(P_M \dot{X}^M - H\right)$$

$$H = \frac{\Lambda}{2M}\left(P_M P_N G^{MN} - M^2\right)$$

Splitting $X^M = (X^\mu, X^{\bar{M}})$

$$I[X^\mu, X^{\bar{M}}, p_\mu, P_{\bar{M}}, \Lambda] = \int d\tau \left[p_\mu \dot{X}^\mu + P_{\bar{M}} \dot{X}^{\bar{M}} - H \right]$$

$$H = \frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu - A_\mu{}^{\bar{J}} P_{\bar{J}}\right)\left(p_\nu - A_\nu{}^{\bar{K}} P_{\bar{K}}\right) + \phi^{\bar{M}\bar{N}} P_{\bar{M}} P_{\bar{N}} - M^2 \right]$$

Hamiltonian

We assume that the extra (or 'internal') space admits isometries given by Killing vector fields $k_\alpha{}^{\bar{J}}$

Projection of momentum onto Killing vector $k_\alpha{}^{\bar{J}} P_{\bar{J}} \equiv p_\alpha$ Charge

$$A_\mu{}^{\bar{J}} = k_\alpha{}^{\bar{J}} A_\mu{}^\alpha$$

$$\phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha{}^{\bar{M}} k_\beta{}^{\bar{N}}$$

$$H = \frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu - A_\mu{}^\alpha p_\alpha\right)\left(p_\nu - A_\nu{}^\beta p_\beta\right) + \varphi^{\alpha\beta} p_\alpha p_\beta - M^2 \right]$$

$$H=\frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu-A_\mu{}^\alpha p_\alpha\right)\left(p_\nu-A_\nu{}^\beta p_\beta\right)+\varphi^{\alpha\beta}p_\alpha p_\beta-M^2\right]$$

$$k_\alpha{}^{\bar{J}}P_{\bar{J}}\equiv p_\alpha$$

$$\dot{p}_\alpha=\{p_\alpha,H\}$$

$$\{p_\alpha,p_\beta\}=\frac{\partial p_\alpha}{\partial X^J}\frac{\partial p_\beta}{\partial X_J}-\frac{\partial p_\beta}{\partial X^J}\frac{\partial p_\alpha}{\partial X_J}=\left(k_{\alpha,J}{}^M k_\beta{}^J-k_{\beta,J}{}^M k_\alpha{}^J\right)p_M=-C_{\alpha\beta}{}^\gamma p_\gamma$$

$$\left(k_{\alpha,J}{}^M k_\beta{}^J-k_{\beta,J}{}^M k_\alpha{}^J\right)=-C_{\alpha\beta}{}^\gamma k_\gamma{}^M$$

$$p_\mu-A_\mu{}^{\bar{J}}P_{\bar{J}}\equiv\pi_\mu,\qquad g^{\mu\nu}\pi_\nu=\frac{M}{\Lambda}\dot{X}^\mu$$

$$\dot{p}_\alpha=C_{\alpha\beta}{}^\gamma\,p_\gamma\,A_\mu{}^\beta\,\dot{X}^\mu-\frac{\Lambda}{2M}\varphi^{\alpha'\beta'}{}_{,\bar{J}}\,p_{\alpha'}p_{\beta'}\,k_\alpha{}^{\bar{J}}$$

Wong equation

One can choose a frame in which

$$k_\alpha{}^M=\left(k_\alpha{}^\mu,k_\alpha{}^{\bar{M}}\right),\quad k_\alpha{}^\mu=0,\quad k_\alpha{}^{\bar{M}}\neq0$$

$$\dot{p}_\mu = \{p_\mu, H\} = -\frac{\partial H}{\partial X^\mu}$$

$$F_{\mu\nu}{}^{\alpha} = \partial_\mu A_\nu{}^{\alpha} - \partial_\nu A_\mu{}^{\alpha} + C_{\alpha'\beta'}{}^{\alpha} A_\mu{}^{\alpha'} A_\nu{}^{\beta'}$$

Yang-Mills field strength

$$g^{\mu\nu}\pi_\nu = \frac{M}{\Lambda}\dot{X}^\mu, \qquad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu}\dot{X}^\nu$$

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu}\pi^\rho\pi^\sigma + F_{\mu\nu}{}^{\alpha} p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta = 0$$

Wong equation
(Equation of geodesic
+ Yang-Mills)

Extra contribution
due to `scalar' fields

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = M^2 - \varphi^{\alpha\beta} p_\alpha p_\beta$$

Four dimensional mass m is given by the higher dimensional mass M and the contribution due to the extra components of momentum $p_{\bar{M}}$

From the perspective of 4-dimensioal spacetime, m has the role of inertial mass. This can be seen if we rewrite the equation of motion

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu} \pi^\rho \pi^\sigma + F_{\mu\nu}{}^\alpha p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M} \left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'} \right) p_\alpha p_\beta = 0$$

$$g^{\mu\nu} \pi_\nu = \frac{M}{\Lambda} \dot{X}^\mu, \qquad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu} \dot{X}^\nu$$

$$\Lambda^2 = \dot{X}^M \dot{X}^N G_{MN}, \qquad \lambda^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

$$\frac{m}{M} = \frac{\lambda}{\Lambda}$$

$$\frac{1}{\lambda} \frac{d}{d\tau} \left(\frac{\dot{X}^\mu}{\lambda} \right) + {}^{(4)}\Gamma^\mu{}_{\rho\sigma} \frac{\dot{X}^\rho \dot{X}^\sigma}{\lambda^2} + \frac{p_\alpha}{m} F_{\mu\nu}{}^\alpha \frac{\dot{X}^\nu}{\lambda}$$

$$+ \frac{1}{2m^2} \left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'} \right) p_\alpha p_\beta + \frac{1}{\lambda m} \frac{dm}{d\tau} = 0$$

Good features of C-space

- No need for extra dimensions of spacetime.
  The extra degrees of freedom are in Clifford space, generated by a basis in $V_{1,3}$.

- No need to compactify the "extra dimensions".
  The extra dimensions of C-space, namely

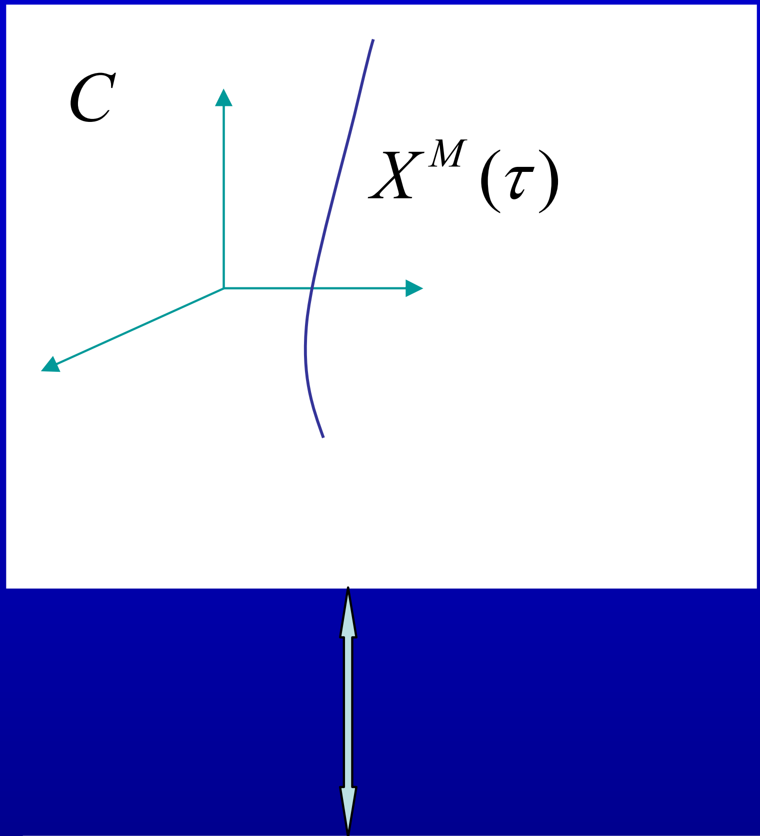
$$s,\ x^{\mu\nu}, x^{\mu\nu\rho}, x^{\mu\nu\rho\sigma}$$

  sample the extended objects. They are physical.

- The number of components $G_{\mu\bar{M}}\ ,\ \bar{M} \neq \mu\ ,\quad \mu$ fixed, is 12. The same as the number of the gauge fields in the Standard model.

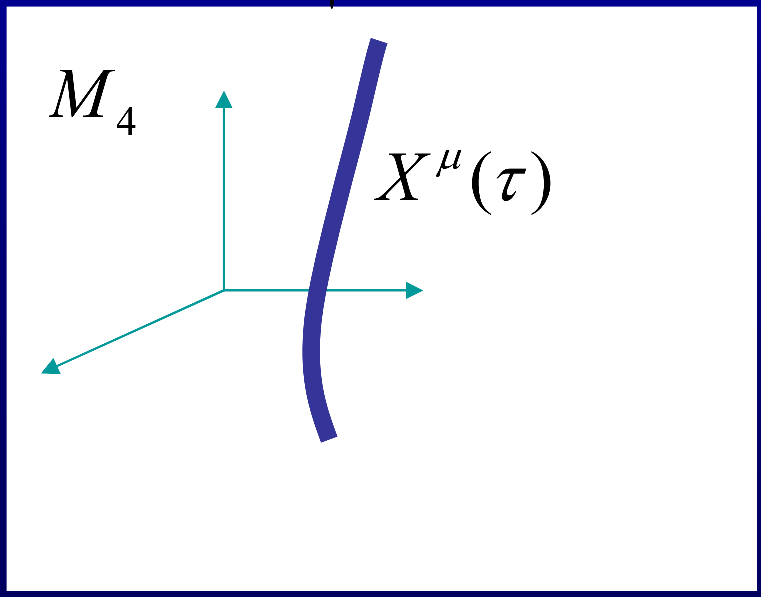
Thick point particles and strings

A world line in *C* represents the evolution of a `thick' particle in spacetime M_4



Thick particle can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.



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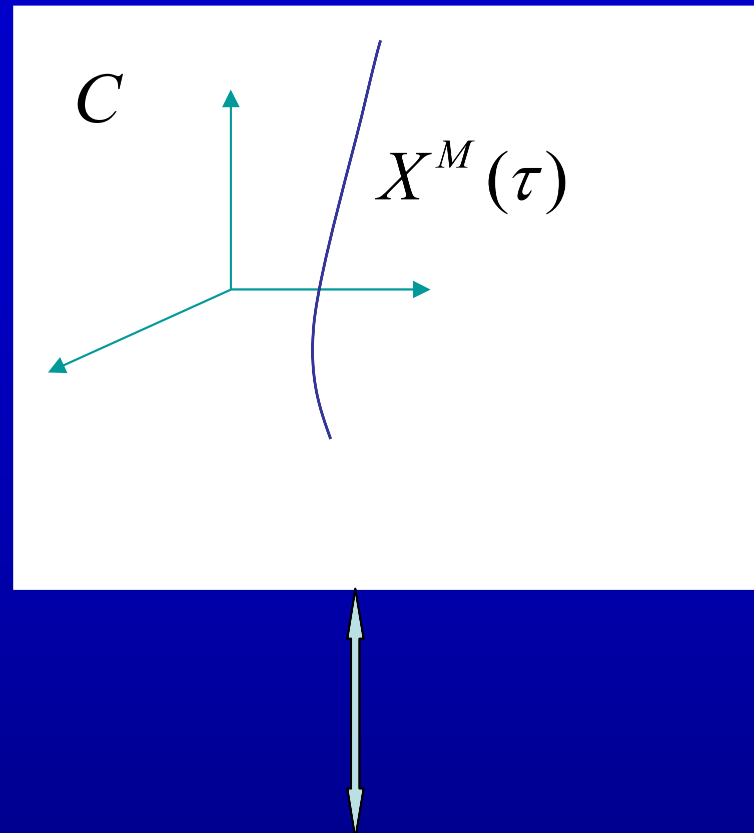
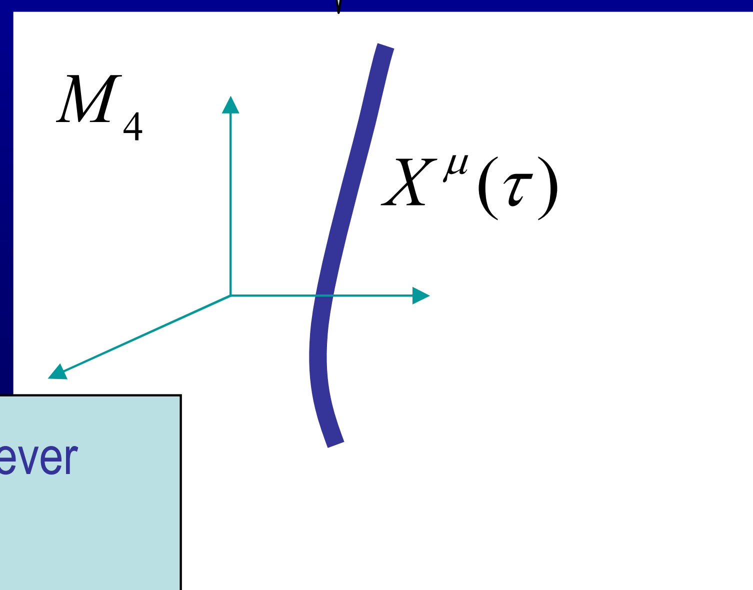


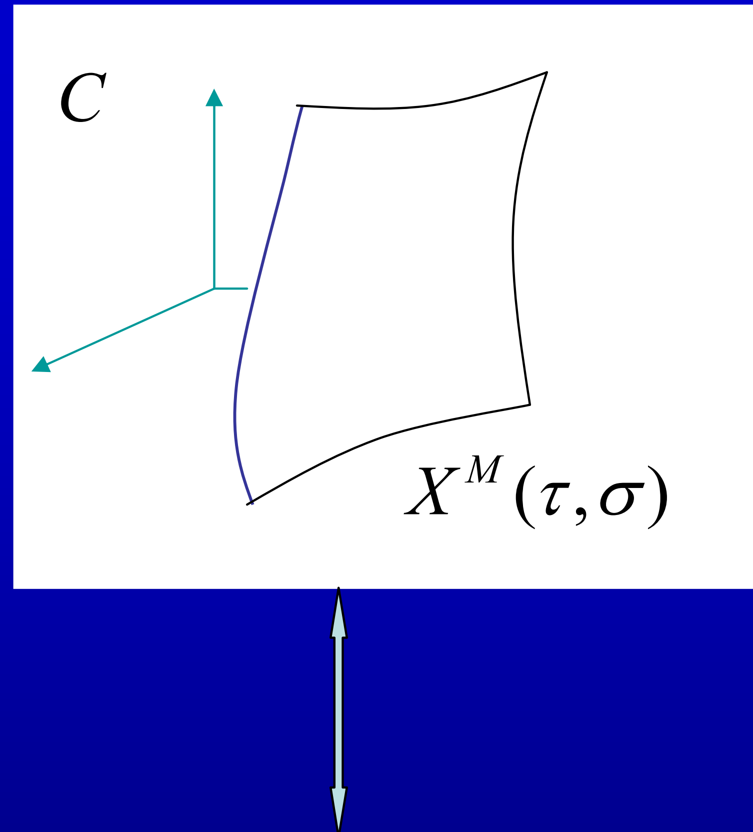
Thick particle can be an aggregate p-branes for various p=0,1,2,…



But such interpretation is not obligatory.

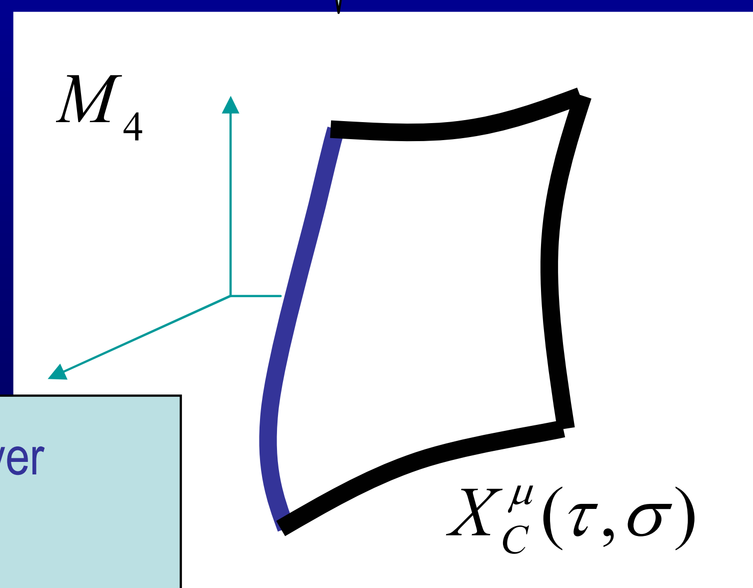
Thick particle may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}$

A world sheet in *C* represents the evolution of a `thick' string in spacetime M_4



Thick string can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.



Thick string may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1\mu_2\ldots\mu_r}$

$X^\mu(\tau)$

Usual strings are infinitely thin object. Although called `extended objects', they are not fully extended.

Instead of infinitely thin strings we thus consider thick strings. Their thickness is encoded in polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}$.

Infinitely thin strings are singular objects

String action

$$I = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^M \dot{X}^N - X'^M X'^N) \, G_{MN}$$

Conformal gauge

No extra dimensions of the spacetime are required

The necessary extra dimensions for consistency of string theory are in 16-dimensional Clifford space.

Jackiw-Kim-Noz definition of vacuum

No central terms in the Virasoro algebra, if the space in which the string lives has signature $(+ + + \ldots - - -)$

The space in which out string lives is Clifford space. Its dimension is 16, and signature (8,8).

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Conformal gauge

No extra dimensions

The necessary extra
are in 16-dimensiona

Jackiw-Kim-Noz defi

No central terms in t
string lives has signa

The space in wh
Its dimension is

$$X^M = (x, x^\mu, x^{\mu\nu}, \ldots)$$

$$\gamma^M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu,}, \ldots)$$

$X^M \gamma_M$ Polyvector

(It contains spinors)

Some quantum issues

Ordering ambiguity resolved

$$\hat{P}^2\Psi = 0$$

$$\hat{P} = -i\gamma^M \partial_M$$

Because momentum operator is defined geometrically, there is no order ambiguity.

An illustration

$$\hat{p}^2\phi = 0$$

$$\phi = \phi(x)$$ scalar field

$$\hat{p} = -i\partial = -i\gamma^\mu \partial_\mu$$ momentum operator in 4D

$$\partial\partial\phi = \gamma^\mu\partial_\mu(\gamma^\nu\partial_\nu\phi) = g^{\mu\nu}\mathrm{D}_\mu\mathrm{D}_\nu\phi = \frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}g^{\mu\nu}\partial_\nu\phi) = 0$$

$$\delta(x, x') = \frac{\delta(x - x')}{|g(x)|}$$

$$\langle x | p | x' \rangle = -i\gamma^\mu(x)\partial_\mu\delta(x, x')$$

$$\langle x' | p | x \rangle^* = \langle x | p | x' \rangle$$

Matrix elements of the vector momentum operator in curved space satisfy the Hermiticity condition

$$\langle x | p^2 | x' \rangle = (-i\gamma^\mu\partial_\mu)(-i\gamma^\nu\partial_\nu)\delta(x, x')$$

Dirac equation in C-space

Geometric form

$$\partial\Psi \equiv \gamma^M \partial_M \Psi = 0$$

Generalized spin connection

$$\partial_M \xi_{\tilde{A}} = \Gamma_M{}^{\tilde{B}}{}_{\tilde{A}}\, \xi_{\tilde{B}}$$

$$\gamma^M (\partial_M \psi^{\tilde{A}} + \Gamma_M{}^{\tilde{A}}{}_{\tilde{B}}\, \psi^{\tilde{B}})\, \xi_{\tilde{A}} = 0$$

$$\left\langle \xi^{\tilde{C}\ddagger} \gamma^M \xi_{\tilde{A}} \right\rangle_S \equiv (\gamma^M)^{\tilde{C}}{}_{\tilde{A}}$$

$$(\gamma^M)^{\tilde{C}}{}_{\tilde{A}} (\partial_M \psi^{\tilde{A}} + \Gamma_M{}^{\tilde{A}}{}_{\tilde{B}}\, \psi^{\tilde{B}}) = 0$$

$$\gamma^M = (\gamma^M)^{\tilde{A}}{}_{\tilde{B}}, \qquad \Gamma_M = \Gamma_M{}^{\tilde{A}}{}_{\tilde{B}}$$

matrices

$$\gamma^M (\partial_M + \Gamma_M)\psi = 0$$

Matrix form

$$\Psi = \psi^{\tilde{A}} \xi_{\tilde{A}}$$

Basis spinors

$$\tilde{A} = 1, 2, 3, \ldots, 16$$

Physical content of the spin connection in C-space

We can write

$$\Gamma_M = \tfrac{1}{4}\Omega^{AB}{}_M \Sigma_{AB} = A_M{}^A \gamma_A$$

$$\Sigma_{AB} = -\Sigma_{BA} = \begin{cases} \gamma_A \gamma_B \,, & \text{if} \quad A < B \\ 0 \quad , & \text{if} \quad A = B \end{cases}$$

$\Sigma_{CD} = f^A_{\ CD} \gamma_A \,,$ $\quad A_M{}^A = \tfrac{1}{4}\Omega^{CD}{}_M \, f^A_{\ CD}$ gauge field

Γ_M contain:

(i) The spin connection of 4-dim. gravity

$$\Gamma^{(4)}_\mu = \frac{1}{8}\Omega^{ab}{}_\mu [\gamma_a, \gamma_b] \,, \qquad a, b = 0, 1, 2, 3$$

(ii) Yang-Mills fields describing other interaction

$$A_\mu{}^{\bar{A}} \gamma_{\bar{A}} \,, \qquad A = (\mu, \bar{A})$$

$$\bar{A} \neq \mu$$

``Internal'' index; assumes 12 values, the same as the number of gauge fields in the standard model

(iii) Antisymmetric potentials

$$A_M{}^o \equiv A_M = (A_\mu, A_{\mu\nu}, A_{\mu\nu\rho}, A_{\mu\nu\rho\sigma})$$

o scalar component

(iv) Non abelian generalization of the antisymmetric potentials $A^{\bar{A}}_{\mu\nu\ldots}$

Conclusion

- Spacetime can be elegantly described by means of γ_μ which generate a Clifford algebra.

- Clifford algebra describes a geometry which goes beyond spacetime: the ingredients are not only points, but also 2-areas, 3-volumes, 4-volumes and scalars.
  All those objects together lead to the concept of a 16-dimensional manifold, called Clifford space (C-space).

- It is quite possible that the arena for physics is not spacetime, but Clifford space.
  And the arena itself can become a part of the play, if we assume that C-space is curved and dynamical.

- We have thus a higher dimensional curved differential manifold, and yet we have not augmented the number of the basic four dimensions. The ``*extra dimensions*'' are related to the physical degrees of freedom due to *the extended nature of physical objects*.
  There is no need to compactify the 12-dimensional ``internal'' part of C-space.

Conclusion

- Spacetime can be elegantly described by means of γ_μ which generate Clifford algebra.

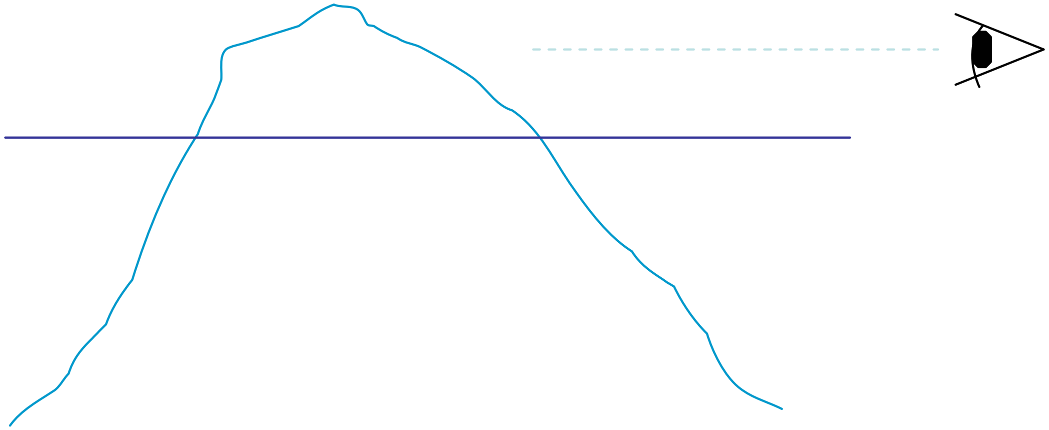
- Clifford algebra describes a geometry which goes beyond spacetime: the ingredients are not only points, but also 2-areas, 3-volumes, 4-volumes and scalars. All those objects together form a 16-dimensional manifold, Clifford space (C-space).

- It is quite possible that the arena for physics is not spacetime, but Clifford space. And the arena itself can become a part of the play, if we assume that C-space is curved and dynamical.

- We have thus a higher dimensional curved differential manifold, and yet we have dimensions. The degrees of freed There is no need part of C-space

-The theory considered here is promising for the unification of fundamental forces. There are possible applications in string theory (thick strings), astrophysics and cosmology.

What I was able to present here was just a tip of an iceberg.



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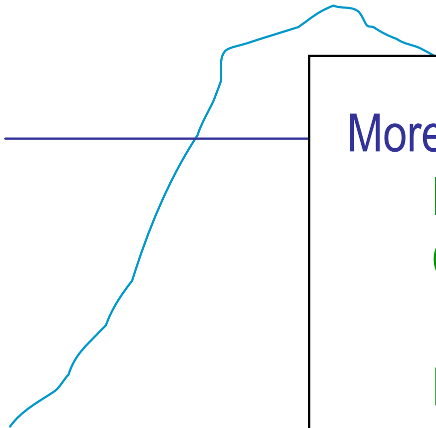

More can be found in the literature:

Hestenes, Crawford, Trayling and Baylis,
Chisholm and Farwell, and many others

Pezzaglia, Castro

M. Pavšič: The Landscape of Theoretical Physics: A Global View;
From Point Particles to the Brane World and Beyond,
in Search of a Unifying principle
(Kluwer Academic, 2001)

and some other related publications:

Class.Quant.Grav.20:2697-2714,2003, gr-qc/0111092

Kaluza-Klein theory without extra dimensions: Curved Clifford space.
Phys.Lett.B614:85-95,2005, hep-th/0412255

Clifford space as a generalization of spacetime: Prospects for QFT of point particles and strings. Found.Phys.35:1617-1642,2005, hep-th/0501222

Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory n 4- dimensional spacetime, Int.J.Mod.Phys.A21:5905-5956,2006, gr-qc/0507053

Auxiliary slides

• Dynamical metric field in M-space

Let us now ascribe the dynamical role to the M-space metric.
M-space perspective: motion of a point "particle" in the presence of the metric field $\rho_{\mu(\phi)\nu(\phi')}$ which is itself dynamical.

$$\phi \equiv \phi^A = (\tau, \xi^A)$$

As a model let us consider

$$I[\rho] = \int \mathcal{D}X \sqrt{|\rho|} \left(\rho_{\mu(\phi)\nu(\phi')} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} + \frac{\varepsilon}{16\pi} \mathcal{R} \right)$$

R Ricci scalar in M

variation with respect to $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$

$$\frac{\mathrm{D}\dot{X}^{\mu(\phi)}}{\mathrm{D}\tau} \equiv \frac{\mathrm{d}\dot{X}^{\mu(\phi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\phi)}_{\alpha(\phi')\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')} = 0$$

geodesic equation in M

$$\dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} + \frac{\varepsilon}{16\pi} \mathcal{R}^{\mu(\phi)\nu(\phi')} = 0$$

Einstein's equations in M

Conserved charges and isometries

Curved Clifford space

K isometries given in terms of Killing fields

$$k^\alpha = k_M^\alpha \gamma^M , \qquad \alpha = 1,2,\ldots,K$$

$$M = 1,2,\ldots,16$$

satisfying

$$D_N k_M^\alpha + D_M k_N^\alpha = 0$$

This index denotes extra dimensions of C-space

Particular coordinate system in which:

$$k^{\alpha\mu} = 0, \quad k^{\alpha\bar{M}} \neq 0, \qquad \mu = 0,1,2,3; \quad \bar{M} \neq \mu$$

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & g_{\mu\bar{M}} \\ g_{\bar{M}\nu} & g_{\bar{M}\bar{N}} \end{pmatrix}, \qquad e^A{}_M = \begin{pmatrix} e^a{}_\mu & e^a{}_{\bar{M}} \\ e^{\bar{A}}{}_\mu & e^{\bar{A}}{}_{\bar{N}} \end{pmatrix}$$

where:

$$e^a{}_{\bar{M}} = 0, \qquad e^{\bar{A}}{}_\mu = e^{\bar{A}}{}_M \, k^{\alpha M} \, W_\mu{}^\alpha , \qquad \partial_{\bar{M}} W_\mu{}^\alpha = 0$$

Inserting this into the spin connection, we obtain:

$$\Omega_{\bar{M}\bar{N}\mu} = \frac{1}{2} k^\alpha_{[\bar{M},\bar{N}]} W_\mu{}^\alpha , \qquad k^\alpha_{[\bar{M},\bar{N}]} = \partial_{\bar{N}} k^\alpha_{\bar{M}} - \partial_{\bar{M}} k^\alpha_{\bar{N}}$$

YM fields $W_\mu{}^\alpha$ occur in C-space vielbein and connection.

Conserved charges and isometries

Curved Clifford space

K isometries given in terms of Killing fields

$$k^{\alpha} = k^{\alpha}_{M}\, \gamma^{M}\,, \qquad \alpha = 1, 2, \ldots, K$$

satisfying

$$D_N k^{\alpha}_{M} + D_M k^{\alpha}_{N} = 0$$

Particular coordinate system in which:

$$k^{\alpha\mu} = 0\,, \qquad k^{\alpha\bar{M}} \neq 0\,,$$

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} & g_{\mu\bar{M}} \\ g_{\bar{M}\nu} & g_{\bar{M}\bar{N}} \end{pmatrix}, \qquad e^{A}{}_{M}$$

where:

$$e^{a}{}_{\bar{M}} = 0\,, \qquad e^{\bar{A}}{}_{\mu} = e^{\bar{A}}{}_{M}\, k^{\alpha M}$$

Inserting this into the spin connection,

$$\Omega_{\bar{M}\bar{N}\mu} = \frac{1}{2} k^{\alpha}_{[\bar{M},\bar{N}]} W_{\mu}{}^{\alpha}\,,$$

YM fields $W_{\mu}{}^{\alpha}$ occur in C-space vielbein and connection.

Connection for local frame field:

From

$$\partial_M \gamma_N = \Gamma^{J}_{MN}\, \gamma_J$$

$$\partial_M \gamma_A = -\Omega_{A}{}^{B}{}_{M}\, \gamma_B$$

$$\gamma_M = e^{A}{}_{M}\, \gamma_A$$

it follows

$$\partial_N e^{C}{}_{M} - \Gamma^{J}_{NM}\, e^{C}{}_{J} - e^{A}{}_{M}\, \Omega_{A}{}^{C}{}_{N} = 0$$

vanishing torsion

$$\Omega_{BCM} = \frac{1}{2} e^{A}{}_{M} \left(\Delta_{[AB]C} - \Delta_{[BC]A} + \Delta_{[CA]B} \right)$$

$$\Delta_{[AB]C} \equiv e_{A}{}^{M}\, e_{B}{}^{N} \left(\partial_M e_{NC} - \partial_N e_{MC} \right)$$

Spinors as members of left ideals of Clifford algebra

$$\Phi = \phi^A \gamma_A$$

Polyvector valued field

γ_A, $A = 1, 2, ..., 16$ Orthonormal basis of C-space

ϕ^A Complex valued scalar components

Φ depends on position in C-space

$\Phi(x^M)$

Another basis

$$\Phi = \psi^{\tilde{A}} \xi_{\tilde{A}} = \Psi$$

$\xi_{\tilde{A}} \equiv \xi_{\alpha i} \in \mathcal{I}_i^L$, $\alpha = 1,2,3,4;\ i = 1,2,3,4$

$\mathcal{I}_i^L$ is the i-th left ideal;

Its elements are spanned by $\gamma_A P_i$

$$P_i = \frac{1}{4}(1 + a_i \gamma_A)(1 + b_i \gamma_B) = \frac{1}{4}(1 + a_i \gamma_A + b_i \gamma_B + c_i \gamma_C)$$

a_i, b_i, c_i complex numbers, such that:

$\gamma_A \gamma_B = \gamma_C$

$P_i^2 = P_i$ idempotent

An example

$$P_1 = \tfrac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_{012})$$

$$P_2 = \tfrac{1}{4}(1 + \gamma_0 - i\gamma_{12} - i\gamma_{012})$$

$$P_3 = \tfrac{1}{4}(1 - \gamma_0 + i\gamma_{12} - i\gamma_{012})$$

$$P_4 = \tfrac{1}{4}(1 - \gamma_0 - i\gamma_{12} + i\gamma_{012})$$

In short:

$$P_i = \tfrac{1}{4}(1 \pm \gamma_0)(1 \pm i\gamma_{12})$$

For instance, the basis of the first left ideal is:

$$\xi_{11} = P_1 = \tfrac{1}{4}(1 + \gamma_0 + i\gamma_{12} + i\gamma_{012})$$

$$\xi_{21} = -\gamma_{13} P_1 = \tfrac{1}{4}(-\gamma_{13} - \gamma_{013} + i\gamma_{23} + i\gamma_{023})$$

$$\xi_{31} = -\gamma_3 P_1 = \tfrac{1}{4}(-\gamma_3 + \gamma_{03} - i\gamma_{123} + i\gamma_{0123})$$

$$\xi_{41} = -\gamma_1 P_1 = \tfrac{1}{4}(-\gamma_1 + \gamma_{01} + i\gamma_2 - i\gamma_{02})$$