THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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Here $F(A)$ is a polyvector-valued function of a polyvector A, and E is an arbitrary polyvector . The star " * " denotes the scalar product

$$A * B = \langle AB \rangle_0 \tag{6.66}$$

of two polyvectors A and B, where $\langle AB \rangle_0$ is the scalar part of the Clifford product AB. Let e_J be a complete set of basis vector of Clifford algebra satisfying[3]

$$e_J * e_K = \delta_{JK}, \tag{6.67}$$

so that any polyvector can be expanded as $A = A^J e_J$. For E in eq. (6.65) we may choose one of the basis vectors. Then

$$\left(e_K * \frac{\partial}{\partial A} \right) F(A) \equiv \frac{\partial F}{\partial A^K} = \lim_{\tau \to 0} \frac{F(A^J e_J + e_K \tau) - F(A^J e_J)}{\tau}. \tag{6.68}$$

This is the partial derivative of F with respect to the multivector components A_K. The derivative with respect to a polyvector A is the sum

$$\frac{\partial F}{\partial A} = e^J \left(e_J * \frac{\partial}{\partial A} \right) F = e^J \frac{\partial F}{\partial A^J}. \tag{6.69}$$

The polyvector A can be a polyvector field $A(X)$ defined over the position polyvector field X which is a generalizatin of the position vector field x defined in (6.61). In particular, the field $A(X)$ can be $A(X) = X$. Then (6.68), (6.69) read

$$\left(e_K * \frac{\partial}{\partial X} \right) F(X) \equiv \frac{\partial F}{\partial X^K} = \lim_{\tau \to 0} \frac{F(X^J e_J + e_K \tau) - F(X^J e_J)}{\tau}, \tag{6.70}$$

$$\frac{\partial F}{\partial X} = e^J \left(e_J * \frac{\partial}{\partial X} \right) F = e^J \frac{\partial F}{\partial X^J} \tag{6.71}$$

which generalizes eqs. (6.63),6.64).

VECTORS IN AN INFINITE-DIMENSIONAL SPACE

In functional analysis functions are considered as vectors in infinite dimensional spaces. As in the case of finite-dimensional spaces one can introduce a basis $h(x)$ in an infinite-dimensional space V_∞ and expand a vector of V_∞ in terms of the basis vectors. The expansion coefficients form a function $f(x)$:

$$f = \int \mathrm{d}x \, f(x) h(x). \tag{6.72}$$

[3]Remember that the set $\{e_J\} = \{1, e_\mu, e_\mu, e_\mu e_\nu, ...\}$.

The basis vectors $h(x)$ are elements of the Clifford algebra $\mathcal{C}_\infty$ of V_∞. The Clifford or geometric product of two vectors is

$$h(x)h(x') = h(x) \cdot h(x') + h(x) \wedge h(x'). \tag{6.73}$$

The symmetric part

$$h(x) \cdot h(x') = \tfrac{1}{2}\left(h(x)h(x') + h(x')h(x)\right) \tag{6.74}$$

is *the inner product* and *the antisymmetric part*

$$h(x) \wedge h(x') = \tfrac{1}{2}\left(h(x)h(x') - h(x')h(x)\right) \tag{6.75}$$

is *the outer, or wedge, product* of two vectors.

The inner product defines the metric $\rho(x, x')$ of V_∞:

$$h(x) \cdot h(x') = \rho(x, x') \tag{6.76}$$

The square or the norm of f is

$$f^2 = f \cdot f = \int \mathrm{d}x\, \mathrm{d}x'\, \rho(x, x') f(x) f(x'). \tag{6.77}$$

It is convenient to introduce notation with upper and lower indices and assume the convention of the integration over the repeated indices. Thus

$$f = f^{(x)} h_{(x)}, \tag{6.78}$$

$$f^2 = f^{(x)} f^{(x')} h_{(x)} \cdot h_{(x')} = f^{(x)} f^{(x')} \rho_{(x)(x')} = f^{(x)} f_{(x)}, \tag{6.79}$$

where (x) is the continuous index.

It is worth stressing here that $h_{(x)} \equiv h(x)$ are abstract elements satisfying the Clifford algebra relation (6.76) for a chosen metric $\rho(x, x') \equiv \rho_{(x)(x')}$. We do not need to worry here about providing an explicit representation of $h(x)$; the requirement that they satisfy the relation (6.76) is all that matters for our purpose[4].

The basis vectors $h_{(x)}$ are generators of Clifford algebra $\mathcal{C}_\infty$ of V_∞. An arbitrary element $F \in \mathcal{C}_\infty$, called a *polyvector*, can be expanded as

$$F = f_0 + f^{(x)} h_{(x)} + f^{(x)(x')} h_{(x)} \wedge h_{(x')} + f^{(x)(x')(x'')} h_{(x)} \wedge h_{(x')} \wedge h_{(x'')} + \ldots, \tag{6.80}$$

[4]Similarly, when introducing the imaginary number i, we do not provide an explicit representation for i. We remain satisfied by knowing that i satisfies the relation $i^2 = -1$.

i.e.,

$$F = f_0 + \int \mathrm{d}x\, f(x)h(x) + \int \mathrm{d}x\,\mathrm{d}x'\, f(x,x')h(x)h(x')$$
$$+ \int \mathrm{d}x\,\mathrm{d}x'\,\mathrm{d}x''\, f(x,x',x'')h(x)h(x')h(x'') + ..., \qquad (6.81)$$

where the wedge product can be replaced by the Clifford product, if $f(x,x')$, $f(x,x',x'')$ are antisymmetric in arguments x, x', ...

We see that once we have a space V_∞ of functions $f(x)$ and basis vectors $h(x)$, we also automatically have a larger space of antisymmetric functions $f(x,x')$, $f(x,x',x'')$. This has far reaching consequences which will be discussed in Sec.7.2.

DERIVATIVE WITH RESPECT TO AN INFINITE-DIMENSIONAL VECTOR

The definition (6.49), (6.50) of the derivative can be straightforwardly generalized to the case of polyvector-valued functions $F(f)$ of an infinite-dimensional vector argument f. The derivative in the direction of a vector g is defined according to

$$\left(g \cdot \frac{\partial}{\partial f}\right) F(f) = \lim_{\tau \to 0} \frac{F(f + g\tau) - F(f)}{\tau}. \qquad (6.82)$$

If $g = h_{(x')}$ then

$$\left(h_{(x')} \cdot \frac{\partial}{\partial f}\right) F = \frac{\partial F}{\partial f^{(x')}} \equiv \frac{\delta F}{\delta f(x')}$$
$$= \lim_{\tau \to 0} \frac{F(f^{(x)}h_{(x)} + h_{(x')}\tau) - F(f^{(x)}h_{(x)})}{\tau}$$
$$= \lim_{\tau \to 0} \frac{F\left[(f^{(x)} + \tau \delta^{(x)}{}_{(x')})h_{(x')}\right] - F[f^{(x)}h_{(x)}]}{\tau}, \qquad (6.83)$$

where $\delta^{(x)}{}_{(x')} \equiv \delta(x - x')$. This is a definition of the *functional derivative.*

A polyvector F can have a definite grade, e.g.,

$$r = 0 \quad : \quad F(f) = F_0[f^{(x)}h_{(x)}] = \Phi_0[f(x)],$$

$$r = 1 \quad : \quad F(f) = F^{(x)}[f^{(x)}h_{(x)}]h_{(x)} = \Phi^{(x)}[f(x)]h_{(x)}, \qquad (6.84)$$

$$r = 2 \quad : \quad F(f) = F^{(x)(x')}[f^{(x)}h_{(x)}]h_{(x)}h_{(x')} = \Phi^{(x)(x')}[f(x)]h_{(x)}h_{(x')}.$$

For a scalar-valued $F(f)$ the derivative (6.83) becomes

$$\frac{\delta \Phi_0}{\delta f(x')} = \lim_{\tau \to 0} \frac{\Phi_0[f(x) + \delta(x - x')\tau] - \phi_0[f(x)]}{\tau}, \tag{6.85}$$

which is the ordinary definition of the functional derivative. Analogously for an arbitrary r-vector valued field $F(f)$.

The derivative with respect to a vector f is then

$$h^{(x)} \left(h_{(x)} \cdot \frac{\partial}{\partial f} \right) F = \frac{\partial F}{\partial f} = h^{(x)} \frac{\partial F}{\partial f^{(x)}}. \tag{6.86}$$

INCLUSION OF DISCRETE DIMENSIONS

Instead of a single function $f(x)$ of a single argument x we can consider a discrete set of functions $f^a(x^\mu)$, $a = 1, 2, ..., N$, of a multiple argument x^μ, $\mu = 1, 2, ..., n$. These functions can be considered as components of a vector f expanded in terms of the basis vectors $h_a(x)$ according to

$$f = \int \mathrm{d}x\, f^a(x) h_a(x) \equiv f^{a(x)} h_{a(x)}. \tag{6.87}$$

Basis vectors $h_{a(x)} \equiv h_a(x)$ and components $f^{a(x)} \equiv f^a(x)$ are now labeled by a set of continuous numbers x^μ, $\mu = 1, 2, ..., n$, and by a set of discrete numbers a, such as, e.g., $a = 1, 2, ..., N$. All equations (6.72)–(6.85) considered before can be straightforwardly generalized by replacing the index (x) with $a(x)$.

6.2. DYNAMICAL VECTOR FIELD IN $\mathcal{M}$-SPACE

We shall now reconsider the action (5.76) which describes a membrane coupled to its own metric field in $\mathcal{M}$-space. The first term is the square of the velocity vector

$$\rho_{\mu(\phi)\nu(\phi)} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} \equiv \dot{X}^2. \tag{6.88}$$

The velocity vector can be expanded in terms of $\mathcal{M}$-space basis vectors $h_{\mu(\phi)}$ (which are a particular example of generic basis vectors $h_a(x)$ considered at the end of Sec. 6.1):

$$\dot{X} = \dot{X}^{\mu(\phi)} h_{\mu(\phi)}. \tag{6.89}$$