THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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For a scalar-valued $F(f)$ the derivative (6.83) becomes

$$\frac{\delta \Phi_0}{\delta f(x')} = \lim_{\tau \to 0} \frac{\Phi_0[f(x) + \delta(x - x')\tau] - \phi_0[f(x)]}{\tau}, \tag{6.85}$$

which is the ordinary definition of the functional derivative. Analogously for an arbitrary r-vector valued field $F(f)$.

The derivative with respect to a vector f is then

$$h^{(x)} \left(h_{(x)} \cdot \frac{\partial}{\partial f} \right) F = \frac{\partial F}{\partial f} = h^{(x)} \frac{\partial F}{\partial f^{(x)}}. \tag{6.86}$$

INCLUSION OF DISCRETE DIMENSIONS

Instead of a single function $f(x)$ of a single argument x we can consider a discrete set of functions $f^a(x^\mu)$, $a = 1, 2, ..., N$, of a multiple argument x^μ, $\mu = 1, 2, ..., n$. These functions can be considered as components of a vector f expanded in terms of the basis vectors $h_a(x)$ according to

$$f = \int \mathrm{d}x \, f^a(x) h_a(x) \equiv f^{a(x)} h_{a(x)}. \tag{6.87}$$

Basis vectors $h_{a(x)} \equiv h_a(x)$ and components $f^{a(x)} \equiv f^a(x)$ are now labeled by a set of continuous numbers x^μ, $\mu = 1, 2, ..., n$, and by a set of discrete numbers a, such as, e.g., $a = 1, 2, ..., N$. All equations (6.72)–(6.85) considered before can be straightforwardly generalized by replacing the index (x) with $a(x)$.

6.2. DYNAMICAL VECTOR FIELD IN $\mathcal{M}$-SPACE

We shall now reconsider the action (5.76) which describes a membrane coupled to its own metric field in $\mathcal{M}$-space. The first term is the square of the velocity vector

$$\rho_{\mu(\phi)\nu(\phi)} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} \equiv \dot{X}^2. \tag{6.88}$$

The velocity vector can be expanded in terms of $\mathcal{M}$-space basis vectors $h_{\mu(\phi)}$ (which are a particular example of generic basis vectors $h_a(x)$ considered at the end of Sec. 6.1):

$$\dot{X} = \dot{X}^{\mu(\phi)} h_{\mu(\phi)}. \tag{6.89}$$

Basis vectors $h_{\mu(\phi)}$ are not fixed, but they depend on the membrane configuration. We must therefore include in the action not only a kinetic term for $\dot{X}^{\mu(\phi)}$, but also for $h_{\mu(\phi)}$. One possibility is just to rewrite the $\mathcal{M}$-space curvature scalar in terms of $h_{\mu(\phi)}$ by exploiting the relations between the metric and the basis vectors,

$$\rho_{\mu(\phi)\nu(\phi')} = h_{\mu(\phi)} \cdot h_{\nu(\xi')}, \tag{6.90}$$

and then perform variations of the action with respect to $h_{\mu(\phi)}$ instead of $\rho_{\mu(\phi)\nu(\phi')}$.

A more direct procedure is perhaps to exploit the formalism of Sec. 6.1. There we had the basis vectors $\gamma_\mu(x)$ which were functions of the coordinates x^μ. Now we have the basis vectors $h_{\mu(\phi)}[X]$ which are functionals of the membrane configuration X (i.e., of the $\mathcal{M}$-space coordinates $X^{\mu(\phi)}$). The relation (6.8) now generalizes to

$$\partial_{\alpha(\phi')} h_{\beta(\phi'')} = \Gamma^{\mu(\phi)}_{\alpha(\phi')\beta(\phi'')} h_{\mu(\phi)} \tag{6.91}$$

where

$$\partial_{\alpha(\phi')} \equiv \frac{\partial}{\partial X^{\alpha(\phi')}} \equiv \frac{\delta}{\delta X^\alpha(\phi')}$$

is the functional derivative. The commutator of two derivatives gives the curvature tensor in $\mathcal{M}$-space

$$[\partial_{\alpha(\phi')}, \partial_{\beta(\phi'')}] h^{\mu(\phi)} = \mathcal{R}^{\mu(\phi)}{}_{\nu(\bar{\phi})\alpha(\phi')\beta(\phi'')} h^{\nu(\bar{\phi})}. \tag{6.92}$$

The inner product of the left and the right hand side of the above equation with $h_{\nu(\bar{\phi}')}$ gives (after renaming the indices)

$$\mathcal{R}^{\mu(\phi)}{}_{\nu(\bar{\phi})\alpha(\phi')\beta(\phi'')} = \left([\partial_{\alpha(\phi')}, \partial_{\beta(\phi'')}] h^{\mu(\phi)} \right) \cdot h_{\nu(\bar{\phi})}. \tag{6.93}$$

The Ricci tensor is then

$$\mathcal{R}_{\nu(\bar{\phi})\beta(\phi'')} = \mathcal{R}^{\mu(\phi)}{}_{\nu(\bar{\phi})\mu(\phi)\beta(\phi'')} = \left([\partial_{\mu(\phi)}, \partial_{\beta(\phi'')}] h^{\mu(\phi)} \right) \cdot h_{\nu(\bar{\phi})}, \tag{6.94}$$

and the curvature scalar is

$$\begin{aligned}
\mathcal{R} &= \rho^{\nu(\bar{\phi})\beta(\phi'')} \mathcal{R}_{\nu(\bar{\phi})\beta(\phi'')} \\
&= \left([\partial_{\mu(\phi)}, \partial_{\nu(\phi')}] \right) \cdot h^{\nu(\phi')} \\
&= \left(\partial_{\mu(\phi)} \partial_{\nu(\phi')} h^{\mu(\phi)} \right) \cdot h^{\nu(\phi')} - \left(\partial_{\nu(\phi')} \partial_{\mu(\phi)} h^{\mu(\phi)} \right) \cdot h^{\nu(\phi')}.
\end{aligned} \tag{6.95}$$

A possible action is then

$$I[X^{\mu(\phi)}, h_{\mu(\phi)}] = \int \mathcal{D}X \sqrt{|\rho|} \left(\dot{X}^2 + \frac{\epsilon}{16\pi} \mathcal{R} \right), \tag{6.96}$$

where ρ is the determinant of $\mathcal{M}$-space metric, and where $\mathcal{R}$ and ρ are now expressed in terms of $h_{\mu(\phi)}$. Variation of (6.96) with respect to $h_{\alpha(\phi)}$ gives their equations of motion. In order to perform such a variation we first notice that $\rho = \det \rho_{\mu(\phi)\nu(\phi')}$, in view of the relation (6.90), is now a function of $h_{\alpha(\phi)}$. Differentiation of $\sqrt{|\rho|}$ with respect to a vector $h_{\alpha(\phi)}$ follows the rules given in Sec. 6.1:

$$\begin{aligned}
\frac{\partial \sqrt{|\rho|}}{\partial h^{\alpha(\phi)}} &= \frac{\partial \sqrt{|\rho|}}{\partial \rho^{\mu(\phi')\nu(\phi'')}} \frac{\partial \rho^{\mu(\phi')\nu(\phi'')}}{\partial h^{\alpha(\phi)}} \\
&= -\tfrac{1}{2} \sqrt{|\rho|} \rho_{\mu(\phi')\nu(\phi'')} (\delta^{\mu(\phi')}{}_{\alpha(\phi)} h^{\nu(\phi'')} + \delta^{\nu(\phi'')}{}_{\alpha(\phi)} h^{\mu(\phi')}) \\
&= -\sqrt{|\rho|} h_{\alpha(\phi)}.
\end{aligned} \tag{6.97}$$

Since the vectors $h_{\alpha(\phi)}$ are functionals of the membrane's configuration $X^{\mu(\phi)}$, instead of the derivative we take the functional derivative

$$\frac{\delta \sqrt{|\rho[X]|}}{\delta h^{\alpha(\phi)}[X']} = -\sqrt{|\rho[X]} h_{\alpha(\phi)}[X] \delta^{(\mathcal{M})}(X - X'), \tag{6.98}$$

where

$$\delta^{(\mathcal{M})}(X - X') \equiv \prod_{\mu(\phi)} (X^{\mu(\phi)} - X'^{\mu(\phi)}) \tag{6.99}$$

is the δ-functional in $\mathcal{M}$-space.

Functional deriative of $\mathcal{R}$ with respect to $h_{\alpha(\phi)}[X]$ gives

$$\begin{aligned}
\frac{\delta \mathcal{R}[X]}{\delta h^{\alpha(\phi)}[X']} &= \left([\partial_{\mu(\phi')}, \partial_{\nu(\phi'')}] \delta^{\mu(\phi')}{}_{\alpha(\phi)} \delta^{(\mathcal{M})}(X - X') \right) h^{\nu(\phi'')}[X] \\
&\quad + [\partial_{\mu(\phi')}, \partial_{\nu(\phi'')}] h^{\mu(\phi')}[X] \delta^{\nu(\phi'')}{}_{\alpha(\phi)} \delta^{(\mathcal{M})}(X - X').
\end{aligned} \tag{6.100}$$

For the velocity term we have

$$\begin{aligned}
&\frac{\delta}{\delta h^{\alpha(\phi)}[X']} \left(\dot{X}_{\mu(\phi')} h^{\mu(\phi')} \dot{X}_{\nu(\phi'')} h^{\nu(\phi'')} \right) \\
&\qquad = 2 \dot{X}_{\alpha(\phi)} \dot{X}_{\nu(\phi'')} h^{\nu(\phi'')} \delta^{(\mathcal{M})}(X - X').
\end{aligned} \tag{6.101}$$

We can now insert eqs. (6.97)–(6.101) into

$$\frac{\delta I}{\delta h^{\alpha(\phi)}[X']} = \int \mathcal{D}X \frac{\delta}{\delta h^{\alpha(\phi)}[X']} \left[\sqrt{|\rho|} \left(\dot{X}^2 + \frac{\epsilon}{16\pi} \mathcal{R} \right) \right] = 0. \tag{6.102}$$

We obtain

$$\dot{X}_{\alpha(\phi)}\dot{X}_{\nu(\phi')}h^{\nu(\phi')} - \tfrac{1}{2}h_{\alpha(\phi)}\dot{X}^2$$
$$+ \frac{\epsilon}{16\pi}\left(-\tfrac{1}{2}h_{\alpha(\phi)}\mathcal{R} + [\partial_{\nu(\phi')}, \partial_{\alpha(\phi'')}]h^{\nu(\phi')}\right) = 0. \qquad (6.103)$$

These are the equations of "motion" for the variables $h_{\mu(\phi)}$. The equations for $X^{\mu(\phi)}$ are the same equations (5.79).

After performing the inner product of eq. (6.103) with a basis vector $h_{\beta(\phi')}$ we obtain the $\mathcal{M}$-space Einstein equations (5.80). This justifies use of $h_{\mu(\phi)}$ as dynamical variables, since their equations of motion contain the equations for the metric $\rho_{\mu(\phi)\nu(\phi')}$.

DESCRIPTION WITH THE VECTOR FIELD IN SPACETIME

The set of $\mathcal{M}$-space basis vectors $h_{\mu(\phi)}$ is an arbitrary solution to the dynamical equations (6.103). In order to find a connection with the usual theory which is formulated, not in $\mathcal{M}$-space, but in a finite-dimensional (spacetime) manifold V_N, we now assume a particular Ansatz for $h^{\mu(\phi)}$:

$$h^{\mu(\phi)} = h^{(\phi)}\gamma^{\mu}(\phi), \qquad (6.104)$$

where

$$h^{(\phi)} \cdot h^{(\phi')} = \frac{\sqrt{\dot{X}^2}}{\kappa\sqrt{|f|}}\,\delta(\phi - \phi') \qquad (6.105)$$

and

$$\gamma^{\mu}(\phi) \cdot \gamma^{\nu}(\phi) = g^{\mu\nu}. \qquad (6.106)$$

Altogether the above Ansatz means that

$$h^{\mu(\phi)} \cdot h^{\nu(\phi')} = \frac{\sqrt{\dot{X}^2}}{\sqrt{|f|}}\,g^{\mu\nu}(\phi)\delta(\phi - \phi'). \qquad (6.107)$$

Here $g^{\mu\nu}(\phi)$ is *the proto-metric*, and $\gamma^{\mu}(\phi)$ are *the proto-vectors* of *spacetime*. The symbol f now meens

$$f \equiv \det f_{ab}\,, \qquad f_{ab} \equiv \partial_a X^{\mu}\partial_b X^{\mu}\gamma_{\nu}\gamma_{\nu} = \partial_a X_{\mu}\,\partial_b X_{\nu}\,\gamma^{\mu}\gamma^{\nu}. \qquad (6.108)$$

Here the $\mathcal{M}$-space basis vectors are factorized into the vectors $h^{(\phi)}$, $\phi = (\phi^A, k)$, $\phi^A \in [0, 2\pi]$, $k = 1, 2, ..., Z$, which are independent for all values of μ, and are functions of parameter ϕ. Loosely speaking, $h(\phi)$ bear the task of being basis vectors of the infinite-dimensional part (index (ϕ) of $\mathcal{M}$-space, while $\gamma^{\mu}(\phi)$ are basis vectors of the finite-dimensional part (index μ)

of $\mathcal{M}$-space. As in Sec. 6.1, γ^μ here also are functions of the membrane's parameters $\phi = (\phi^A, k)$.

The functional derivative of $h^{\mu(\phi)}$ with respect to the membrane coordinates $X^{\mu(\phi)}$ is

$$\partial_{\nu(\phi')} h^{\mu(\phi)} = \frac{\delta h(\phi)}{\delta X^\nu(\phi')} \gamma^\mu(\phi) + h(\phi) \frac{\delta \gamma^\mu(\phi)}{\delta X^\nu(\phi')}. \tag{6.109}$$

Let us assume a particular case where

$$\begin{aligned} [\partial_{\mu(\phi)}, \partial_{\nu(\phi')}] \gamma^\mu(\phi'') &= \left[\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu}\right] \gamma^\mu(\phi'') \delta(\phi - \phi'') \delta(\phi' - \phi'') \\ &\neq 0 \end{aligned} \tag{6.110}$$

and

$$[\partial_{\mu(\phi)}, \partial_{\nu(\phi')}] h(\phi'') = 0. \tag{6.111}$$

Inserting the relations (6.104)–(6.111) into the action (6.96), and omitting the integration over $\mathcal{D}X$ we obtain

$$\begin{aligned} \epsilon \mathcal{R} &= \epsilon \int \frac{\sqrt{\dot{X}^2}}{\kappa \sqrt{|f(\phi)|}} \delta^2(0) \mathrm{d}\phi \left(\left[\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu}\right] \gamma^\mu(\phi) \right) \cdot \gamma^\nu(\phi) \\ &= \epsilon \int \frac{\sqrt{\dot{X}^2}}{\kappa \sqrt{|f(\phi)|}} \delta^2(0) \mathrm{d}\phi \\ &\qquad \times \left(\left[\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu}\right] \gamma^\mu(\phi) \right) \cdot \gamma^\nu(\phi) \frac{\delta\left(x - X(\phi)\right)}{\sqrt{|g|}} \sqrt{|g|}\, \mathrm{d}^N x \\ &= \frac{1}{G} \int \mathrm{d}^N x \sqrt{|g|}\, \widetilde{\mathcal{R}}, \end{aligned} \tag{6.112}$$

where we have set

$$\begin{aligned} \epsilon \int \mathrm{d}\phi\, \delta^2(0) \frac{\sqrt{\dot{X}^2}}{\kappa \sqrt{|f(\phi)|}} \left(\left[\frac{\partial}{\partial X^\mu}, \frac{\partial}{\partial X^\nu}\right] \gamma^\mu(\phi) \right) \cdot \gamma^\nu(\phi) \frac{\delta\left(x - X(\phi)\right)}{\sqrt{|g|}} \\ = \frac{1}{G} \widetilde{\mathcal{R}}(x), \end{aligned} \tag{6.113}$$

G being the gravitational constant. The expression $\widetilde{\mathcal{R}}(x)$ is defined formally at all points x, but because of $\delta(x - X(\phi))$ it is actually different from zero only on the set of membranes. If we have a set of membranes filling spacetime, then $\widetilde{\mathcal{R}}(x)$ becomes a continuous function of x:

$$\widetilde{\mathcal{R}}(x) = ([\partial_\mu, \partial_\nu] \gamma^\mu(x)) \cdot \gamma^\nu(x) = R(x) \tag{6.114}$$

which is actually a *Ricci scalar*.

For the first term in the action (6.96) we obtain

$$
\begin{aligned}
h_{\mu(\phi)}h_{\nu(\phi')}\dot{X}^{\mu(\phi)}\dot{X}^{\nu(\phi')} \\
&= \int \frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\mathrm{d}\phi\,\gamma_\mu(\phi)\gamma_\nu(\phi)\dot{X}^\mu(\phi)\dot{X}^\nu(\phi) \\
&= \int \mathrm{d}^n\phi\,\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\gamma_\mu(x)\gamma_\nu(x)\dot{X}^\mu(\phi)\dot{X}^\nu(\phi)\,\delta\left(x - X(\phi)\right)\mathrm{d}^N x \\
&= \int \mathrm{d}^n\phi\,\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,g_{\mu\nu}(x)\dot{X}^\mu(\phi)\dot{X}^\nu(\phi)\,\delta\left(x - X(\phi)\right)\mathrm{d}^N x.
\end{aligned}
\tag{6.115}
$$

Altogether we have

$$
\begin{aligned}
I[X^\mu(\phi),\gamma^\mu(x)] = \kappa \int \mathrm{d}\phi\,\sqrt{|f|}\sqrt{\gamma_\mu(x)\gamma_\nu(x)\dot{X}^\mu(\phi)\dot{X}^\nu(\phi)}\,\delta^N\left(x - X(\phi)\right)\mathrm{d}^N x \\
+ \frac{1}{16\pi G}\int \mathrm{d}^N x\sqrt{|g|}\,\left([\partial_\mu,\partial_\nu]\gamma^\mu(x)\right)\cdot\gamma^\nu(x).
\end{aligned}
\tag{6.116}
$$

This is an action for the spacetime vector field $\gamma^\mu(x)$ in the presence of a membrane configuration filling spacetime. It was derived from the action (6.96) in which we have omitted the integration over $\mathcal{D}X\sqrt{\rho}$.

Since $\gamma_\mu \cdot \gamma_\nu = g_{\mu\nu}$, and since according to (6.13)

$$
\left([\partial_\mu,\partial_\nu]\gamma^\mu\right)\cdot\gamma^\nu = R,
\tag{6.117}
$$

the action (6.116) is equivalent to

$$
\begin{aligned}
I[X^\mu(\phi),g_{\mu\nu}] = \kappa \int \mathrm{d}\phi\,\sqrt{|f|}\sqrt{\dot{X}^2}\,\delta\left(x - X(\phi)\right)\,\mathrm{d}^N x \\
+ \frac{1}{16\pi G}\int \mathrm{d}^N x\sqrt{|g|}\,R
\end{aligned}
\tag{6.118}
$$

which is an action for the gravitational field $g_{\mu\nu}$ in the presence of membranes.

Although (6.118) formally looks the same as the usual gravitational action in the presence of matter, there is a significant difference. In the conventional general relativity the matter part of the action may vanish and we thus obtain the Einstein equations in vacuum. On the contrary, in the theory based on $\mathcal{M}$-space, the metric of $\mathcal{M}$-space is intimately connected to the existence of a membrane configuration. Without membranes there is no $\mathcal{M}$-space and no $\mathcal{M}$-space metric. When considering the $\mathcal{M}$-space action (5.76) or (6.96) from the point of view of an effective spacetime (defined in our case by the Ansatz (6.104)–(6.107), we obtain the spacetime action (6.118) in which the matter part cannot vanish. There is always present a set of membranes filling spacetime. Actually, the points of spacetime are identified with the points on the membranes.

The need to fill spacetime with a reference fluid (composed of a set of reference particles) has been realized recently by Rovelli [26], following an earlier work by DeWitt [25]. According to Rovelli and DeWitt, because of the Einstein "hole argument" [42], spacetime points cannot be identified at all. This is a consequence of the invariance of the Einstein equations under active diffeomorphisms. One can identify spacetime points if there exists a material reference fluid with respect to which spacetime points are identified.

We shall now vary the action (6.116) with respect to the vector field $\gamma^\alpha(x)$:

$$\frac{\delta I}{\delta\gamma^\alpha(x)} = \int \mathrm{d}\phi\, \frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\, \dot{X}_\alpha \dot{X}_\nu\, \gamma^\nu \delta\left(x - X(\phi)\right) + \frac{1}{16\pi G} \int \mathrm{d}x' \left[\frac{\delta\sqrt{|g(x')|}}{\delta\gamma^\alpha(x)}\, R(x') + \sqrt{|g(x')|}\, \frac{\delta R(x')}{\delta\gamma^\alpha(x)} \right]. \tag{6.119}$$

For this purpose we use

$$\frac{\delta\gamma^\mu(x)}{\delta\gamma^\nu(x')} = \delta^\mu{}_\nu \delta(x - x') \tag{6.120}$$

and

$$\frac{\delta\sqrt{|g(x)|}}{\delta\gamma^\nu(x')} = -\sqrt{|g(x)|}\,\gamma_\nu(x)\delta(x - x'). \tag{6.121}$$

In (6.121) we have taken into account that $g \equiv \det g_{\mu\nu}$ and $g_{\mu\nu} = \gamma_\mu \cdot \gamma_\nu$. Using (6.120), we have for the gravitational part

$$\begin{aligned}\frac{\delta I_g}{\delta\gamma^\alpha(x)} &= \frac{1}{16\pi G} \int \mathrm{d}^N x'\, \sqrt{|g(x')|} \Big[-\gamma_\alpha(x') R(x') \delta(x - x') \\ &\quad + \Big([\partial'_\mu, \partial'_\nu] \delta^\mu{}_\alpha \delta(x - x') \Big)\, \gamma^\nu(x') + [\partial'_\mu, \partial'_\nu] \gamma^\mu(x') \delta^\nu{}_\alpha \delta(x - x') \Big] \\ &= \frac{1}{16\pi} \sqrt{|g|}\, (-\gamma_\alpha R + 2[\partial_\mu, \partial_\alpha]\gamma^\mu)\,. \end{aligned} \tag{6.122}$$

The equations of motion for $\gamma_\alpha(x)$ are thus

$$[\partial_\mu, \partial_\alpha]\gamma^\mu - \tfrac{1}{2}R\,\gamma_\alpha$$
$$= -8\pi G \int \mathrm{d}\phi \sqrt{|f|}\sqrt{\dot{X}^2}\left(\frac{\dot{X}_\alpha \dot{X}_\nu}{\dot{X}^2} + \partial^a X_\alpha\, \partial_a X_\nu\right)$$
$$\times\, \gamma^\nu \frac{\delta\,(x - X(\phi))}{\sqrt{|g|}}\ , \tag{6.123}$$

where now $\dot{X}^2 \equiv \dot{X}^\mu \dot{X}_\mu$.

After performing the inner product with γ_β the latter equations become the Einstein equations

$$R_{\alpha\beta} - \tfrac{1}{2}R\, g_{\alpha\beta} = -8\pi G T_{\alpha\beta}\ , \tag{6.124}$$

where

$$R_{\alpha\beta} = ([\partial_\mu, \partial_\alpha])\gamma^\mu \cdot \gamma_\beta \tag{6.125}$$

and

$$T_{\alpha\beta} = \kappa \int \mathrm{d}\phi\, \sqrt{|f|}\sqrt{\dot{X}^2}\left(\frac{\dot{X}_\alpha \dot{X}_\beta}{\dot{X}^2} + \partial^a X_\alpha \partial_a X_\beta\right)\frac{\delta(x - X(\phi)}{\sqrt{|g|}} \tag{6.126}$$

is the stress–energy tensor of the membrane configuration. It is the ADW split version of the full stress–energy tensor

$$T_{\mu\nu} = \kappa \int \mathrm{d}\phi\, (\det \partial_A X^\alpha \partial_B X_\alpha)^{1/2}\, \partial^A X_\mu \partial_A X_\nu\, \frac{\delta\,(x - X(\phi))}{\sqrt{|g|}}. \tag{6.127}$$

The variables $\gamma^\mu(x)$ appear much easier to handle than the variables $g_{\mu\nu}$. The expressions for the curvature scalar (6.117) and the Ricci tensor (6.125) are very simple, and it is easy to vary the action (6.116) with respect to $\gamma(x)$.[5]

We should not forget that the matter stress–energy tensor on the right hand side of the Einstein equations (6.124) is present everywhere in spacetime and it thus represents a sort of background matter[6] whose origin is in the original $\mathcal{M}$-space formulation of the theory. The ordinary matter is then expected to be present in addition to the background matter. This will be discussed in Chapter 8.

[5] At this point we suggest the interested reader study the Ashtekar variables [74], and compare them with $\gamma(x)$.

[6] It is tempting to speculate that this background is actually *the hidden mass* or *dark matter* postulated in astrophysics and cosmology [73].

6.3. FULL COVARIANCE IN THE SPACE OF PARAMETERS ϕ^A

So far we have exploited the fact that, according to (4.21), a membrane moves as a point particle in an infinite-dimensional $\mathcal{M}$-space. For a special choice of $\mathcal{M}$-space metric (5.4) which involves the membrane velocity we obtain equations of motion which are identical to the equations of motion of the conventional constrained membrane, known in the literature as the *p-brane*, or simply the *brane*[7]. A moving brane sweeps a surface V_n which incorporates not only the brane parameters ξ^a, $a = 1, 2, ..., p$, but also an extra, time-like parameter τ. Altogether there are $n = p + 1$ parameters $\phi^A = (\tau, \xi^a)$ which denote a point on the surface V_n. The latter surface is known in the literature under names such as *world surface*, *world volume* (now the most common choice) and *world sheet* (my favorite choice).

Separating the parameter τ from the rest of the parameters turns out to be very useful in obtaining *the unconstrained membrane* out of the Clifford algebra based polyvector formulation of the theory.

On the other hand, when studying interactions, the separate treatment of τ was a nuisance, therefore in Sec. 5.1 we switched to a description in terms of the variables $X^\mu(\tau, \xi^a) \equiv X^\mu(\phi^A)$ which were considered as $\mathcal{M}$-space coordinates $X^\mu(\phi^A) \equiv X^{\mu(\phi)}$. So $\mathcal{M}$-space was enlarged from that described by coordinates $X^{\mu(\xi)}$ to that described by $X^{\mu(\phi)}$. In the action and in the equations of motion there occurred the $\mathcal{M}$-space velocity vector $\dot{X}^{\mu(\phi)} \equiv \partial X^{\mu(\phi)}/\partial\tau$. Hence manifest covariance with respect to reparametrizations of ϕ^A was absent in our formulation. In my opinion such an approach was good for introducing the theory and fixing the development of the necessary concepts. This is now to be superceded. We have learnt enough to be able to see a way how a fully reparametrization covariant theory should possibly be formulated.

DESCRIPTION IN SPACETIME

As a first step I now provide a version of the action (6.116) which is invariant under arbitrary reparametrizations $\phi^A \to \phi'^A = f^A(\tau)$. In the form as it stands (6.116) (more precisely, its "matter" term) is invariant under reparametrizations of ξ^a and τ separately. A fully invariant action

[7]For this reason I reserve the name *p-brane* or *brane* for the extended objetcs described by the conventional theory, while the name *membrane* stands for the extended objects of the more general, $\mathcal{M}$-space based theory studied in this book (even if the same name "membrane" in the conventional theory denotes 2-branes, but this should not cause confusion).