THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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Chapter 9

THE EINSTEIN–HILBERT ACTION ON THE BRANE AS THE EFFECTIVE ACTION

After so many years of intensive research the quantization of gravity is still an unfinished project. Amongst many approaches followed, there is the one which seems to be especially promising. This is the so called induced gravity proposed by Sakharov [96]. His idea was to treat the metric not as a fundamental field but as one induced from more basic fields. The idea has been pursued by numerous authors [97]; especially illuminating are works by Akama, Terazawa and Naka [98]. Their basic action contains N scalar fields and it is formally just a slight generalization of the well-known Dirac–Nambu–Goto action for an n-dimensional world sheet swept by an $(n-1)$-dimensional membrane.

Here we pursue such an approach and give a concrete physical interpretation of the N scalar fields which we denote $\eta^a(x)$. We assume that the spacetime is a surface V_4, called the spacetime sheet, embedded in a higher-dimensional space V_N, and $\eta^a(x)$ are the embedding functions. An embedding model has been first proposed by Regge and Teitelboim [99] and investigated by others [100]. In that model the action contains the Ricci scalar expressed in terms of the embedding functions. In our present model [88, 90, 89], on the contrary, we start from an action which is essentially the minimal surface action[1] weighted with a function $\omega(\eta)$ in V_N. For a suitably chosen ω, such that it is singular (δ-function like) on certain surfaces $\widehat{V}_m$, also embedded in V_N, we obtain on V_4 a set of world lines. In the previous chapter it was shown that these worldlines are geodesics of V_4, provided that V_4 described by $\eta^a(x)$ is a solution to our variational

[1] Recently Bandos [91] has considered a string-like description of gravity by considering bosonic p-branes coupled to an antisymmetric tensor field.

procedure. I will show that after performing functional integrations over $\eta^a(x)$ we obtain two contributions to the path integral. One contribution comes from all possible $\eta^a(x)$ not intersecting $\widehat{V}_m$, and the other from those $\eta^a(x)$ which do intersect the surfaces $\widehat{V}_m$. In the effective action so obtained the first contribution gives the Einstein–Hilbert term R plus higher-order terms like R^2. The second contribution can be cast into the form of a path integral over all possible worldlines $X^\mu(\tau)$. Thus we obtain an action which contains matter sources and a kinetic term for the metric field (plus higher orders in R). So in the proposed approach both the metric field and the matter field are induced from more basic fields $\eta^a(x)$.

9.1. THE CLASSICAL MODEL

Let us briefly summarize the discussion of Chapter 8. We assume that the arena where physics takes place is an N-dimensional space V_N with $N \geq 10$. Next we assume that an n-dimensional surface V_n living in V_N represents a possible spacetime. The parametric equation of such a 'spacetime sheet' V_n is given by the embedding functions $\eta^a(x^\mu)$, $a = 0, 1, 2, ..., N$, where x^μ, $\mu = 0, 1, 2, \ldots, n-1$, are coordinates (parameters) on V_n. We assume that the action is just that for a minimal surface V_n

$$I[\eta^a(x)] = \int (\det \partial_\mu \eta^a \, \partial_\nu \eta^b \gamma_{ab})^{1/2} \mathrm{d}^n x, \tag{9.1}$$

where γ_{ab} is the metric tensor of V_N. The dimension of the spacetime sheet V_n is taken here to be arbitrary, in order to allow for the Kaluza–Klein approach. In particular, we may take $n = 4$. We admit that the embedding space is curved in general. In particular, let us consider the case of a conformally flat V_N, such that $\gamma_{ab} = \omega^{2/n} \eta_{ab}$, where η_{ab} is the N-dimensional Minkowski tensor. Then Eq. (9.1) becomes

$$I[\eta^a(x)] = \int \omega(\eta) (\det \partial_\mu \eta^a \, \partial_\nu \eta^b \eta_{ab})^{1/2} \mathrm{d}^n x. \tag{9.2}$$

From now on we shall forget about the origin of $\omega(\eta)$ and consider it as a function of position in a *flat* embedding space. The indices a, b, c will be raised and lowered by η^{ab} and η_{ab}, respectively.

In principle $\omega(\eta)$ is arbitrary. But it is very instructive to choose the following function:

$$\omega(\eta) = \omega_0 + \sum_i \int m_i \frac{\delta^N(\eta - \hat{\eta}_i)}{\sqrt{|\gamma|}} \, \mathrm{d}^m \hat{x} \sqrt{|\hat{f}|} \, , \tag{9.3}$$

where $\eta^a = \hat{\eta}^a_i(\hat{x})$ is the parametric equation of an m-dimensional surface $\widehat{V}^{(i)}_m$, called the *matter sheet*, also embedded in V_N, $\hat{f}$ is the determinant of the induced metric on $\widehat{V}^{(i)}_m$, and $\sqrt{|\gamma|}$ allows for taking curved coordinates in otherwise flat V_N. If we take $m = N - n + 1$ then the intersection of V_n and $\widehat{V}^{(i)}_m$ can be a (one-dimensional) line, i.e., a worldline C_i on V_n. In general, when $m = N - n + (p+1)$ the intersection can be a $(p+1)$-dimensional world sheet representing the motion of a p-dimensional membrane (also called p-brane). In this chapter we confine our consideration to the case $p = 0$, that is, to the motion of a point particle.

Inserting (9.3) into (9.2) and writing $f_{\mu\nu} \equiv \partial_\mu \eta^a \partial_\nu \eta_a$, $f \equiv \det f_{\mu\nu}$ we obtain

$$I[\eta] = \omega_0 \int \mathrm{d}^n x \sqrt{|f|} \; + \int \mathrm{d}^n x \sum_i m_i \, \delta^n(x - X_i)(f_{\mu\nu}\dot{X}^\mu_i \dot{X}^\nu_i)^{1/2} \, \mathrm{d}\tau. \quad (9.4)$$

As already explained in Sec. 8.1, the result above was obtained by writing

$$\mathrm{d}^m \hat{x} = \mathrm{d}^{m-1}\hat{x}\, \mathrm{d}\tau \; , \qquad \hat{f} = \hat{f}^{(m-1)}(\dot{X}^\mu_i \dot{X}_{i\mu})^{1/2}$$

and taking the coordinates η^a such that $\eta^a = (x^\mu, \eta^n, ..., \eta^{N-1})$, where x^μ are (curved) coordinates on V_n. The determinant of the metric of the embedding space V_N in such curvilinear coordinates is then $\gamma = \det \partial_\mu \eta^a \, \partial_\nu \eta_a$ $= f$.

If we vary the action (9.4) with respect to $\eta^a(x)$ we obtain

$$\partial_\mu \left[\sqrt{|f|}\, (\omega_0 f^{\mu\nu} + T^{\mu\nu}) \partial_\nu \eta_a \right] = 0, \quad (9.5)$$

where

$$T^{\mu\nu} = \frac{1}{\sqrt{|f|}} \sum_i \int \mathrm{d}^n x \, m_i \, \delta^n(x - X_i) \, \frac{\dot{X}^\mu_i \dot{X}^\nu_i}{(\dot{X}^\alpha_i \dot{X}_{i\alpha})^{1/2}} \, \mathrm{d}\tau \quad (9.6)$$

is the stress–energy tensor of dust. Eq. (9.5) can be rewritten in terms of the covariant derivative D_μ on V_n:

$$D_\mu \left[(\omega_0 f^{\mu\nu} + T^{\mu\nu}) \partial_\nu \eta_a \right] = 0. \quad (9.7)$$

The latter equation gives

$$\partial_\nu \eta_a \, D_\mu T^{\mu\nu} + (\omega_0 f^{\mu\nu} + T^{\mu\nu}) D_\mu D_\nu \eta_a = 0, \quad (9.8)$$

where we have taken into account that a covariant derivative of metric is zero, i.e., $D_\alpha f_{\mu\nu} = 0$ and $D_\alpha f^{\mu\nu} = 0$, which implies also $\partial_\alpha \eta^c \, D_\mu D_\nu \eta_c = 0$, since $f_{\mu\nu} \equiv \partial_\mu \eta^a \partial_\nu \eta_a$. Contracting Eq.(9.8) by $\partial^\alpha \eta^a$ we have

$$D_\mu T^{\mu\nu} = 0. \quad (9.9)$$

The latter are the well known equations of motion for the sources. In the case of dust (9.9) implies that dust particles move along geodesics of the spacetime V_n. We have thus obtained the very interesting result that the worldlines C_i which are obtained as intersections $V_n \cap \hat{V}_m^{(i)}$ are geodesics of the spacetime sheet V_n. The same result is also obtained directly by varying the action (9.4) with respect to the variables $X_i^\mu(\tau)$.

A solution to the equations of motion (9.5) (or (9.7)) gives both: the spacetime sheet $\eta^a(x)$ and the worldlines $X_i^\mu(\tau)$. Once $\eta^a(x)$ is determined the induced metric $g_{\mu\nu} = \partial_\mu \eta^a \partial_\nu \eta_a$ is determined as well. But such a metric in general does not satisfy the Einstein equations. In the next section we shall see that quantum effects induce the necessary Einstein–Hilbert term $(-g)^{1/2} R$.

9.2. THE QUANTUM MODEL

For the purpose of quantization we shall use the classical action [88], that is, a generalization of the well known Howe–Tucker action [31], which is equivalent to (9.2) (see also Sec. 4.2):

$$I[\eta^a, g^{\mu\nu}] = \tfrac{1}{2} \int \mathrm{d}^n x \sqrt{|g|}\, \omega(\eta)(g^{\mu\nu}\partial_\mu \eta^a \partial_\nu \eta_a + 2 - n). \tag{9.10}$$

It is a functional of the embedding functions $\eta^a(x)$ and the Lagrange multipliers $g^{\mu\nu}$. Varying (9.10) with respect to $g^{\mu\nu}$ gives the constraints

$$-\frac{\omega}{4} \sqrt{|g|}\, g_{\alpha\beta}(g^{\mu\nu}\partial_\mu \eta^a \partial_\nu \eta_a + 2 - n) + \frac{\omega}{2} \sqrt{|g|}\, \partial_\alpha \eta^a \partial_\beta \eta_a = 0. \tag{9.11}$$

Contracting (9.11) with $g^{\alpha\beta}$, we find $g^{\mu\nu}\partial_\mu \eta^a \partial_\nu \eta_a = n$, and after inserting the latter relation back into (9.11) we find

$$g_{\alpha\beta} = \partial_\alpha \eta^a \partial_\beta \eta_a, \tag{9.12}$$

which is the expression for an induced metric on the surface V_n. In the following paragraphs we shall specify $n = 4$; however, whenever necessary, we shall switch to the generic case of arbitrary n.

In the classical theory we may say that a 4-dimensional spacetime sheet is swept by a 3-dimensional space-like hypersurface Σ which moves forward in time. The latter surface is specified by initial conditions, and the equations of motion then determine Σ at every value of a time-like coordinate $x^0 = t$. Knowledge of a particular hypersurface Σ implies knowledge of the corresponding intrinsic 3-geometry specified by the 3-metric $g_{ij} = \partial_i \eta^a \partial_j \eta_a$ induced on Σ $(i, j = 1, 2, 3)$. However, knowledge of the data $\eta^a(t, x^i)$ on an entire infinite Σ is just a mathematical idealization which cannot be realized in a practical situation by an observer because of the finite speed of light.

In quantum theory a state of a surface Σ is not specified by the coordinates $\eta^a(t, x^i)$, but by a wave functional $\psi[t, \eta^a(x^i)]$. The latter represents the probability amplitude that at time t an observer would obtain, as a result of measurement, a particular surface Σ.

The probability amplitude for the transition from a state with definite Σ_1 at time t_1 to a state Σ_2 at time t_2 is given by the Feynman path integral

$$K(2,1) = \langle \Sigma_2, t_2 | \Sigma_1, t_1 \rangle = \int \mathrm{e}^{i\, I[\eta,g]} \mathcal{D}\eta \mathcal{D}g. \tag{9.13}$$

Now, if in Eq.(9.13) we perform integration only over the embedding functions $\eta^a(x^\mu)$, then we obtain the so called *effective action* $I_{\rm eff}$

$$\mathrm{e}^{i I_{\rm eff}[g]} \equiv \int \mathrm{e}^{i\, I[\eta,g]} \mathcal{D}\eta, \tag{9.14}$$

which is a functional of solely the metric $g^{\mu\nu}$. From eq. (9.14) we obtain by functional differentiation

$$\frac{\delta I_{\rm eff}[g]}{\delta g^{\mu\nu}} = \frac{\int \frac{\delta I[\eta,g]}{\delta g^{\mu\nu}}\, \mathrm{e}^{i\, I[\eta,g]} \mathcal{D}\eta}{\int \mathrm{e}^{i\, I[\eta,g]} \mathcal{D}\eta} \equiv \left\langle \frac{\delta I[\eta, g]}{\delta g^{\mu\nu}} \right\rangle = 0. \tag{9.15}$$

On the left hand side of eq. (9.15) we have taken into account the constraints $\delta I[\eta, g]/\delta g^{\mu\nu} = 0$ (explicitly given in eq. (9.11)).

The expression $\delta I_{\rm eff}[\eta, g]/\delta g^{\mu\nu} = 0$ gives the classical equations for the metric $g_{\mu\nu}$, derived from the effective action.

Let us now consider a specific case in which we take for $\omega(\eta)$ the expression (9.3). Then our action (9.10) splits into two terms

$$I[\eta, g] = I_0[\eta, g] + I_m[\eta, g] \tag{9.16}$$

with

$$I_0[\eta, g] = \frac{\omega_0}{2} \int \mathrm{d}^n\, x \sqrt{|g|}\, (g^{\mu\nu} \partial_\mu \eta^a \partial_\nu \eta_a + 2 - n), \tag{9.17}$$

$$\begin{aligned} I_m[\eta, g] = \frac{1}{2} \int \mathrm{d}^n x \, \sqrt{|g|}\, \sum_i m_i \, \frac{\delta^N(\eta - \hat{\eta}_i)}{\sqrt{|\gamma|}}\, \mathrm{d}^m \hat{x} \\ \times \sqrt{|\hat{f}|}\, (g^{\mu\nu} \partial_\mu \eta^a \partial_\nu \eta_a + 2 - n). \end{aligned} \tag{9.18}$$

The last expression can be integrated over $m{-}1$ coordinates $\hat{x}^\mu$, while $\hat{x}^0$ is chosen so to coincide with the parameter τ of a worldline C_i. We also split the metric as $g^{\mu\nu} = n^\mu n^\nu / n^2 + \bar{g}^{\mu\nu}$, where n^μ is a time-like vector and $\bar{g}^{\mu\nu}$ is the projection tensor, giving $\bar{g}^{\mu\nu} \partial_\mu \eta^a \partial_\nu \eta_a = n - 1$. So we obtain

$$
\begin{aligned}
I_m[\eta, g] = \frac{1}{2} \int \mathrm{d}^n x \sqrt{|g|} \sum_i \frac{\delta^n(x - X_i(\tau))}{\sqrt{|g|}} \\
\times \left(\frac{g_{\mu\nu} \dot{X}_i^\mu \dot{X}_i^\nu}{\mu_i} + \mu_i \right) \mathrm{d}\tau = I_m[X_i, g]. \qquad (9.19)
\end{aligned}
$$

Here $\mu_i \equiv 1/\sqrt{n^2}|_{C_i}$ are the Lagrange multipliers giving, after variation, the worldline constraints $\mu_i^2 = \dot{X}_i^2$. Eq. (9.19) is the well known Howe-Tucker action [31] for point particles.

Now let us substitute our specific action (9.16)–(9.19) into the expression (9.14) for the effective action. The functional integration now runs over two distinct classes of spacetime sheets V_n [represented by $\eta^a(x)$]:

(a) those V_n which either *do not intersect* the matter sheets $\widehat{V}_m^{(i)}$ [represented by $\hat{\eta}_i^a(\hat{x})$] and do not self-intersect, or, if they do, the intersections are just single points, and

(b) those V_n which *do intersect* $\widehat{V}_m^{(i)}$ and/or self-intersect, the intersections being worldlines C_i.

The sheets V_n which correspond to the case (b) have two distinct classes of points (events):

(b1) *the points outside the intersection*, i.e., outside the worldlines C_i, and

(b2) *the points on the intersection*, i.e., the events belonging to C_i.

The measure $\mathcal{D}\eta^a(x)$ can be factorized into the contribution which corresponds to the case (a) or (b1) ($x \notin C_i$), and the contribution which corresponds to the case (b2) ($x \in C_i$):

$$
\begin{aligned}
\mathcal{D}\eta &= \prod_{a,x} (|g(x)|)^{1/4} \mathrm{d}\eta^a(x) \\
&= \prod_{a,x \notin C_i} (|g(x)|)^{1/4} \mathrm{d}\eta^a(x) \prod_{a,x \in C_i} (|g(x)|)^{1/4} \mathrm{d}\eta^a(x) \\
&\equiv \mathcal{D}_0\eta \, \mathcal{D}_m. \qquad (9.20)
\end{aligned}
$$

The additional factor $(|g(x)|)^{1/4}$ comes from the requirement that the measure be invariant under reparametrizations of x^μ (see Ref. [102] and Sec. 4.1 for details). From the very definition of $\prod_{a,x \in C_i} (|g(x)|)^{1/4} \mathrm{d}\eta^a(x)$ as the measure of the set of points on the worldlines C_i (each C_i being represented by an equation $x = X_i^\mu(\tau)$) we conclude that

$$
\mathcal{D}_m \eta^a(x) = \mathcal{D} X_i^\mu(\tau). \qquad (9.21)
$$

The effective action then satisfies [owing to (9.16)–(9.21)]

$$\mathrm{e}^{iI_{\rm eff}[g]} = \int \mathrm{e}^{iI_0[\eta,g]}\, \mathcal{D}_0\eta\, \mathrm{e}^{iI_m[X_i,g]}\, \mathcal{D}X_i \equiv \mathrm{e}^{iW_0}\, \mathrm{e}^{iW_{\rm m}}, \tag{9.22}$$

$$I_{\rm eff} = W_0 + W_{\rm m}. \tag{9.23}$$

The measure $\mathcal{D}_0\eta$ includes all those sheets V_n that do not intersect the matter sheet and do not self-intersect [case (a)], and also all those sheets which do intersect and/or self-intersect [case (b1)], apart from the points on the intersections.

The first factor in the product (9.22) contains the action (9.17). The latter has the same form as the action for N scalar fields in a curved background spacetime with the metric $g_{\mu\nu}$. The corresponding effective action has been studied and derived in Refs. [103]. Using the same procedure and substituting our specific constants $\omega_0/2$ and $(n-1)$ occurring in eq. (9.17), we find the following expression for the effective action:

$$I_{\rm eff} = \lim_{\mu^2\to 0} \int \mathrm{d}^n x \sqrt{|g|} \left(N\, \omega_0^{-1} (4\pi)^{-n/2} \sum_{j=0}^{\infty} (\mu^2)^{n/2-j}\, a_j(x)\, \Gamma(j-\frac{n}{2}) \right.$$
$$\left. + \frac{\omega_0}{2}(2-n) \right) \tag{9.24}$$

with

$$a_0(x) = 1 \tag{9.25}$$

$$a_1(x) = R/6 \tag{9.26}$$

$$a_2(x) = \frac{1}{12}R^2 + \frac{1}{180}(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} - R_{\alpha\beta}R^{\alpha\beta}) - \frac{1}{30}D_\mu D^\mu R \tag{9.27}$$

where R, $R_{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}$ are the Ricci scalar, the Ricci tensor and the Riemann tensor, respectively. The function $\Gamma(y) = \int_0^\infty \mathrm{e}^{-t}\, t^{y-1}\, \mathrm{d}t$; it is divergent at negative integers y and finite at $y = \frac{3}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, -\frac{5}{2}, \ldots$. The effective action (9.24) is thus divergent in even-dimensional spaces V_n. For instance, when $n = 4$, the argument in eq. (9.24) is $j-2$, which for $j = 0, 1, 2, \ldots$ is indeed a negative integer. Therefore, in order to obtain a finite effective action one needs to introduce a suitable cut-off parameter Λ and replace

$$(\mu^2)^{n/2-j}\Gamma(j-\frac{n}{2}) = \int_0^\infty t^{j-1-n/2} e^{-\mu^2 t}\mathrm{d}t$$

with

$$\int_{1/\Lambda^2}^\infty t^{j-1-n/2} e^{-\mu^2 t}\mathrm{d}t$$

Then we find

$$I_{\text{eff}} = \int \mathrm{d}^n x \sqrt{|g|} \Big(\lambda_0 + \lambda_1 R + \lambda_2 R^2 + \lambda_3 (R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} - R_{\alpha\beta} R^{\alpha\beta}) + \lambda_4 D_\mu D^\mu R + O(\Lambda^{n-6}) \Big) , \tag{9.28}$$

where[2], for $n > 4$,

$$\begin{aligned}
\lambda_0 &= \frac{N\omega_0}{4(4\pi)^{n/2}} \frac{2\Lambda^n}{n} + \frac{\omega_0}{2}(2-n), \\
\lambda_1 &= \frac{N\omega_0}{4(4\pi)^{n/2}} \frac{2\Lambda^{n-2}}{6(n-2)}, \\
\lambda_2 &= \frac{N\omega_0}{4(4\pi)^{n/2}} \frac{2\Lambda^{n-4}}{12(n-4)}, \\
\lambda_3 &= \frac{N\omega_0}{4(4\pi)^{n/2}} \frac{2\Lambda^{n-4}}{180(n-4)}, \\
\lambda_4 &= -\frac{N\omega_0}{4(4\pi)^{n/2}} \frac{2\Lambda^{n-4}}{30(n-4)}.
\end{aligned} \tag{9.29}$$

Here λ_0 is the cosmological constant, whilst λ_1 is related to the gravitational constant G in n-dimensions according to

$$\lambda_1 \equiv (16\pi G)^{-1}. \tag{9.30}$$

This last relation shows how the induced gravitational constant is calculated in terms of ω_0, which is a free parameter of our embedding model, and the cutoff parameter Λ. According to Akama [98], et al., and Sugamoto [92], we consider the cutoff Λ to be a physical quantity, the inverse thickness of a membrane, because the original action (9.17) describes an idealized theory of extended objects with vanishing thickness, but the real extended objects have non-vanishing thickness playing a role of the ultraviolet cutoff. In the case of thin extended objects, we can ignore the $O(\Lambda^{n-4})$ terms in eq. (9.28).

In the above calculation of the effective action we have treated all functions $\eta^a(x)$ entering the path integral (9.14) as those representing distinct spacetime sheets V_n. However, owing to the reparametrization invariance, there exist equivalence classes of functions representing the same V_n. This complication must be taken into account when calculating the entire amplitude (9.13). The conventional approach is to introduce ghost fields which

[2]In the case $n = 4$ it is convenient to use Akama's regularization [98]. Similarly for the problematic coefficients in higher dimensions.

cancel the non-physical degrees of freedom. An alternative approach, first explored in ref. [53]–[55] and much discussed in this book, is to assume that all possible embedding functions $\eta^a(x)$ can nevertheless be interpreted as describing physically distinct spacetime sheets $\mathcal{V}_n$. This is possible if the extra degrees of freedom in $\eta^a(x)$ describe tangent deformations of $\mathcal{V}_n$. Such a deformable surface $\mathcal{V}_n$ is then a different concept from a non-deformable surface V_n. The path integral can be performed in a straightforward way in the case of $\mathcal{V}_n$, as was done in arriving at the result (9.24). However, even from the standard point of view [93], gauge fixing is not required for the calculation of the effective action, since in the η^a integration, $g_{\mu\nu}$ is treated as a fixed background.

Let us now return to eq. (9.22). In the second factor of eq. (9.22) the functional integration runs over all possible worldlines $X^\mu(\tau)$. Though they are obtained as intersections of *various* V_n with $\widehat{V}_m^{(i)}$, we may consider all those worldlines to be lying in *the same* effective spacetime $V_n^{(\text{eff})}$ with the intrinsic metric $g_{\mu\nu}$. In other words, in the effective theory we identify all those various V_n's, having the same induced (intrinsic) metric $g_{\mu\nu}$, as one and the same spacetime. If one considers the embedding space V_N of sufficiently high dimension N, then there is enough freedom to obtain as an intersection any possible worldline in the effective spacetime $V_n^{(\text{eff})}$.

This is even more transparent when the spacetime sheet self-intersects. Then we can have a situation in which various sheets coincide in all the points, apart from the points in the vicinity of the intersections

When the condition for the classical approximation is satisfied, i.e., when $I_m \gg \hbar = 1$, then only those trajectories X_i^μ which are close to the classically allowed ones effectively contribute:

$$\mathrm{e}^{iW_{\mathrm{m}}} = \mathrm{e}^{iI_m[X_i,g]} \,, \qquad W_{\mathrm{m}} = I_m \tag{9.31}$$

The effective action is then a sum of the gravitational kinetic term W_0 given in eq. (9.28) and the source term I_m given in (9.19). Variation of I_{eff} with respect to $g^{\mu\nu}$ then gives the gravitational field equation in the presence of point particle sources with the stress–energy tensor $T^{\mu\nu}$ as given in Eq. (9.6):

$$R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} + \lambda_0 g^{\mu\nu} + (\text{higher order terms}) = -8\pi G\, T^{\mu\nu} \,. \tag{9.32}$$

However, in general the classical approximation is not satisfied, and in the evaluation of the matter part W_{m} of the effective action one must take into account the contributions of all possible paths $X_i^\mu(\tau)$. So we have

(confining ourselves to the case of only one particle, omitting the subscript i, and taking $\mu = 1$)

$$\begin{aligned} \mathrm{e}^{iW_\mathrm{m}} &= \int_{x_a}^{x_b} \mathcal{D}X \exp\left(\frac{i}{2}\int_{\tau_a}^{\tau_b} \mathrm{d}\tau \, m\,(g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu + 1)\right) \\ &= \mathcal{K}(x_b, \tau_b; x_a, \tau_a) \equiv \mathcal{K}(b, a) \end{aligned} \tag{9.33}$$

which is a propagator or a Green's function satisfying (for $\tau_b \geq \tau_a$) the equation

$$\left(i\,\frac{\partial}{\partial \tau_b} - H\right) \mathcal{K}(x_b, \tau_b; x_a, \tau_a) = -\frac{1}{\sqrt{|g|}}\,\delta^n(x_b - x_a)\,\delta(\tau_b - \tau_a)\,, \tag{9.34}$$

where $H = (|g|)^{-1/2}\partial_\mu((|g|)^{1/2}\partial^\mu)$. From (9.34) it follows that

$$\mathcal{K}(b, a) = -\left[i\frac{\partial}{\partial\tau} - H\right]^{-1}_{x_b, \tau_b; x_a, \tau_a} \tag{9.35}$$

where the inverse Green's function is treated as a matrix in the (x, τ) space.

Using the following relation [32] for Gaussian integration

$$\int y_m y_n \prod_{i=1}^{N} \mathrm{d}y_i \, \mathrm{e}^{-\sum_{ij} y_i A_{ij} y_j} \propto \frac{(A^{-1})_{mn}}{(\det|A_{ij}|)^{1/2}}, \tag{9.36}$$

we can rewrite the Green's function in terms of the second quantized field:

$$\begin{aligned} \mathcal{K}(a, b) = \int \psi^*(x_b, \tau_b)\,\psi(x_b, \tau_b)\,\mathcal{D}\psi^*\,\mathcal{D}\psi \\ \times \exp\left[-i\int \mathrm{d}\tau\,\mathrm{d}^n x\,\sqrt{|g|}\,\psi^*(i\partial_\tau - H)\psi\right]. \end{aligned} \tag{9.37}$$

If the conditions for the "classical" approximation are satisfied, so that the phase in (9.37) is much greater than $\hbar = 1$, then only those paths $\psi(\tau, x)\,, \quad \psi^*(\tau, x)$ which are close to the extremal path, along which the phase is zero, effectively contribute to $\mathcal{K}(a, b)$. Then the propagator is simply

$$\mathcal{K}(b, a) \propto \exp\left[-i\int \mathrm{d}\tau\,\mathrm{d}^n x \sqrt{|g|}\,\psi^*(i\partial_\tau - H)\psi\right]. \tag{9.38}$$

The effective one-particle "matter" action W_m is then

$$W_\mathrm{m} = -\int \mathrm{d}\tau\,\mathrm{d}^n x\,\sqrt{|g|}\,\psi^*(\tau, x)(i\partial_\tau - H)\psi(\tau, x). \tag{9.39}$$

If we assume that the τ-dependence of the field $\psi(\tau, x)$ is specified[3] by $\psi(\tau, x) = \mathrm{e}^{im\tau}\, \phi(x)$, then eq. (9.39) simplifies to the usual well known expression for a scalar field:

$$W_{\mathrm{m}} = \int \mathrm{d}^n x \sqrt{|g|}\phi^*(x)\Big[\frac{1}{\sqrt{|g|}}\partial_\mu(\sqrt{|g|}\partial^\mu) + m^2\Big)\phi(x)$$
$$= -\tfrac{1}{2}\int \mathrm{d}^n x\, \sqrt{|g|}\, (g^{\mu\nu}\, \partial_\mu\phi^*\, \partial_\nu\phi - m^2)\,, \tag{9.40}$$

where the surface term has been omitted.

Thus, starting from our basic fields $\eta^a(x)$, which are the embedding functions for a spacetime sheet V_n, we have arrived at the effective action I_{eff} which contains the kinetic term W_0 for the metric field $g^{\mu\nu}$ (see Eq. (9.28)) and the source term W_{m} (see eq. (9.32) and (9.18), or (9.40)). Both the metric field $g_{\mu\nu}$ and the bosonic matter field ϕ are induced from the basic fields $\eta^a(x)$.

9.3. CONCLUSION

We have investigated a model which seems to be very promising in attempts to find a consistent relation between quantum theory and gravity. Our model exploits the approach of induced gravity and the concept of spacetime embedding in a higher-dimensional space and has the following property of interest: what appear as worldlines in, e.g., a 4-dimensional spacetime, are just the intersections of a spacetime sheet V_4 with "matter" sheets $\widehat{V}_{\mathrm{m}}^{(i)}$, or self-intersections of V_4. Various choices of spacetime sheets then give various configurations of worldlines. Instead of V_4 it is convenient to consider a spacetime sheet V_n of arbitrary dimension n. When passing to the quantized theory a spacetime sheet is no longer definite. All possible alternative spacetime sheets are taken into account in the expression for a wave functional or a Feynman path integral. The intersection points of V_n with itself or with a matter sheet $\widehat{V}_m^{(i)}$ are treated specially, and it is found that their contribution to the path integral is identical to the contribution of a point particle path. We have paid special attention to the effective action which results from functionally integrating out all possible embeddings with the same induced metric tensor. We have found that the effective ac-

[3] By doing so we in fact project out the so called physical states from the set of all possible states. Such a procedure, which employs a "fictitious" evolution parameter τ is often used (see Chapter 1). When gauge fixing the action (9.33) one pretends that such an action actually represents an "evolution" in parameter τ. Only later, when all the calculations (e.g., path integral) are performed, one integrates over τ and thus projects out the physical quantities.

tion, besides the Einstein–Hilbert term and the corresponding higher-order terms, also contains a source term. The expression for the latter is equal to that of a classical (when such an approximation can be used) or quantum point particle source described by a scalar (bosonic) field.

In other words, we have found that the (n-dimensional) Einstein equations (including the R^2 and higher derivative terms) with classical or quantum point-particle sources are effective equations resulting from performing a quantum average over all possible embeddings of the spacetime. Gravity —as described by Einstein's general relativity— is thus considered not as a fundamental phenomenon, but as something induced quantum-mechanically from more fundamental phenomena.

In our embedding model of gravity with bosonic sources, new and interesting possibilities are open. For instance, instead of a 4-dimensional spacetime sheet we can consider a sheet which possesses additional dimensions, parametrized either with usual or Grassmann coordinates. In such a way we expect to include, on the one hand, via the Kaluza–Klein mechanism, other interactions as well besides the gravitational one, and, on the other hand, fermionic sources. The latter are expected also to result from the polyvector generalization of the action, as discussed in Sec. 2.5.

It is well known that the quantum field theory based on the action (9.40) implies the infinite vacuum energy density and consequently the infinite (or, more precisely, the Planck scale cutoff) cosmological constant. This is only a part of the total cosmological constant predicted by the complete theory, as formulated here. The other part comes from eqs. (9.28), (9.29). The parameters N, ω_0, Λ and n could in principle be adjusted so to give a small or vanishing total cosmological constant. If carried out successfully this would be then an alternative (or perhaps a complement) to the solution of the cosmological constant problem as suggested in Chapter 3.

Finally, let us observe that within the "brane world" model of Randall and Sundrum [95], in which matter fields are localized on a 3-brane, whilst gravity propagates in the bulk, it was recently proposed [104] to treat the Einstein–Hilbert term on the brane as being induced in the quantum theory of the brane. It was found that the localized matter on a brane can induce via loop correction a 4D kinetic term for gravitons. This also happens in our quantum model if we consider the effective action obtained after integrating out the second quantized field in (9.37). (The procedure expounded in refs. [21] is then directly applicable.) In addition, in our model we obtain, as discussed above, a kinetic term for gravity on the brane after functionally integrating out the embedding functions.