THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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a distinguished feature of our approach and we have reasons to expect that also the p-brane gauge field theory —not yet a completely solved problem— can be straightforwardly formulated along the lines indicated here.

7.2. CLIFFORD ALGEBRA AND QUANTIZATION

PHASE SPACE

Let us first consider the case of a 1-dimensional coordinate variable q and its conjugate momentum p. The two quantities can be considered as coordinates of a point in the 2-dimensional *phase space*. Let e_q and e_p be the basis vectors satisfying the Clifford algebra relations

$$e_q \cdot e_p \equiv \tfrac{1}{2}(e_q e_p + e_p e_q) = 0, \tag{7.97}$$

$$e_q^2 = 1 \ , \qquad e_p^2 = 1. \tag{7.98}$$

An arbitrary vector in phase space is then

$$Q = q e_q + p e_p. \tag{7.99}$$

The product of two vectors e_p and e_q is the unit bivector in phase space and it behaves as the imaginary unit

$$i = e_p e_q \ , \qquad i^2 = -1. \tag{7.100}$$

The last relation immediately follows from (7.97), (7.98): $i^2 = e_p e_q e_p e_q = -e_p^2 e_q^2 = -1$.

Multiplying (7.99) respectively from the right and from the left by e_q we thus introduce the quantities Z and Z^*:

$$Q e_q = q + p e_p e_q = q + pi = Z, \tag{7.101}$$

$$e_q Q = q + p e_q e_p = q - pi = Z^*. \tag{7.102}$$

For the square we have

$$Q e_q e_q Q = Z Z^* = q^2 + p^2 + i(pq - qp), \tag{7.103}$$

$$e_q Q Q e_q = Z^* Z = q^2 + p^2 - i(pq - qp). \tag{7.104}$$

Upon quantization q, p do not commute, but satisfy

$$[q, p] = i, \tag{7.105}$$

therefore (7.103), (7.104) become

$$ZZ^* = q^2 + p^2 + 1, \tag{7.106}$$

$$Z^*Z = q^2 + p^2 - 1, \tag{7.107}$$

$$[Z, Z^*] = 1. \tag{7.108}$$

Even before quantization the natural variables for describing physics are *the complex quantity* Z and its conjugate Z^*. *The imaginary unit is the bivector of the phase space, which is 2-dimensional.*

Writing $q = \rho \cos\phi$ and $p = \rho \sin\phi$ we find

$$Z = \rho(\cos\phi + i\sin\phi) = \rho\, \mathrm{e}^{i\phi}\ , \tag{7.109}$$

$$Z^* = \rho(\cos\phi - i\sin\phi) = \rho\, \mathrm{e}^{-i\phi}\ , \tag{7.110}$$

where ρ and ϕ are real numbers. Hence taking into account that physics takes place in the phase space and that the latter can be described by complex numbers, we automatically introduce complex numbers into both the classical and quantum physics. And what is nice here is that *the complex numbers are nothing but the Clifford numbers of the 2-dimensional phase space.*

What if the configuration space has more than one dimension, say n? Then with each spatial coordinate is associated a 2-dimensional phase space. The dimension of the total phase space is then $2n$. A phase space vector then reads

$$Q = q^\mu e_{q\mu} + p^\mu e_{p\mu}. \tag{7.111}$$

The basis vectors have now two indices q, p (denoting the direction in the 2-dimensional phase space) and $\mu = 1, 2, ..., n$ (denoting the direction in the n-dimensional configuration space).

The basis vectors can be written as the product of the configuration space basis vectors e_μ and the 2-dimensional phase space basis vectors e_q, e_p:

$$e_{q\mu} = e_q e_\mu\ , \qquad e_{p\mu} = e_p e_\mu\ . \tag{7.112}$$

A vector Q is then

$$Q = (q^\mu e_q + p^\mu e_p) e_\mu = Q^\mu e_\mu\ , \tag{7.113}$$

where

$$Q^\mu = q^\mu e_q + p^\mu e_p. \tag{7.114}$$

Eqs. (7.101), (7.102) generalize to

$$Q^\mu e_q = q^\mu + p^\mu e_p e_q = q^\mu + p^\mu i = Z^\mu\ , \tag{7.115}$$

$$e_q Q^\mu = q^\mu + p^\mu e_q e_p = q^\mu - p^\mu i = Z^{*\mu}. \tag{7.116}$$

Hence, even if configuration space has many dimensions, the imaginary unit i in the variables X^μ comes from the bivector $e_q e_p$ of the 2-dimensional phase space which is associated with every direction μ of the configuration space.

When passing to quantum mechanics it is then natural that in general the wave function is complex-valued. *The imaginary unit is related to the phase space which is the direct product of the configuration space and the 2-dimensional phase space.*

At this point let us mention that Hestenes was one of the first to point out clearly that imaginary and complex numbers need not be postulated separately, but they are automatically contained in the geometric calculus based on Clifford algebra. When discussing quantum mechanics Hestenes ascribes the occurrence of the imaginary unit i in the Schrödinger and especially in the Dirac equation to a chosen configuration space Clifford number which happens to have the square -1 and which commutes with all other Clifford numbers within the algebra. This brings an ambiguity as to which of several candidates should serve as the imaginary unit i. In this respect Hestenes had changed his point of view, since initially he proposed that one must have a 5-dimensional space time whose pseudoscalar unit $I = \gamma_0\gamma_1\gamma_2\gamma_3\gamma_4$ commutes with all the Clifford numbers of $\mathcal{C}_5$ and its square is $I^2 = -1$. Later he switched to 4-dimensional space time and chose the bivector $\gamma_1\gamma_2$ to serve the role of i. I regard this as unsatisfactory, since $\gamma_2\gamma_3$ or $\gamma_1\gamma_3$ could be given such a role as well. In my opinion it is more natural to ascribe the role of i to the bivector of the 2-dimensional phase space sitting at every coordinate of the configuration space. A more detailed discussion about the relation between the geometric calculus in a generic 2-dimensional space (not necessarily interpreted as phase space) and complex number is to be found in Hestenes' books [22].

WAVE FUNCTION AS A POLYVECTOR

We have already seen in Sec. 2.5 that a wave function can in general be considered as a polyvector, i.e., as a Clifford number or Clifford aggregate generated by a countable set of basis vectors e_μ. Such a wave function contains spinors, vectors, tensors, etc., all at once. In particular, it may contain only spinors, or only vectors, etc. .

Let us now further generalize this important procedure. In Sec. 6.1 we have discussed vectors in an infinite-dimensional space V_∞ from the point of view of geometric calculus based on the Clifford algebra generated by the uncountable set of basis vectors $h(x)$ of V_∞. We now apply that procedure to the case of the wave function which, in general, is complex-valued.

For an arbitrary complex function we have

$$f(x) = \frac{1}{\sqrt{2}}(f_1(x) + i f_2(x)) \, , \qquad f^*(x) = \frac{1}{\sqrt{2}}(f_1(x) - i f_2(x)) \, , \quad (7.117)$$

where $f_1(x)$, $f_2(x)$ are real functions. From (7.117) we find

$$f_1(x) = \frac{1}{\sqrt{2}}(f(x) + f^*(x)) \, , \qquad f_2(x) = \frac{1}{i\sqrt{2}}(f(x) - f^*(x)). \quad (7.118)$$

Hence, instead of a complex function we can consider a set of two independent real functions $f_1(x)$ and $f_2(x)$.

Introducing the basis vectors $h_1(x)$ and $h_2(x)$ satisfying the Clifford algebra relations

$$h_i(x) \cdot h_j(x') \equiv \tfrac{1}{2}(h_i(x) h_j(x') + h_j(x') h_i(x)) = \delta_{ij} \delta(x - x') \, , \quad i, j = 1, 2, \quad (7.119)$$

we can expand an arbitrary vector F according to

$$F = \int \mathrm{d}x (f_1(x) h_1(x) + f_2(x) h_2(x)) = f^{i(x)} h_{i(x)} \, , \quad (7.120)$$

where $h_{i(x)} \equiv h_i(x)$, $f^{i(x)} \equiv f_i(x)$. Then

$$F \cdot h_1(x) = f_1(x) \, , \qquad F \cdot h_2(x) = f_2(x) \quad (7.121)$$

are components of F.

Introducing the imaginary unit i which commutes[7] with $h_i(x)$ we can form a new set of basis vectors

$$h(x) = \frac{h_1(x) + ih_2(x)}{\sqrt{2}} \,, \qquad h^*(x) = \frac{h_1(x) - ih_2(x)}{\sqrt{2}} \,, \tag{7.122}$$

the inverse relations being

$$h_1(x) = \frac{h(x) + h^*(x)}{\sqrt{2}} \,, \qquad h_2(x) = \frac{h(x) - h^*(x)}{i\sqrt{2}}. \tag{7.123}$$

Using (7.118), (7.123) and (7.120) we can re-express F as

$$F = \int \mathrm{d}x(f(x)h^*(x) + f^*(x)h(x)) = f^{(x)}h_{(x)} + f^{*(x)}h^*_{(x)} \,, \tag{7.124}$$

where

$$f^{(x)} \equiv f^*(x) \,, \quad f^{*(x)} \equiv f(x) \,, \qquad h_{(x)} \equiv h(x) \,, \quad h^*_{(x)} \equiv h^*(x). \tag{7.125}$$

From (7.119) and (7.123) we have

$$h(x) \cdot h^*(x) \equiv \tfrac{1}{2}(h(x)h^*(x') + h^*(x')h(x)) = \delta(x - x') \,, \tag{7.126}$$

$$h(x) \cdot h(x') = 0 \,, \qquad h^*(x) \cdot h^*(x') = 0 \,, \tag{7.127}$$

which are *the anticommutation relations for a fermionic field.*

A vector F can be straightforwardly generalized to a *polyvector*:

$$\begin{aligned} F &= f^{i(x)}h_{i(x)} + f^{i(x)j(x')}h_{i(x)}h_{j(x')} + f^{i(x)j(x')k(x'')}h_{i(x)}h_{j(x')}h_{k(x'')} + ... \\ &= f^{(x)}h_{(x)} + f^{(x)(x')}h_{(x)}h_{(x')} + f^{(x)(x')(x'')}h_{(x)}h_{(x')}h_{(x'')} + ... \\ &\quad + f^{*(x)}h^*_{(x)} + f^{*(x)(x')}h^*_{(x)}h^*_{(x')} + f^{*(x)(x')(x'')}h^*_{(x)}h^*_{(x')}h^*_{(x'')} + ... \end{aligned} \tag{7.128}$$

[7]Now, the easiest way to proceed is in forgetting how we have obtained the imaginary unit, namely as a bivector in 2-dimensional phase space, and define all the quantities i, $h_1(x)$, $h_2(x)$, etc., in such a way that i commutes with everything. If we nevertheless persisted in maintaining the geometric approach to i, we should then take $h_1(x) = e(x)$, $h_2(x) = e(x)e_pe_q$, satisfying

$$\begin{aligned} h_1(x) \cdot h_1(x') &= e(x) \cdot e(x') = \delta(x - x'), \\ h_2(x) \cdot h_2(x) &= -\delta(x - x'), \\ h_1(x) \cdot h_2(x') &= \delta(x - x')1 \cdot (e_pe_q) = 0, \end{aligned}$$

where according to Hestenes the inner product of a scalar with a multivector is zero. Introducing $h = (h_1 + h_2)/\sqrt{2}$ and $h^* = (h_1 - h_2)/\sqrt{2}$ one finds $h(x) \cdot h^*(x') = \delta(x - x')$, $h(x) \cdot h(x') = 0$, $h^*(x) \cdot h^*(x') = 0$.

where $f^{(x)(x')(x'')\dots}$ are scalar coefficients, *antisymmetric* in $(x)(x')(x'')\dots$

We have exactly the same expression (7.128) in the usual quantum field theory (QFT), where $f^{(x)}$, $f^{(x)(x')}$,..., are 1-particle, 2-particle,..., wave functions (wave packet profiles). Therefore a natural interpretation of the polyvector F is that it represents a superposition of multi-particle states.

In the usual formulation of QFT one introduces a vacuum state $|0\rangle$, and interprets $h(x)$, $h^*(x)$ as the operators which create or annihilate a particle or an antiparticle at x, so that (roughly speaking) e.g. $h^*(x)|0\rangle$ is a state with a particle at position x.

In the geometric calculus formulation (based on the Clifford algebra of an infinite-dimensional space) the Clifford numbers $h^*(x)$, $h(x)$ already represent vectors. At the same time $h^*(x)$, $h(x)$ also behave as operators, satisfying (7.126), (7.127). When we say that a state vector is expanded in terms of $h^*(x)$, $h(x)$ we mean that it is a superposition of states in which a particle has a definite position x. The latter states are just $h^*(x)$, $h(x)$. Hence the Clifford numbers (operators) $h^*(x)$, $h(x)$ need not act on a vacuum state in order to give the one-particle states. They are already the one-particle states. Similarly the products $h(x)h(x')$, $h(x)h(x')h(x'')$, $h^*(x)h^*(x')$, etc., already represent the multi-particle states.

When performing quantization of a classical system we arrived at *the wave function.* The latter can be considered as an uncountable (infinite) set of scalar components of a vector in an infinite-dimensional space, spanned by the basis vectors $h_1(x)$, $h_2(x)$. Once we have basis vectors we automatically have not only arbitrary vectors, but also arbitrary polyvectors which are *Clifford numbers* generated by $h_1(x)$, $h_2(x)$ (or equivalently by $h(x)$, $h^*(x)$. Hence *the procedure in which we replace infinite-dimensional vectors with polyvectors is equivalent to the second quantization.*

If one wants to consider *bosons* instead of *fermions* one needs to introduce a new type of fields $\xi_1(x)$, $\xi_2(x)$, satisfying *the commutation relations*

$$\tfrac{1}{2}[\xi_i(x), \xi_j(x')] \equiv \tfrac{1}{2}[\xi_i(x)\xi_j(x') - \xi_j(x')\xi_i(x)] = \epsilon_{ij}\Delta(x - x')\tfrac{1}{2}, \tag{7.129}$$

with $\epsilon_{ij} = -\epsilon_{ji}$, $\Delta(x - x') = -\Delta(x' - x)$, which stay instead of the anticommutation relations (7.119). Hence the numbers $\xi(x)$ are not Clifford numbers. By (7.129) the $\xi_i(x)$ generate a new type of algebra, which could be called an *anti-Clifford algebra.*

Instead of $\xi_i(x)$ we can introduce the basis vectors

$$\xi(x) = \frac{\xi_1(x) + i\xi_2}{\sqrt{2}}\,, \qquad \xi^*(x) = \frac{\xi_1(x) - i\xi_2}{\sqrt{2}} \tag{7.130}$$

which satisfy the commutation relations

$$[\xi(x), \xi^*(x')] = -i\Delta(x - x'), \tag{7.131}$$

$$[\xi(x),\xi(x')] = 0\ , \qquad [\xi^*(x),\xi^*(x')] = 0. \tag{7.132}$$

A polyvector representing a superposition of bosonic multi-particle states is then expanded as follows:

$$\begin{aligned} B &= \phi^{i(x)}\xi_{i(x)} + \phi^{i(x)j(x')}\xi_{i(x)}\xi_{j(x')} + ... \qquad (7.133) \\ &= \phi^{(x)}\xi_{(x)} + \phi^{(x)(x')}\xi_{(x)}\xi_{(x')} + \phi^{(x)(x')(x'')}\xi_{(x)}\xi_{(x')}\xi_{(x'')} + ... \\ &\quad + \phi^{*(x)}\xi^*_{(x)} + \phi^{*(x)(x')}\xi^*_{(x)}\xi^*_{(x')} + \phi^{*(x)(x')(x'')}\xi^*_{(x)}\xi^*_{(x')}\xi^*_{(x'')} + ...\ , \end{aligned}$$

where $\phi^{i(x)j(x')...}$ and $\phi^{(x)(x')...}$, $\phi^{*(x)(x')(x'')...}$ are scalar coefficients, *symmetric* in $i(x)j(x')...$ and $(x)(x')...$, respectively. They can be interpreted as representing 1-particle, 2-particle,..., wave packet profiles. Because of (7.131) $\xi(x)$ and $\xi^*(x)$ can be interpreted as creation operators for *bosons*. Again, *a priori* we do not need to introduce a vacuum state. However, whenever convenient we may, of course, define a vacuum state and act on it by the operators $\xi(x)$, $\xi^*(x)$.

EQUATIONS OF MOTION FOR BASIS VECTORS

In the previous subsection we have seen how the geometric calculus naturally leads to the second quantization which incorporates superpositions of multi-particle states. We shall now investigate what are the equations of motion that the basis vectors satisfy.

For illustration let us consider the action for a *real scalar field* $\phi(x)$:

$$I[\phi] = \tfrac{1}{2}\int \mathrm{d}^4x\,(\partial_\mu\phi\partial^\mu\phi - m^2). \tag{7.134}$$

Introducing the metric

$$\rho(x,x') = h(x)\cdot h(x') \equiv \tfrac{1}{2}(h(x)h(x') + h(x')h(x)) \tag{7.135}$$

we have

$$I[\phi] = \tfrac{1}{2}\int \mathrm{d}x\,\mathrm{d}x'\,\Big(\partial_\mu\phi(x)\partial'^\mu\phi(x') - m^2\phi(x)\phi(x')\Big)\,h(x)h(x'). \tag{7.136}$$

If, in particular,

$$\rho(x,x') = h(x)\cdot h(x') = \delta(x-x') \tag{7.137}$$

then the action (7.136) is equivalent to (7.134).

In general, $\rho(x,x')$ need not be equal to $\delta(x-x')$, and (7.136) is then a generalization of the usual action (7.134) for the scalar field. An action which is invariant under field redefinitions ('coordinate' transformations in the space of fields) has been considered by Vilkovisky [85]. Integrating (7.136) *per partes* over x and x' and omitting the surface terms we obtain

$$I[\phi] = \tfrac{1}{2} \int \mathrm{d}x \, \mathrm{d}x' \, \phi(x)\phi(x') \left(\partial_\mu h(x) \partial'^\mu h(x') - m^2 h(x) h(x') \right) . \tag{7.138}$$

Derivatives no longer act on $\phi(x)$, but on $h(x)$. If we fix $\phi(x)$ then instead of an action for $\phi(x)$ we obtain an action for $h(x)$.

For instance, if we take

$$\phi(x) = \delta(x - y) \tag{7.139}$$

and integrate over y we obtain

$$I[h] = \frac{1}{2} \int \mathrm{d}y \left(\frac{\partial h(y)}{\partial y^\mu} \frac{\partial h(y)}{\partial y_\mu} - m^2 h^2(y) \right) . \tag{7.140}$$

The same equation (7.140), of course, follows directly from (7.136) in which we fix $\phi(x)$ according to (7.139).

On the other hand, if instead of $\phi(x)$ we fix $h(x)$ according to (7.137), then we obtain the action (7.134) which governs the motion of $\phi(x)$.

Hence the same basic expression (7.136) can be considered either as an action for $\phi(x)$ or an action for $h(x)$, depending on which field we consider as fixed and which one as a variable. If we consider the basis vector field $h(x)$ as a variable and $\phi(x)$ as fixed according to (7.139), then we obtain the action (7.140) for $h(x)$. The latter field is actually an operator. The procedure from now on coincides with the one of quantum field theory.

Renaming y^μ as x^μ (7.140) becomes an action for a bosonic field:

$$I[h] = \tfrac{1}{2} \int \mathrm{d}x \, (\partial_\mu h \partial^\mu h - m^2 h^2). \tag{7.141}$$

The canonically conjugate variables are

$$h(t, \mathbf{x}) \qquad \text{and} \qquad \pi(t, \mathbf{x}) = \partial \mathcal{L} / \partial \dot{h} = \dot{h}(t, \mathbf{x}).$$

They satisfy the *commutation relations*

$$[h(t, \mathbf{x}), \pi(t, \mathbf{x}') = i\delta^3(\mathbf{x} - \mathbf{x}') \, , \qquad [h(t, \mathbf{x}), h(t, \mathbf{x}')] = 0. \tag{7.142}$$

At different times $t' \neq t$ we have

$$[h(x), h(x')] = i\Delta(x - x') \, , \tag{7.143}$$

where $\Delta(x-x')$ is the well known covariant function, antisymmetric under the exchange of x and x'.

The geometric product of two vectors can be decomposed as

$$h(x)h(x') = \tfrac{1}{2}\left(h(x)h(x') + h(x')h(x)\right) + \tfrac{1}{2}\left(h(x)h(x') - h(x')h(x)\right). \tag{7.144}$$

In view of (7.143) we have that *the role of the inner product is now given to the antisymmetric part*, whilst *the role of the outer product is given to the symmetric part*. This is characteristic for *bosonic vectors*; they generate what we shall call the *anti-Clifford algebra*. In other words, when the basis vector field $h(x)$ happens to satisfy the *commutation relation*

$$[h(x), h(x')] = f(x, x'), \tag{7.145}$$

where $f(x,x')$ is a scalar two point function (such as $i\Delta(x-x')$), it behaves as a *bosonic field*. On the contrary, when $h(x)$ happens to satisfy the *anticommutation relation*

$$\{h(x), h(x')\} = g(x, x'), \tag{7.146}$$

where $g(x,x')$ is also a scalar two point function, then it behaves as a *fermionic field*[8]

The latter case occurs when instead of (7.134) we take the action for the *Dirac field*:

$$I[\psi, \bar{\psi}] = \int \mathrm{d}^4x\, \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{7.147}$$

Here we are using the usual spinor representation in which the spinor field $\psi(x) \equiv \psi_\alpha(x)$ bears the spinor index α. A generic vector is then

$$\begin{aligned} \Psi &= \int \mathrm{d}x\, \left(\bar{\psi}_\alpha(x) h_\alpha(x) + \psi_\alpha(x)\bar{h}_\alpha(x)\right) \\ &\equiv \int \mathrm{d}x\, \left(\bar{\psi}(x)h(x) + \psi(x)\bar{h}(x)\right). \end{aligned} \tag{7.148}$$

Eq. (7.147) is then equal to the scalar part of the action

$$\begin{aligned} I[\psi, \bar{\psi}] &= \int \mathrm{d}x\, \mathrm{d}x'\, \bar{\psi}(x')\bar{h}(x)h(x')(i\gamma^\mu\partial_\mu - m)\psi(x) \\ &= \int \mathrm{d}x\, \mathrm{d}x'\, \bar{\psi}(x')\left[\bar{h}(x)(i\gamma^\mu - m)h(x')\right]\psi(x), \end{aligned} \tag{7.149}$$

where $h(x)$, $\bar{h}(x)$ are assumed to satisfy

$$\bar{h}(x)\cdot h(x') \equiv \tfrac{1}{2}\left(\bar{h}(x)h(x') + h(x')\bar{h}(x)\right) = \delta(x-x') \tag{7.150}$$

[8] In the previous section the bosonic basis vectors were given a separate name $\xi(x)$. Here we retain the same name $h(x)$ both for bosonic and fermionic basis vectors.

The latter relation follows from the Clifford algebra relations amongst the basis fields $h_i(x)$, $i = 1, 2$,

$$h_i(x) \cdot h_j(x') = \delta_{ij}(x - x') \tag{7.151}$$

related to $\bar{h}(x)$, $h(x)$ accroding to

$$h_1(x) = \frac{h(x) + \bar{h}(x)}{\sqrt{2}}\,, \qquad h_2(x) = \frac{h(x) - \bar{h}(x)}{i\sqrt{2}}. \tag{7.152}$$

Now we relax the condition (7.150) and (7.149) becomes a generalization of the action (7.147).

Moreover, if in (7.149) we fix the field ψ according to

$$\psi(x) = \delta(x - y), \tag{7.153}$$

integrate over y, and rename y back into x, we find

$$I[h, \bar{h}] = \int \mathrm{d}x\, \bar{h}(x)(i\gamma^\mu \partial_\mu - m)h(x). \tag{7.154}$$

This is an action for the basis vector field $h(x)$, $\bar{h}(x)$, which are *operators*.

The canonically conjugate variables are now

$$h(t, \mathbf{x}) \qquad \text{and} \qquad \pi(t, \mathbf{x}) = \partial\mathcal{L}/\partial\dot{h} = i\bar{h}\gamma^0 = ih^\dagger.$$

They satisfy the anticommutation relations

$$\{h(t, \mathbf{x}), h^\dagger(t, \mathbf{x}'\} = \delta^3(\mathbf{x} - \mathbf{x}'), \tag{7.155}$$

$$\{h(t, \mathbf{x}), h(t, \mathbf{x}')\} = \{h^\dagger(t, \mathbf{x}), h^\dagger(t, \mathbf{x}')\} = 0. \tag{7.156}$$

At different times $t' \neq t$ we have

$$\{h(x), \bar{h}(x')\} = (i\gamma^\mu + m)i\Delta(x - x'), \tag{7.157}$$

$$\{h(x), h(x')\} = \{\bar{h}(x), \bar{h}(x')\} = 0. \tag{7.158}$$

The basis vector fields $h_i(x)$, $i = 1, 2$, defined in (7.152) then satisfy

$$\{h_i(x), h_j(x')\} = \delta_{ij}(i\gamma^\mu + m)i\Delta(x - x'), \tag{7.159}$$

which can be written as *the inner product*

$$h_i(x) \cdot h_j(x') = \tfrac{1}{2}\delta_{ij}(i\gamma^\mu + m)i\Delta(x - x') = \rho(x, x') \tag{7.160}$$

with the metric $\rho(x,x')$. We see that our procedure leads us to a metric which is different from the metric assumed in (7.151).

Once we have basis vectors we can form an arbitrary *vector* according to

$$\Psi = \int \mathrm{d}x \ \left(\psi(x)\bar{h}(x) + \bar{\psi}(x)h(x)\right) = \psi^{(x)}h_{(x)} + \bar{\psi}^{(x)}\bar{h}_{(x)}. \tag{7.161}$$

Since the $h(x)$ generates a Clifford algebra we can form not only a vector but also an arbitrary multivector and a superposition of multivectors, i.e., a *polyvector* (or *Clifford aggregate*):

$$\begin{aligned} \Psi &= \int \mathrm{d}x \ \left(\psi(x)\bar{h}(x) + \bar{\psi}(x)h(x)\right) \\ &\quad + \int \mathrm{d}x\,\mathrm{d}x' \ \left(\psi(x,x')\bar{h}(x)\bar{h}(x') + \bar{\psi}(x,x')h(x)h(x')\right) + ... \\ &= \psi^{(x)}h_{(x)} + \psi^{(x)(x')}h_{(x)}h_{(x')} + ... \\ &\quad + \bar{\psi}^{(x)}\bar{h}_{(x)} + \bar{\psi}^{(x)(x')}\bar{h}_{(x)}\bar{h}_{(x')} + ... \,, \end{aligned} \tag{7.162}$$

where $\psi(x,x',...) \equiv \bar{\psi}^{(x)(x')...}$, $\bar{\psi}(x,x',...) \equiv \psi^{(x)(x')...}$ are *antisymmetric* functions, interpreted as wave packet profiles for a system of free *fermions.*

Similarly we can form an arbitrary polyvector

$$\begin{aligned} \Phi &= \int \mathrm{d}x\, \phi(x)h(x) + \int \mathrm{d}x\,\mathrm{d}x'\, \phi(x,x')h(x)h(x') + ... \\ &\equiv \phi^{(x)}h_{(x)} + \phi^{(x)(x')}h_{(x)}h_{(x')} + ... \end{aligned} \tag{7.163}$$

generated by *the basis vectors* which happen to satisfy the *commutation relations* (7.142). In such a case the uncountable set of basis vectors behaves as a *bosonic field.* The corresponding multi-particle wave packet profiles $\phi(x,x',...)$ are *symmetric* functions of x, x',... . If one considers a complex field, then the equations (7.141)–(7.142) and (7.163) are generalized in an obvious way.

As already mentioned, within the conceptual scheme of Clifford algebra and hence also of anti-Clifford algebra we do not need, if we wish so, to introduce a vacuum state[9], since the operators $h(x)$, $\bar{h}(x)$ already represent states. From the actions (7.141), (7.147) we can derive the corresponding Hamiltonian, and other relevant operators (e.g., the generators of spacetime translations, Lorentz transformations, etc.). In order to calculate their expectation values in a chosen multi-particle state one may simply sandwich those operators between the state and its Hermitian conjugate (or Dirac

[9]Later, when discussing the states of the quantized p-brane, we nevertheless introduce a vacuum state and the set of orthonormal basis states spanning the Fock space.

conjugate) and take the scalar part of the expression. For example, the expectation value of the Hamiltonian H in a bosonic 2-particle state is

$$\langle H \rangle = \langle \int \mathrm{d}x\,\mathrm{d}x'\,\phi(x,x')h(x)h(x')H \int \mathrm{d}x''\,\mathrm{d}x'''\,\phi(x'',x''')h(x'')h(x''')\rangle_0, \tag{7.164}$$

where, in the case of the real scalar field,

$$H = \tfrac{1}{2} \int \mathrm{d}^3\mathbf{x}(\dot{h}^2(x) - \partial^i h \partial_i h + m^2 h^2). \tag{7.165}$$

Instead of performing the operation $\langle\ \rangle_0$ (which means taking the scalar part), in the conventional approach to quantum field theory one performs the operation $\langle 0|....|0\rangle$ (i.e., taking the vacuum expectation value). However, instead of writing, for instance,

$$\langle 0|a(\mathbf{k})a^*(\mathbf{k}')|0\rangle = \langle 0|[a(\mathbf{k}), a^*(\mathbf{k}')]|0\rangle = \delta^3(\mathbf{k}-\mathbf{k}'), \tag{7.166}$$

we can write

$$\begin{aligned} \langle a(\mathbf{k})a^*(\mathbf{k}')\rangle_0 &= \tfrac{1}{2}\langle a(\mathbf{k})a^*(\mathbf{k}') + a^*(\mathbf{k}')a(\mathbf{k})\rangle_0 \\ &\quad + \tfrac{1}{2}\langle a(\mathbf{k})a^*(\mathbf{k}') - a^*(\mathbf{k}')a(\mathbf{k})\rangle_0 \\ &= \tfrac{1}{2}\delta^3(\mathbf{k}-\mathbf{k}'), \end{aligned} \tag{7.167}$$

where we have taken into account that for a bosonic operator the symmetric part is not a scalar. Both expressions (7.166) and (7.167) give the same result, up to the factor $\frac{1}{2}$ which can be absorbed into the normalization of the states.

We leave to the interested reader to explore in full detail (either as an exercise or as a research project), for various operators and kinds of field, how much the results of the above procedure (7.164) deviate, if at all, from those of the conventional approach. Special attention should be paid to what happens with the vacuum energy (the cosmological constant problem) and what remains of the anomalies. According to a very perceptive explanation provided by Jackiw [86], anomalies are the true physical effects related to the choice of vacuum (see also Chapter 3). So they should be present, at least under certain circumstances, in the procedure like (7.164) which does not explicitly require a vacuum. I think that, e.g. for the Dirac field our procedure, in the language of QFT means dealing with bare vacuum. In other words, the momentum space Fourier transforms of the vectors $h(x)$, $\bar{h}(x)$ represent states which in QFT are created out of the bare vacuum. For consistency reasons, in QFT the bare vacuum is replaced by the Dirac vacuum, and the creation and annihilation operators are redefined accordingly. Something analogous should also be done in our procedure.

QUANTIZATION OF THE STUECKELBERG FIELD

In Part I we have paid much attention to the unconstrained theory which involves a *Lorentz invariant evolution parameter* τ. We have also seen that such an unconstrained Lorentz invariant theory is embedded in a polyvector generalization of the theory. Upon quantization we obtain the Schrödinger equation for the wave function $\psi(\tau, x^\mu)$:

$$i\frac{\partial\psi}{\partial\tau} = \frac{1}{2\Lambda}(-\partial_\mu\partial^\mu - \kappa^2)\psi. \tag{7.168}$$

The latter equation follows from the action

$$I[\psi,\psi^*] = \int \mathrm{d}\tau\,\mathrm{d}^4x \left(i\psi^*\frac{\partial\psi}{\partial\tau} - \frac{\Lambda}{2}(\partial_\mu\psi^*\partial^\mu\psi - \kappa^2\psi^*\psi)\right). \tag{7.169}$$

This is equal to the scalar part of

$$\begin{aligned} I[\psi,\psi^*] = \int \mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}x\,\mathrm{d}x' &\left[i\psi^*(\tau',x')\frac{\partial\psi(\tau,x)}{\partial\tau} - \frac{\Lambda}{2}(\partial'_\mu\psi^*(\tau',x')\partial^\mu\psi(\tau,x)\right.\\ &\left.-\kappa^2\psi^*(\tau',x')\psi(\tau,x)\right]h^*(\tau,x)h(\tau',x')\\ = \int \mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}x\,\mathrm{d}x'\,&\psi^*(\tau',x')\psi(\tau,x)\\ &\times\left[-i\frac{\partial h^*(\tau,x)}{\partial\tau}h(\tau',x')\right.\\ &\left.-\frac{\Lambda}{2}\left(\partial_\mu h^*(\tau,x)\partial'^\mu h(\tau',x') - \kappa^2 h^*(\tau,x)h(\tau',x')\right)\right] \end{aligned} \tag{7.170}$$

where $h(\tau',x')$, $h^*(\tau,x)$ are assumed to satisfy

$$\begin{aligned} h(\tau',x')\cdot h^*(\tau,x) &\equiv \tfrac{1}{2}\left(h(\tau',x')h^*(\tau,x) + h^*(\tau,x)h(\tau',x')\right),\\ &= \delta(\tau-\tau')\delta^4(x-x') \end{aligned}$$

$$h(\tau,x)\cdot h(\tau',x') = 0\ , \qquad h^*(\tau,x)\cdot h^*(\tau',x') = 0. \tag{7.171}$$

The relations above follow, as we have seen in previous subsection (eqs. (7.119)–(7.127)) from the Clifford algebra relations amongst the basis fields $h_i(x)$, $i = 1, 2$,

$$h_i(\tau, x) \cdot h_j(\tau', x') = \delta_{ij}\delta(\tau - \tau')\delta(x - x') \tag{7.172}$$

related to $h(\tau, x)$, $h^*(\tau, x)$ according to

$$h_1 = \frac{h + h^*}{\sqrt{2}}\,, \qquad h_2 = \frac{h - h^*}{i\sqrt{2}}. \tag{7.173}$$

Let us now relax the condition (7.171) so that (7.170) becomes a generalization of the original action (7.169).

Moreover, if in (7.170) we fix the field ψ according to

$$\psi(\tau, x) = \delta(\tau - \tau')\delta^4(x - x'), \tag{7.174}$$

integrate over τ', x', and rename τ', x' back into τ, x, we find

$$I[h, h^*] = \int \mathrm{d}\tau\, \mathrm{d}^4x \left[ih^* \frac{\partial h}{\partial \tau} - \frac{\Lambda}{2}(\partial_\mu h^* \partial^\mu h - \kappa^2 h^* h)\right], \tag{7.175}$$

which is an action for basis vector fields $h(\tau, x)$, $h^*(\tau, x)$. The latter fields are *operators*.

The usual canonical procedure then gives that the field $h(x)$ and its conjugate momentum $\pi = \partial\mathcal{L}/\partial\dot{h} = ih^*$, where $\dot{h} \equiv \partial h/\partial\tau$, satisfy *the commutation relations*

$$\begin{aligned}
[h(\tau, x), \pi(\tau', x')]|_{\tau'=\tau} &= i\delta(x - x'),\\
[h(\tau, x), h(\tau', x')]_{\tau'=\tau} = [h^*(\tau, x), h^*(\tau', x')]_{\tau'=\tau} &= 0.
\end{aligned} \tag{7.176}$$

From here on the procedure goes along the same lines as discussed in Chapter 1, Section 4.

QUANTIZATION OF THE PARAMETRIZED DIRAC FIELD

In analogy with the Stueckelberg field we can introduce an invariant evolution parameter for the Dirac field $\psi(\tau, x^\mu)$. Instead of the usual Dirac equation we have

$$i\frac{\partial \psi}{\partial \tau} = -i\gamma^\mu \partial_\mu \psi. \tag{7.177}$$

The corresponding action is

$$I[\psi, \bar{\psi}] = \int \mathrm{d}\tau\, \mathrm{d}^4x \left(i\bar{\psi}\frac{\partial \psi}{\partial \tau} + i\bar{\psi}\gamma^\mu \partial_\mu \psi\right). \tag{7.178}$$

Introducing a basis $h(\tau, x)$ in function space so that a generic vector can be expanded according to

$$\begin{aligned}\Psi &= \int \mathrm{d}\tau\,\mathrm{d}x\ \left(\bar{\psi}_\alpha(\tau,x)h_\alpha(\tau,x) + \psi_\alpha(\tau,x)\bar{h}_\alpha(\tau,x)\right)\\ &\equiv \int \mathrm{d}\tau\,\mathrm{d}x\ \left(\bar{\psi}(\tau,x)h(\tau,x) + \psi(\tau,x)\bar{h}(\tau,x)\right). \end{aligned} \tag{7.179}$$

We can write (7.178) as the scalar part of

$$I[\psi,\bar{\psi}] = \int \mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}x\,\mathrm{d}x'\ i\bar{\psi}(\tau',x')\bar{h}(\tau,x)h(\tau',x')\left(\frac{\partial\psi(\tau,x)}{\partial\tau} + i\gamma^\mu\partial_\mu\psi(\tau,x)\right), \tag{7.180}$$

where we assume

$$\begin{aligned}\bar{h}(\tau,x)\cdot h(\tau',x') &\equiv \tfrac{1}{2}\left(\bar{h}(\tau,x)h(\tau',x') + h(\tau',x')\bar{h}(\tau,x)\right)\\ &= \delta(\tau-\tau')\delta(x-x')\end{aligned} \tag{7.181}$$

For simplicity, in the relations above we have suppressed the spinor indices.

Performing partial integrations in (7.180) we can switch the derivatives from ψ to h, as in (7.149):

$$\begin{aligned}I = \int &\mathrm{d}\tau\,\mathrm{d}\tau'\,\mathrm{d}x\,\mathrm{d}x'\ \bar{\psi}(\tau,x)\\ &\times\left[-i\frac{\bar{h}(\tau,x)}{\partial\tau}h(\tau',x') - i\gamma^\mu\partial_\mu\bar{h}(\tau,x)h(\tau',x')\right]\psi(\tau',x').\end{aligned} \tag{7.182}$$

We now relax the condition (7.181). Then eq. (7.182) is no longer equivalent to the action (7.178). Actually we shall no more consider (7.182) as an action for ψ. Instead we shall fix[10] ψ according to

$$\psi(\tau,x) = \delta(\tau-\tau')\delta^4(x-x'). \tag{7.183}$$

Integrating (7.182) over τ'', x'' and renaming τ'', x'' back into τ, x, we obtain an action for basis vector fields $h(\tau,x)$, $\bar{h}(\tau,x)$:

$$I[h,\bar{h}] = \int \mathrm{d}\tau\,\mathrm{d}^4x\left[i\bar{h}\frac{\partial h}{\partial\tau} + i\bar{h}\gamma^\mu\partial_\mu h\right]. \tag{7.184}$$

Derivatives now act on h, since we have performed additional partial integrations and have omitted the surface terms.

[10]Taking also the spinor indices into account, instead of (7.183) we have

$$\psi_\alpha(\tau,x) = \delta_{\alpha,\alpha'}\delta(\tau-\tau'')\delta^4(x-x'').$$

Again we have arrived at an action for field operators h, $\bar{h}$. The equations of motion (the field equations) are

$$i\frac{\partial h}{\partial \tau} = -i\gamma^\mu \partial_\mu h. \tag{7.185}$$

The canonically conjugate variables are h and $\pi = \partial \mathcal{L}/\partial \dot{h} = i\bar{h}$, and they satisfy *the anticommutation relations*

$$\{h(\tau, x), \pi(\tau', x')\}|_{\tau'=\tau} = i\delta(x - x'), \tag{7.186}$$

or

$$\{h(\tau, x), \bar{h}(\tau', x')\}|_{\tau'=\tau} = \delta(x - x'), \tag{7.187}$$

and

$$\{h(\tau, x), h(\tau', x')\}|_{\tau'=\tau} = \{\bar{h}(\tau, x), \bar{h}(\tau', x')\}|_{\tau'=\tau} = 0. \tag{7.188}$$

The anticommutation relations above being satisfied, the Heisenberg equation

$$\frac{\partial h}{\partial \tau} = i[h, H] \, , \qquad H = \int \mathrm{d}x \, i\bar{h}\gamma^\mu \partial_\mu h \, , \tag{7.189}$$

is equivalent to the field equation (7.185).

QUANTIZATION OF THE p-BRANE: A GEOMETRIC APPROACH

We have seen that a field can be considered as an uncountable set of components of an infinite-dimensional vector. Instead of considering the action which governs the dynamics of components, we have considered the action which governs the dynamics of the basis vectors. The latter behave as operators satisfying the Clifford algebra. The quantization consisted of the crucial step in which we abolished the requirement that the basis vectors satisfy the Clifford algebra relations for a "flat" metric in function space (which is proportional to the δ-function). We admitted an arbitrary metric in principle. The action itself suggested which are the (commutation or anti commutation) relations the basis vectors (operators) should satisfy. Thus we arrived at the conventional procedure of the field quantization.

Our geometric approach brings a new insight about the nature of field quantization. In the conventional approach classical fields are replaced by operators which satisfy the canonical commutation or anti-commutation relations. In the proposed geometric approach we observe that *the field operators are, in fact, the basis vectors $h(x)$.* By its very definition a basis vector $h(x)$, for a given x, "creates" a particle at the position x. Namely, an arbitrary vector Φ is written as a superposition of basis vectors

$$\Phi = \int \mathrm{d}x' \, \phi(x') h(x') \tag{7.190}$$

and $\phi(x)$ is "the wave packet" profile. If in particular $\phi(x') = \delta(x' - x)$, i.e., if the "particle" is located at x, then

$$\Phi = h(x). \tag{7.191}$$

We shall now explore further the possibilities brought by such a geometric approach to quantization. Our main interest is to find out how it could be applied to the quantization of strings and p-branes in general. In Sec. 4.2, we have found out that a conventional p-brane can be described by the following action

$$I[X^{\alpha(\xi)}(\tau)] = \int \mathrm{d}\tau' \, \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} = \rho_{\alpha(\phi')\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')}, \tag{7.192}$$

where

$$\rho_{\alpha(\phi')\beta(\phi'')} = \frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}} \delta(\tau' - \tau'')\delta(\xi' - \xi'') g_{\alpha\beta}. \tag{7.193}$$

Here $\dot{X}^{\alpha(\phi)} \equiv \dot{X}^{\alpha(\tau,\xi)} \equiv \dot{X}^{\alpha(\xi)}(\tau)$, where $\xi \equiv \xi^a$ are the p-brane coordinates, and $\phi \equiv \phi^A = (\tau, \xi^a)$ are coordinates of the world surface which I call worldsheet.

If the $\mathcal{M}$-space metric $\rho_{\alpha(\phi')\beta(\phi'')}$ is different from (7.193), then we have a deviation from the usual Dirac–Nambu–Goto p-brane theory. Therefore in the classical theory $\rho_{\alpha(\phi')\beta(\phi'')}$ was made dynamical by adding a suitable kinetic term to the action.

Introducing the basis vectors $h_{\alpha(\phi)}$ satisfying

$$h_{\alpha(\phi')} \cdot h_{\beta(\phi'')} = \rho_{\alpha(\phi')\beta(\phi'')} \tag{7.194}$$

we have

$$I[X^{\alpha(\phi)}] = h_{\alpha(\phi')} h_{\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')}. \tag{7.195}$$

Here $h_{\alpha(\phi')}$ are fixed while $X^{\alpha(\phi')}$ are variables. If we now admit that $h_{\alpha(\phi')}$ also change with τ, we can perform the partial integrations over τ' and τ'' so that eq. (7.195) becomes

$$I = \dot{h}_{\alpha(\phi')} \dot{h}_{\beta(\phi'')} X^{\alpha(\phi')} X^{\beta(\phi'')}. \tag{7.196}$$

We now assume that $X^{\alpha(\phi')}$ is an arbitrary configuration, not necessarily the one that solves the variational principle (7.192). In particular, let us take

$$X^{\alpha(\phi')} = \delta^{\alpha(\phi')}{}_{\mu(\phi)} \ , \qquad X^{\beta(\phi'')} = \delta^{\beta(\phi'')}{}_{\mu(\phi)}, \tag{7.197}$$

which means that our p-brane is actually a *point* at the values of the parameters $\phi \equiv (\tau, \xi^a)$ and the value of the index μ. So we have

$$I_0 = \dot{h}_{\mu(\phi)} \dot{h}_{\mu(\phi)} \quad \text{no sum and no integration.} \tag{7.198}$$

By taking (7.197) we have in a sense "quantized" the classical action. The above expression is a "quantum" of (7.192).

Integrating (7.198) over ϕ and summing over μ we obtain

$$I[h_{\mu(\phi)}] = \int \mathrm{d}\phi \sum_\mu \dot{h}_\mu(\phi) \dot{h}_\mu(\phi). \tag{7.199}$$

The latter expression can be written as[11]

$$\begin{aligned} I[h_{\mu(\phi)}] &= \int \mathrm{d}\phi \, \mathrm{d}\phi' \, \delta(\phi - \phi') \eta^{\mu\nu} \dot{h}(\phi) \dot{h}_\nu(\phi') \\ &\equiv \eta^{\mu(\phi)\nu(\phi')} \dot{h}_{\mu(\phi)} \dot{h}_{\nu(\phi')}, \end{aligned} \tag{7.200}$$

where

$$\eta^{\mu(\phi)\nu(\phi')} = \eta^{\mu\nu} \delta(\phi - \phi') \tag{7.201}$$

is the flat $\mathcal{M}$-space metric. In general, of course, $\mathcal{M}$-space is not flat, and we have to use arbitrary metric. Hence (7.200) generalizes to

$$I[h_{\mu(\phi)}] = \rho^{\mu(\phi)\nu(\phi')} \dot{h}_{\mu(\phi)} \dot{h}_{\nu(\phi')}, \tag{7.202}$$

where

$$\rho^{\mu(\phi)\nu(\phi')} = h^{\mu(\phi)} \cdot h^{\nu(\phi')} = \tfrac{1}{2}(h^{\mu(\phi)} h^{\nu(\phi')} + h^{\nu(\phi')} h^{\mu(\phi)}). \tag{7.203}$$

Using the expression (7.203), the action becomes

$$I[h_{\mu(\phi)}] = h^{\mu(\phi)} h^{\nu(\phi')} \dot{h}_{\mu(\phi)} \dot{h}_{\nu(\phi')}. \tag{7.204}$$

[11] Again summation and integration convention is assumed.

METRIC IN THE SPACE OF OPERATORS

In the definition of the action (7.202), or (7.204), we used the relation (7.203) in which the $\mathcal{M}$-space metric is expressed in terms of the basis vectors $h_{\mu(\phi)}$. In order to allow for a more general case, we shall introduce the *metric* $Z^{\mu(\phi)\nu(\phi')}$ in the "space" of operators. In particular it can be

$$\tfrac{1}{2} Z^{\mu(\phi)\nu(\phi')} = h^{\mu(\phi)} h^{\nu(\phi')}, \tag{7.205}$$

or

$$\tfrac{1}{2} Z^{\mu(\phi)\nu(\phi')} = \tfrac{1}{2}(h^{\mu(\phi)} h^{\nu(\phi')} + h^{\nu(\phi')} h^{\mu(\phi)}), \tag{7.206}$$

but in general, $Z^{\mu(\phi)\nu(\phi')}$ is expressed arbitrarily in terms of $h^{\mu(\phi)}$. Then instead of (7.204), we have

$$I[h] = \tfrac{1}{2} Z^{\mu(\phi)\nu(\phi')} \dot{h}_{\mu(\phi)} \dot{h}_{\nu(\phi')} = \tfrac{1}{2} \int \mathrm{d}\tau \, Z^{\mu(\xi)\nu(\xi')} \dot{h}_{\mu(\xi)} \dot{h}_{\nu(\xi')}. \tag{7.207}$$

The factor $\frac{1}{2}$ is just for convenience; it does not influence the equations of motion.

Assuming $Z^{\mu(\phi)\nu(\phi')} = Z^{\mu(\xi)\nu(\xi')} \delta(\tau - \tau')$ we have

$$I[h] = \tfrac{1}{2} \int \mathrm{d}\tau \, Z^{\mu(\xi)\nu(\xi')} \dot{h}_{\mu(\xi)} \dot{h}_{\nu(\xi')}. \tag{7.208}$$

Now we could continue by assuming the validity of the scalar product relations (7.194) and explore the equations of motion derived from (7.207) for a chosen $Z^{\mu(\phi)\nu(\phi')}$. This is perhaps a possible approach to geometric quantization, but we shall not pursue it here.

Rather we shall forget about (7.194) and start directly from the action (7.208), considered as an action for the operator field $h_{\mu(\xi)}$ where the commutation relations should now be determined. The canonically conjugate variables are

$$h_{\mu(\xi)} \, , \qquad \pi^{\mu(\xi)} = \partial L / \partial \dot{h}_{\mu(\xi)} = Z^{\mu(\xi)\nu(\xi')} \dot{h}_{\nu(\xi')}. \tag{7.209}$$

They are assumed to satisfy the equal τ commutation relations

$$[h_{\mu(\xi)}, h_{\nu(\xi')}] = 0 \, , \qquad [\pi^{\mu(\xi)}, \pi^{\nu(\xi')}] = 0 \, ,$$
$$[h_{\mu(\xi)}, \pi^{\nu(\xi')}] = i \delta_{\mu(\xi)}{}^{\nu(\xi')}. \tag{7.210}$$

By imposing (7.210) we have abolished the Clifford algebra relation (7.194) in which the inner product (defined as the symmetrized Clifford product) is equal to a *scalar valued* metric.

The Heisenberg equations of motion are

$$\dot{h}_{\mu(\xi)} = -i[h_{\mu(\xi)}, H], \tag{7.211}$$

$$\dot{\pi}^{\mu(\xi)} = -i[\pi^{\mu(\xi)}, H], \tag{7.212}$$

where the Hamiltonian is

$$H = \tfrac{1}{2} Z_{\mu(\xi)\nu(\xi')} \pi^{\mu(\xi)} \pi^{\nu(\xi')}. \tag{7.213}$$

In particular, we may take a trivial metric which does not contain $h_{\mu(\xi)}$, e.g.,

$$Z^{\mu(\xi)\nu(\xi')} = \eta^{\mu\nu} \delta(\xi - \xi'). \tag{7.214}$$

Then the equation of motion resulting from (7.212) or directly from the action (7.208) is

$$\dot{\pi}_{\mu(\xi)} = 0. \tag{7.215}$$

Such a dynamical system cannot describe the usual p-brane, since the equations of motion is too simple. It serves here for the purpose of demonstrating the procedure. In fact, in the quantization procedures for the Klein–Gordon, Dirac, Stueckelberg field, etc., we have in fact used a fixed prescribed metric which was proportional to the δ-function.

In general, the metric $Z^{\mu(\xi)\nu(\xi')}$ is an expression containing $h_{\mu(\xi)}$. The variation of the action (7.207) with respect to $h_{\mu(\xi)}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}\tau}(Z^{\mu(\phi)\nu(\phi')} \dot{h}_{\nu(\phi)}) - \frac{1}{2} \frac{\delta Z^{\alpha(\phi')\beta(\phi'')}}{\delta h_{\mu(\phi)}} \dot{h}_{\alpha(\phi')} \dot{h}_{\beta(\phi'')} = 0. \tag{7.216}$$

Using (7.210) one finds that the Heisenberg equation (7.212) is equivalent to (7.216).

THE STATES OF THE QUANTIZED BRANE

According to the traditional approach to QFT one would now introduce a vacuum *state vector* $|0\rangle$ and define

$$h_{\alpha(\xi)}|0\rangle \quad , \quad h_{\alpha(\xi)} h_{\beta(\xi)}|0\rangle \ , ... \tag{7.217}$$

as *vectors* in Fock space. Within our geometric approach we can do something quite analogous. First we realize that because of the commutation relations (7.210) $h_{\mu(\xi)}$ are in fact not elements of the Clifford algebra. Therefore they are not vectors in the usual sense. In order to obtain vectors we introduce an object v_0 which, by definition, is a Clifford number satisfying[12]

$$v_0 v_0 = 1 \tag{7.218}$$

[12]The procedure here is an alternative to the one considered when discussing quantization of the Klein–Gordon and other fields.

and has the property that the products

$$h_{\mu(\xi)}v_0 \quad , \quad h_{\mu(\xi)}h_{\nu(\xi)}v_0 \; ... \tag{7.219}$$

are also Clifford numbers. Thus $h_{\mu(\xi)}v_0$ behaves as a vector. The inner product between such vectors is defined as usually in Clifford algebra:

$$\begin{aligned}(h_{\mu(\xi)}v_0)\cdot(h_{\nu(\xi')}v_0) \quad &\equiv \quad \tfrac{1}{2}\left[(h_{\mu(\xi)})v_0)(h_{\nu(\xi')}v_0) + (h_{\nu(\xi')}v_0)(h_{\mu(\xi)})v_0)\right] \\ &= \quad \rho_{0\mu(\xi)\nu(\xi')} \; , \end{aligned}\tag{7.220}$$

where $\rho_{0\mu(\xi)\nu(\xi')}$ is a scalar-valued metric. The choice of $\rho_{0\mu(\xi)\nu(\xi')}$ is determined by the choice of v_0. We see that the vector v_0 corresponds to the vacuum state vector of QFT. The vectors (7.219) correspond to the other basis vectors of Fock space. And we see here that choice of the vacuum vector v_0 determines the metric in Fock space. Usually basis vector of Fock space are orthonormal, hence we take

$$\rho_{0\mu(\xi)\nu(\xi')} = \eta_{\mu\nu}\delta(\xi - \xi'). \tag{7.221}$$

In the conventional field-theoretic notation the relation (7.220) reads

$$\langle 0|\tfrac{1}{2}(h_{\mu(\xi)}h_{\nu(\xi')} + h_{\nu(\xi')}h_{\mu(\xi)})|0\rangle = \rho_{0\mu(\xi)\nu(\xi')}. \tag{7.222}$$

This is the vacuum expectation value of the operator

$$\hat{\rho}_{\mu(\xi)\nu(\xi')} = \tfrac{1}{2}(h_{\mu(\xi)}h_{\nu(\xi')} + h_{\nu(\xi')}h_{\mu(\xi)}), \tag{7.223}$$

which has the role of the $\mathcal{M}$-*space metric operator*.

In a generic state $|\Psi\rangle$ of Fock space the expectation value of the operator $\hat{\rho}_{\mu(\xi)\nu(\xi')}$ is

$$\langle\Psi|\hat{\rho}_{\mu(\xi)\nu(\xi')}|\Psi\rangle = \rho_{\mu(\xi)\nu(\xi')}. \tag{7.224}$$

Hence, in a given state, for the expectation value of the metric operator we obtain a certain scalar valued $\mathcal{M}$*-space metric* $\rho_{\mu(\xi)\nu(\xi')}$.

In the geometric notation (7.224) reads

$$\langle V h_{\mu(\xi)}h_{\nu(\xi')}V\rangle_0 = \rho_{\mu(\xi)\nu(\xi')}. \tag{7.225}$$

This means that we choose a Clifford number (Clifford aggregate) V formed from (7.219)

$$V = (\phi^{\mu(\xi)}h_{\mu(\xi)} + \phi^{\mu(\xi)\nu(\xi')}h_{\mu(\xi)}h_{\nu(\xi')} + ...)v_0, \tag{7.226}$$

where $\phi^{\mu(\xi)}$, $\phi^{\mu(\xi)\nu(\xi')}$, ... , are the wave packet profiles, then we write the expression $V h_{\mu(\xi)}h_{\nu(\xi')}V$ and take its *scalar part*.

Conceptually our procedure appears to be very clear. We have an action (7.208) for operators $h_{\mu(\xi)}$ satysfying the commutation relations (7.210) and the Heisenberg equation of motion (7.212). With the aid of those operators we form a Fock space of states, and then we calculate the expectation value of the metric operator $\hat{\rho}_{\mu(\xi)\nu(\xi')}$ in a chosen state. *We interpret this expectation value as the classical metric of $\mathcal{M}$-space.* This is justified because there is a correspondence between the operators $h_{\mu(\xi)}$ and the $\mathcal{M}$-space basis vectors (also denoted $h_{\mu(\xi)}$, but obeying the Clifford algebra relations (7.198)). The operators create, when acting on $|0\rangle$ or v_0, the many brane states and it is natural to interpret the expectation value of $\hat{\rho}_{\mu(\xi)\nu(\xi')}$ as the classical $\mathcal{M}$-space metric for such a many brane configuration.

WHICH CHOICE FOR THE OPERATOR METRIC $Z^{\mu(\xi)\nu(\xi')}$?

A question now arise of how to choose $Z^{\mu(\xi)\nu(\xi')}$. In principle any combination of operators $h_{\mu(\xi)}$ is good, provided that $Z^{\mu(\xi)\nu(\xi')}$ has its inverse defined according to

$$Z^{\mu(\xi)\alpha(\xi'')}Z_{\alpha(\xi'')\nu(\xi')} = \delta^{\mu(\xi)}{}_{\nu(\xi')} \equiv \delta^{\mu}{}_{\nu}\delta(\xi - \xi'). \tag{7.227}$$

Different choices of $Z^{\mu(\xi)\nu(\xi')}$ mean different membrane theories, and hence different expectation values $\rho_{\mu(\xi)\nu(\xi')} = \langle\hat{\rho}_{\mu(\xi)\nu(\xi')}\rangle$ of the $\mathcal{M}$-space metric operator $\hat{\rho}_{\mu(\xi)\nu(\xi')}$. We have already observed that different choices of $\rho_{\mu(\xi)\nu(\xi')}$ correspond to different classical membrane theories. In order to get rid of a fixed background we have given $\rho_{\mu(\xi)\nu(\xi')}$ the status of a dynamical variable and included a kinetic term for $\rho_{\mu(\xi)\nu(\xi')}$ in the action (or equivalently for $h_{\mu(\xi)}$, which is the "square root" of $\rho_{\mu(\xi)\nu(\xi')}$, since classically $\rho_{\mu(\xi)\nu(\xi')} = h_{\mu(\xi)} \cdot h_{\nu(\xi')}$). In performing the quantization we have seen that to the classical vectors $h_{\mu(\xi)}$ there correspond quantum operators[13] $\hat{h}_{\mu(\xi)}$ which obey the equations of motion determined by the action (7.208). Hence in the quantized theory we do not need a separate kinetic term for $\hat{h}_{\mu(\xi)}$. But now we have something new, namely, $Z^{\mu(\xi)\nu(\xi')}$ which is a background metric in the space of operators $\hat{h}_{\mu(\xi)}$. In order to obtain a background independent theory we need a kinetic term for $Z^{\mu(\xi)\nu(\xi')}$. The search for such a kinetic term will remain a subject of future investigations. It has its parallel in the attempts to find a background independent string or p-brane theory. However, it may turn out that we do not need a kinetic term for $Z^{\mu(\xi)\nu(\xi')}$ and that it is actually given by the expression (7.205) or (7.206), so that (7.204) is already the "final" action for the quantum p-brane.

[13]Now we use hats to make a clear distinction between the classical vectors $h_{\mu(\xi)}$, satisfying the Clifford algebra relations (7.194), and the quantum operators $\hat{h}_{\mu(\xi)}$ satisfying (7.210).

In the field theory discussed previously we also have a fixed metric $Z^{\mu(\xi)\nu(\xi')}$, namely, $Z^{(\xi)(\xi')} = \delta(\xi - \xi')$ for the Klein–Gordon and similarly for the Dirac field. Why such a choice and not some other choice? This clearly points to the plausible possibility that the usual QFT is not complete. That QFT is not yet a finished story is clear from the occurrence of infinities and the need for "renormalization"[14].

[14]An alternative approach to the quantization of field theories, also based on Clifford algebra, has been pursued by Kanatchikov [87].