TOWARDS THE UNIFICATION OF GRAVITY WITH OTHER INTERACTIONS: WHAT HAS BEEN MISSED?

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Contents

- Introduction

Why a new paradigm?

Persisting puzzles: quantum gravity, problem of time, unification of interactions, the nature of dark matter, dark energy, etc.

From history we know that such situation calls for `paradigm shift'

We also know that often a formalism is more powerful than initially envisaged

Examples: Hamilton-Jacobi function (hints of wave mechanics)
Clifford algebra (implies spin)
Line element in Minkowski spacetime
(suggests generalization to curved spacetime)

Common: In all those cases the formalism itself pointed to its own generalization!
This introduced important new physics

Proposal: To do something analogous with the formalism describing configurations of physical systems

- Generalising relativity
Configuration space replaces spacetime

Action for a system of point particles

$$I[\dot{X}_i^{\mu}] = \sum_i m_i \int d\tau \, [\dot{X}_i^{\mu} \dot{X}_i^{\nu} \; g_{\mu\nu}(X_i^{\mu})]^{1/2}$$

The Schild action for a system of point particles

$$I[\dot{X}_i^{\mu}] = \int \mathrm{d}\tau \sum_i \dot{X}_i^{\mu} \dot{X}_i^{\nu} \frac{m_i}{k_i} g_{\mu\nu}(X_i^{\mu})$$

$$\dot{X}_i^{\mu} \equiv \dot{X}^{(i\mu)} \equiv \dot{X}^{M}, \qquad M = (i\mu)$$

$$\frac{m_i}{k_i} g_{\mu\nu} \equiv \frac{M}{K} g_{(i\mu)(j\nu)} \equiv \frac{M}{K} g_{MN}$$

$$I[X^M] = \int \mathrm{d}\tau \; \dot{X}^M \dot{X}^N \frac{M}{K} g_{MN}(X^M)$$

The Schild action in configuration space C.

It is equivalent to the reparametrization invariant action in C:

$$I[X^M] = M \int \mathrm{d}\tau \; [\dot{X}^M \dot{X}^N g_{MN}(X^M)]^{1/2}$$

$$M^2 = \sum_i m_i^2$$

Action for a point particle:

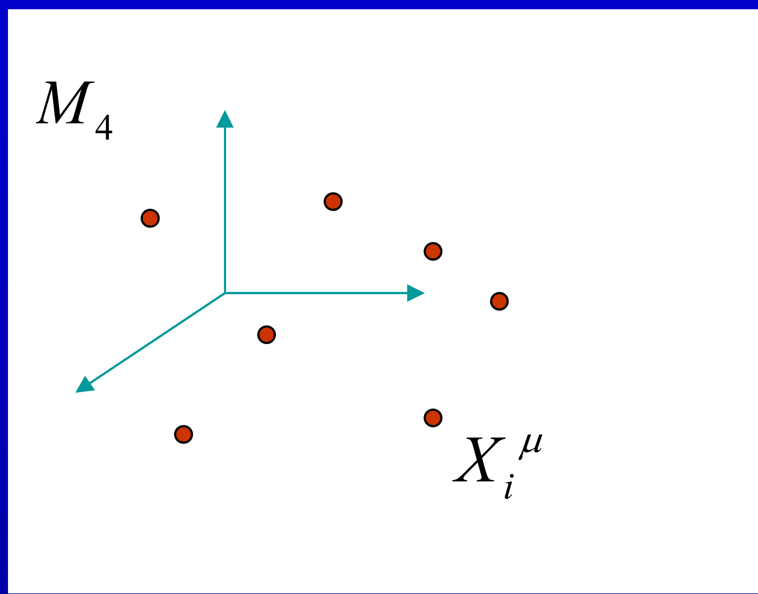
$$I[X^{\mu}] = \int \mathrm{d}\tau \, m \, (\dot{X}^{\mu} \dot{X}^{\nu} \, g_{\mu\nu})^{1/2}$$

Gauge fixed action (the Schild action):

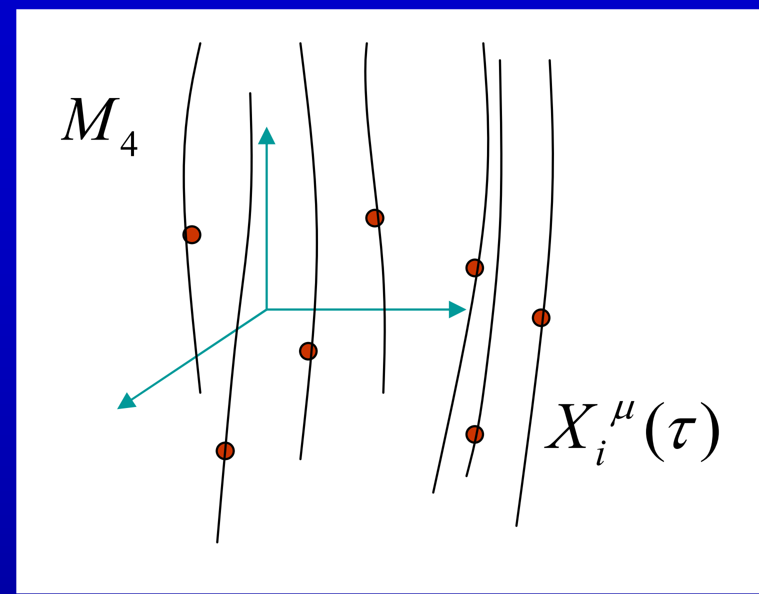
$$I[X^{\mu}] = \int \mathrm{d}\tau \frac{m}{k} \dot{X}^{\mu} \dot{X}^{\nu} \, g_{\mu\nu}$$

$$\dot{X}^{\mu} \dot{X}^{\nu} g_{\mu\nu} = k^2$$ Is constant

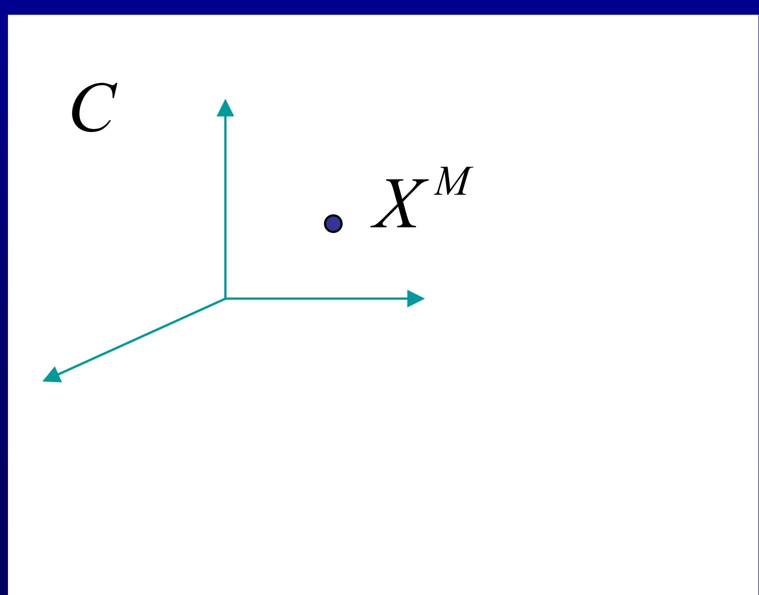
‘Instantaneous’ configuration in M_4

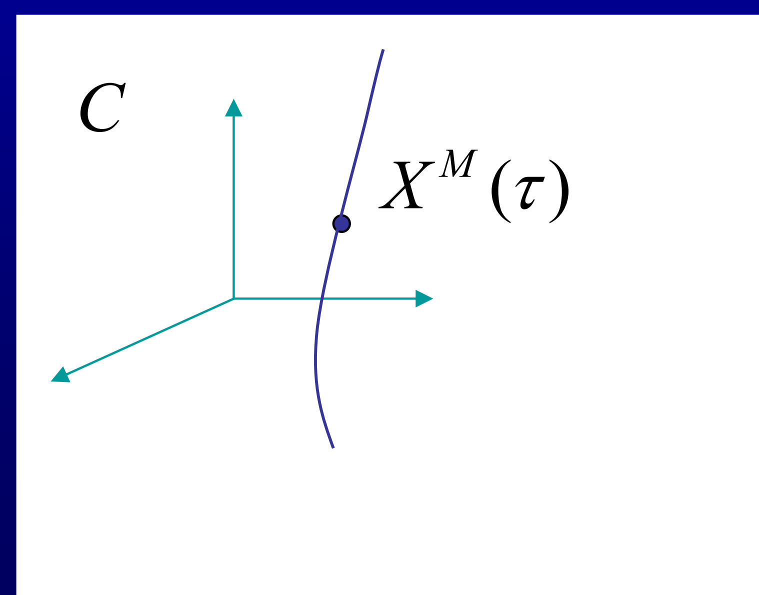


‘Evolution’ of configuration in M_4

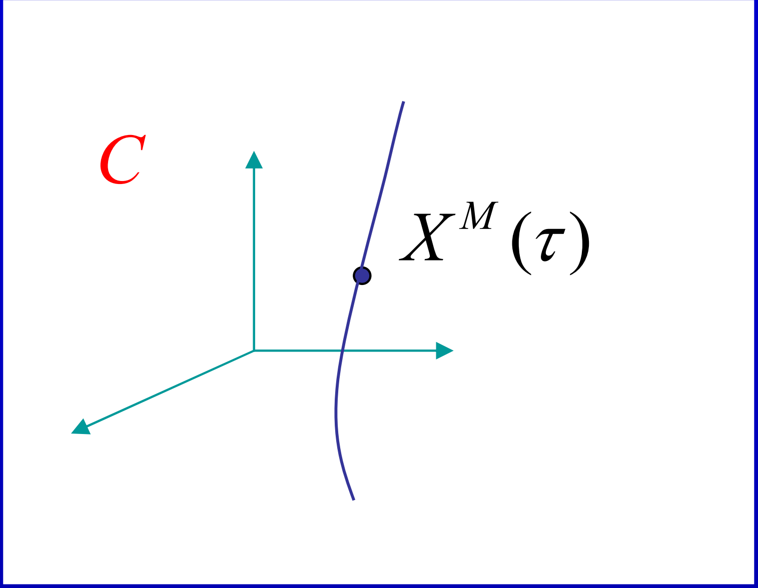


Representation in configuration space C





A given configuration, described by coordinates X^M, traces a world line $X^M(\tau)$ in configuration space C.



We assume that, in general, the metric g_{MN} of C can be arbitrary. The world line is a geodesic of C.

In particular, for the block diagonal metric

$$G_{MN} \equiv G_{(i\mu)(j\nu)} = \begin{pmatrix} g_{\mu\nu}(x_1) & 0 & 0 & \dots \\ 0 & g_{\mu\nu}(x_2) & 0 & \dots \\ 0 & 0 & g_{\mu\nu}(x_3) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

we obtain the ordinary relativistic theory for a many particle system in a gravitational field.

By allowing for more general metric, we go beyond the ordinary theory.
Now we have general relativity in configuration space, given by the action

$$I[X^M, G_{MN}] = I_m + I_g$$

$$I_m = \int \mathrm{d}\tau \; M \, (G_{MN} \dot{X}^M \dot{X}^N)^{(1/2)} = \int \mathrm{d}\tau \; M \, (G_{MN} \dot{X}^M \dot{X}^N)^{(1/2)} \, \delta^D(x - X(\tau)) \, \mathrm{d}^D x$$

$$I_g = \frac{1}{16\pi G_D} \int d^D x \sqrt{|g^{(C)}|} \, R^{(C)}$$

Formally we arrived at a theory which is analogous to Kaluza-Klein theory.

Configuration space C is a higher dimensional space.

A 4-dimensional subspace, associated with a chosen particle, is spacetime M_4

The concept of configuration space can take place:

In macrophysics: The theory predicts deviations from the conventional theory. New effects. Possible theoretical basis for MONDs.

In microphysics: We can consider a configuration space associated with extended objects, e.g., strings and branes. This gives fundamental interactions (gravity + YM)

Configuration space can be:

Finite dimensional: Associated with a system of point particles, dust,

Infinite dimensional: Associated with a fluid, string, brane

Equations of motion for a configuration of point particles

Quadratic form in C

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \text{extra terms}$$

$$X^M = (X^\mu, X^{\bar{M}}), \qquad X^\mu \equiv X^{1\mu}$$

Ansatz for the metric

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu{}^{\bar{M}} A_\nu{}^{\bar{N}} \phi_{\bar{M}\bar{N}}, & A_\mu{}^{\bar{N}} \phi_{\bar{M}\bar{N}} \\ A_\nu{}^{\bar{N}} \phi_{\bar{M}\bar{N}}, & \phi_{\bar{M}\bar{N}} \end{pmatrix}$$

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$\dot{X}_{\bar{M}} = G_{\bar{M}N} \dot{X}^N = A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{N}} \dot{X}^{\bar{N}}$$

Split action

$$I = M \int d\tau \left[\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \phi^{\bar{M}\bar{N}} (A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{J}} \dot{X}^{\bar{J}})(A_{\bar{N}\nu} \dot{X}^\nu + \phi_{\bar{N}\bar{K}} \dot{X}^{\bar{K}}) \right]^{1/2}$$

Variation with respect to X^μ

$$\frac{1}{\left(\dot{X}^2\right)^{1/2}} \frac{d}{d\tau} \left(\frac{\dot{X}^\mu}{\left(\dot{X}^2\right)^{1/2}} \right) + \frac{1}{\dot{X}^2} \Gamma^\mu{}_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma + \text{extra terms} = 0$$

$$\dot{X}^2 \equiv g_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma$$

Relation between higher dimensional and 4-dimensional mass

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$g_{\mu\nu} = G_{\mu\nu} - \phi^{\bar{M}\bar{N}} A_{\mu\bar{M}} A_{\nu\bar{N}}$$

$$\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = \dot{X}^M \dot{X}^N G_{MN} - \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$\frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} = 1 - \frac{\dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}}{\dot{X}^M \dot{X}^N G_{MN}}$$

Multiplying by M^2

$$M^2 \frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = g^{\mu\nu} p_\mu p_\nu = m^2$$

$$\frac{m}{M} = \left(\frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} \right)^{1/2}$$

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}}$$

Four dimensional mass m is given by the higher dimensional mass M and the contribution due to the extra components of momentum $p_{\bar{M}}$

From the perspective of 4-dimensioal spacetime, m has the role of inertial mass.

We assume that the extra (or 'internal') space admits isometries given by Killing vector fields $k_\alpha{}^{\bar{J}}$

Projection of momentum onto Killing vector $k_\alpha{}^{\bar{J}} P_{\bar{J}} \equiv p_\alpha$ Charge

$$A_\mu{}^{\bar{J}} = k_\alpha{}^{\bar{J}} A_\mu{}^\alpha \qquad \phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha{}^{\bar{M}} k_\beta{}^{\bar{N}}$$

$$F_{\mu\nu}{}^\alpha = \partial_\mu A_\nu{}^\alpha - \partial_\nu A_\mu{}^\alpha + C_{\alpha'\beta'}{}^\alpha A_\mu{}^{\alpha'} A_\nu{}^{\beta'} \qquad \left(k_{\alpha,J}{}^M k_\beta{}^J - k_{\beta,J}{}^M k_\alpha{}^J\right) = -C_{\alpha\beta}{}^\gamma k_\gamma{}^M$$

$$\frac{1}{\lambda}\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\lambda}\right) + {}^{(4)}\Gamma^\mu{}_{\rho\sigma}\frac{\dot{X}^\rho \dot{X}^\sigma}{\lambda^2} + \frac{p_\alpha}{m} F_{\mu\nu}{}^\alpha \frac{\dot{X}^\nu}{\lambda}$$
$$+ \frac{1}{2m^2}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta + \frac{1}{\lambda m}\frac{dm}{d\tau} = 0$$

$$\lambda^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}}$$

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$$+ \frac{1}{2m^2}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta + \frac{1}{\lambda m}\frac{dm}{d\tau} = 0$$

$$\lambda^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

4-dimensional mass

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = M^2 - \varphi^{\alpha\beta} p_\alpha p_\beta + \text{extra terms}$$

$$p_\alpha \equiv k_\alpha{}^{\bar{M}} P_{\bar{M}}$$

In general m is not constant. In particular, when the extra terms vanish, m may be constant.

A configuration under consideration can be the universe.

According to this theory, the motion of a subsystem, approximated as a point particle, obeys the law of motion given in previous slide.

Besides the usual 4-dimensional gravity, there are extra forces. They come from the generalized metric, i.e., the metric of configuration space.

The inertial mass depends on momenta of all other particles within the configuration. This is reminiscent of the Mach principle.

Such approach opens a Pandora's box of possibilities to revise our current views on the universe, dark matter, dark energy, MOND, the Pioneer effect, the horizon problem, etc.

Configuration space for infinite dimensional objetcs - branes

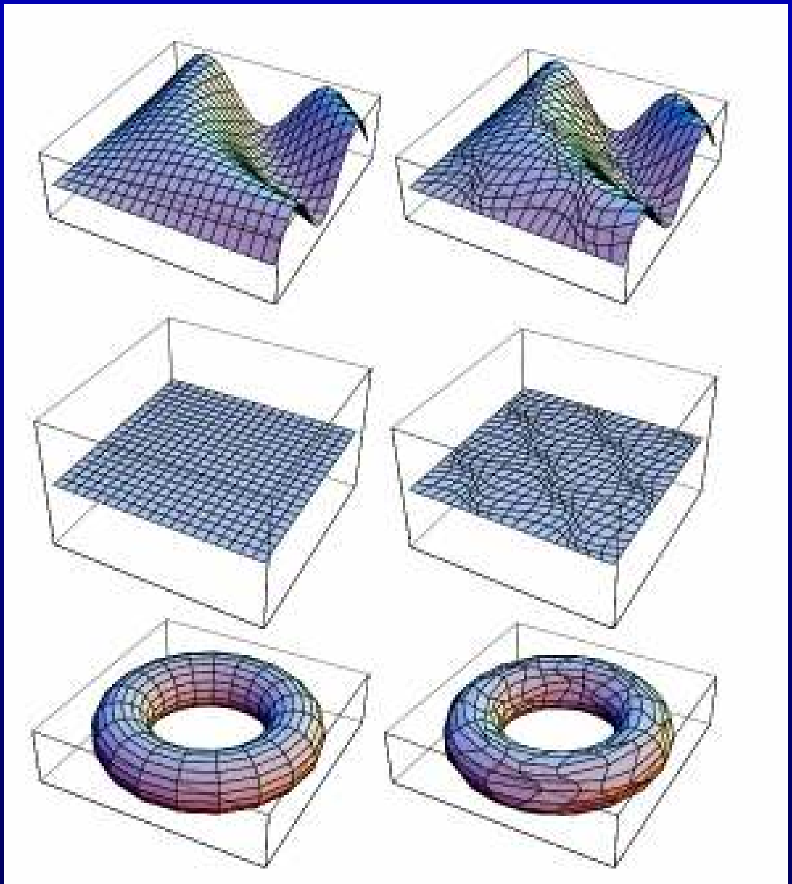
A brane can be considered as a point in infinite dimensional space with coordinates

$$X^{\mu}(\xi^{a}) \equiv X^{\mu(\xi)} \equiv X^{M}$$

This includes classes of tangentially deformed branes which we can interpret as physically different objects, not just reparametrizations.

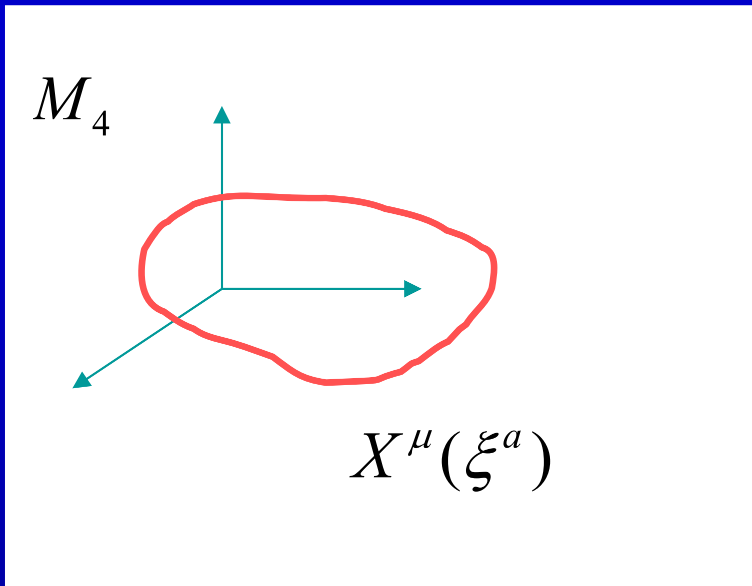
Mathematically the surfaces on the left and the right are the same.
Physically they are different.

They are represented by two different points in C

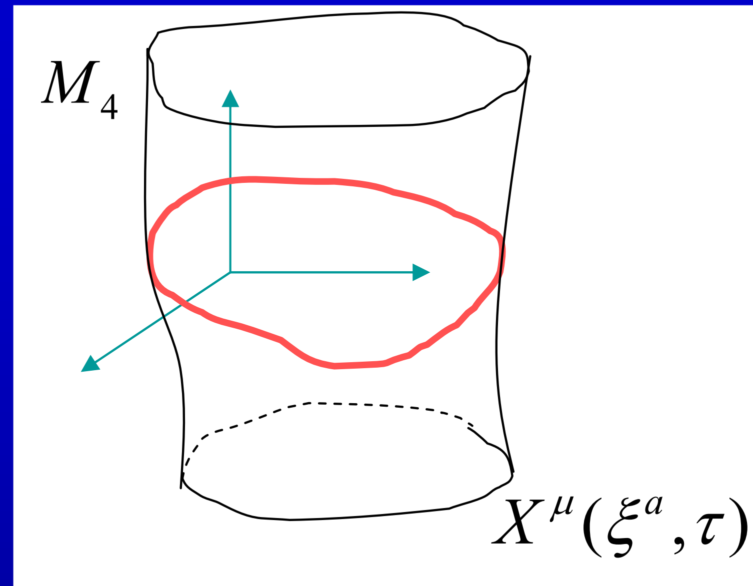


For the configuration space associated with a brane we will also use the name brane space $\mathcal{M}$

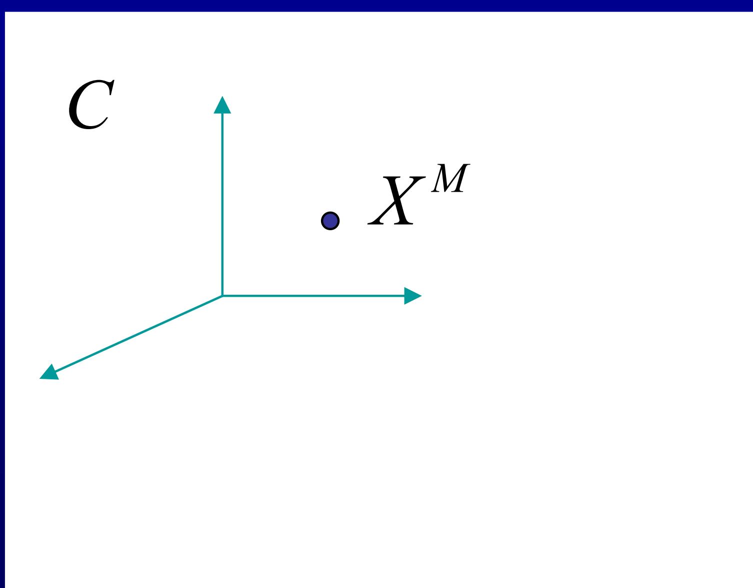
‘Instantaneous’ brane configuration in M_4

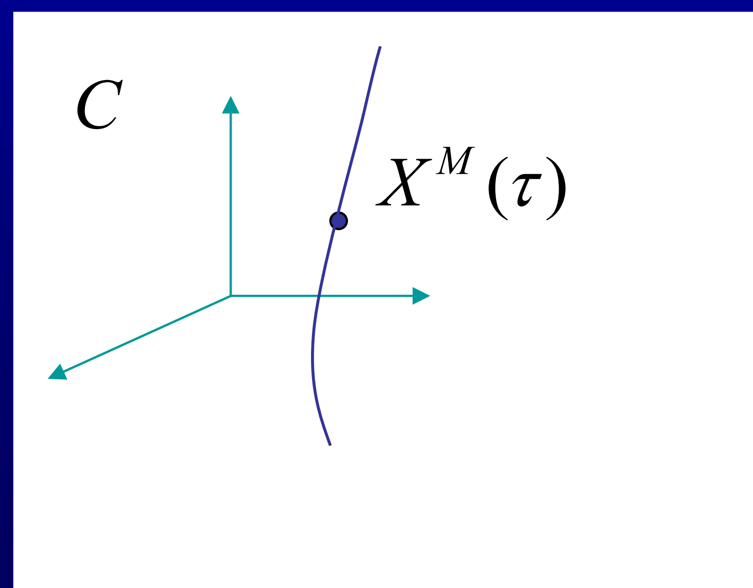


‘Evolution’ of a brane configuration in M_4



Representation in configuration space C





Action in the brane space $\mathcal{M}$

$$I[X^M] = \int \mathrm{d}\tau \, (\rho_{MN} \dot{X}^M \dot{X}^N)^{(1/2)}$$

Short hand notation

$$M \equiv \mu(\xi) \, , \qquad X^M \equiv X^{\mu(\xi)} \equiv X^\mu(\xi)$$

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} \right)^{1/2}$$

More explicit notation

If metric is given by

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \eta_{\alpha\beta}$$

$$f \equiv \det f_{ab} \, , \qquad f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

$$\dot{X}^2 \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

then the corresponding equations of motion are precisely those of a Dirac-Nambu-Goto brane!

In this theory we assume that the metric above is just one particular chose amongst many other possible metrics that are solution to the Einstein equations in the configuration space.

For more details see:

M. Pavšič: The Landscape of theoretical Physics (Kluwer, 2001), gr-qc/0610061 ; hep-th/0311060

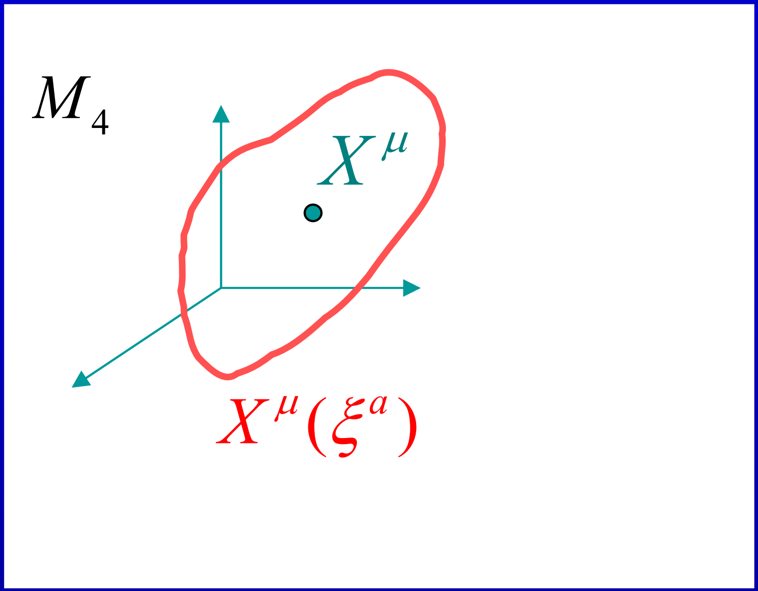
Finite dimensional description of extended objects

The Earth has a huge (practically infinite) number of degree of freedom. And yet, when describing the motion of the Earth around the Sun, we neglect them all, except for the coordinates of the centre of mass.

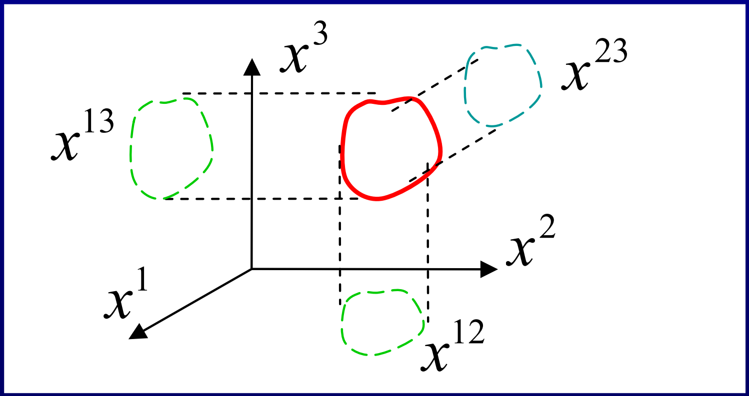
Instead of infinitely many degrees of freedom associated with an extended object, we may consider a finite number of degrees of freedom.

Strings and branes have infinitely many degrees of freedom.

But at first approximation we can consider just the centre of mass.

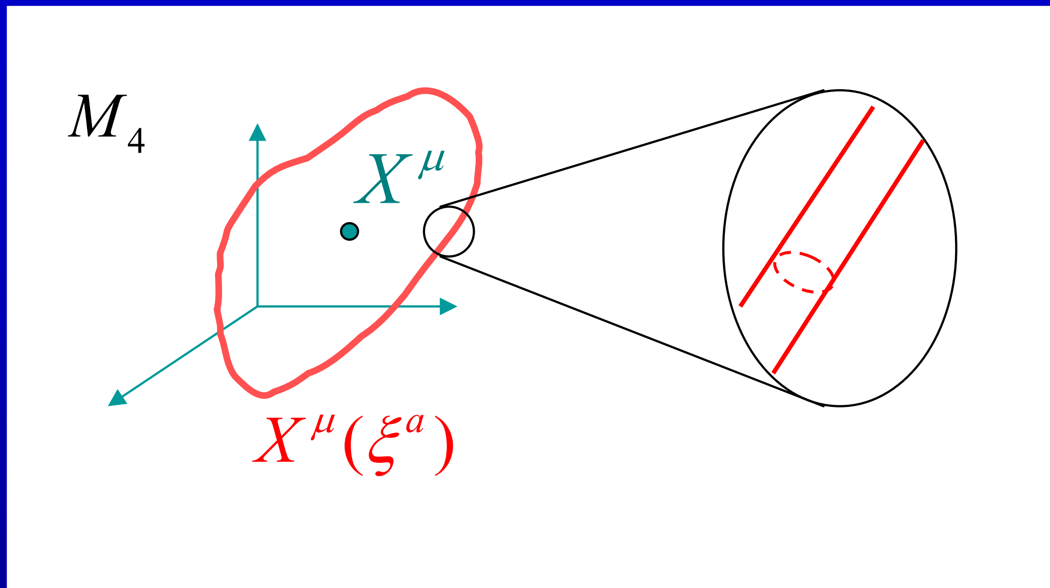


Next approximation is in considering the holographic coordinates of the oriented area enclosed by the string.



We may go further and search for eventual thickness of the object.

If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom.



X^{123}

In general, for an extended object in M_4, we have 16 coordinates

$$x^M \equiv x^{\mu_1 \ldots \mu_r} , \qquad r = 0,1,2,3,4$$

They are the projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra.

Instead of the usual relativity formulated in spacetime in which the interval is

$$\mathrm{d}s^2 = \eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$\mathrm{d}S^2 = G_{MN}\,\mathrm{d}x^M \mathrm{d}x^N \qquad \mathrm{d}x^M \equiv \mathrm{d}x^{\mu_1 \ldots \mu_r}, \quad r = 0, 1, 2, 3, 4$$

Coordinates of Clifford space can be used to model extended objects.
They are a generalization of the concept of center of mass.

Instead of describing an extended object in ``full detail'', we can describe it in terms of the center of mass, area and volume coordinates.

In particular, the extended object can be a fundamental string or brane.

Dynamics

Action:

$$I = \int d\tau \, (\eta_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Generalization of ordinary relativity

Equations of motion:

$$\ddot{X}^M \equiv \frac{\mathrm{d}^2 X^M}{\mathrm{d}\tau^2} = 0$$

These equations imply area (volume) motion

Metric:

$$\eta_{MN}$$

Diagonal metric

Signature:

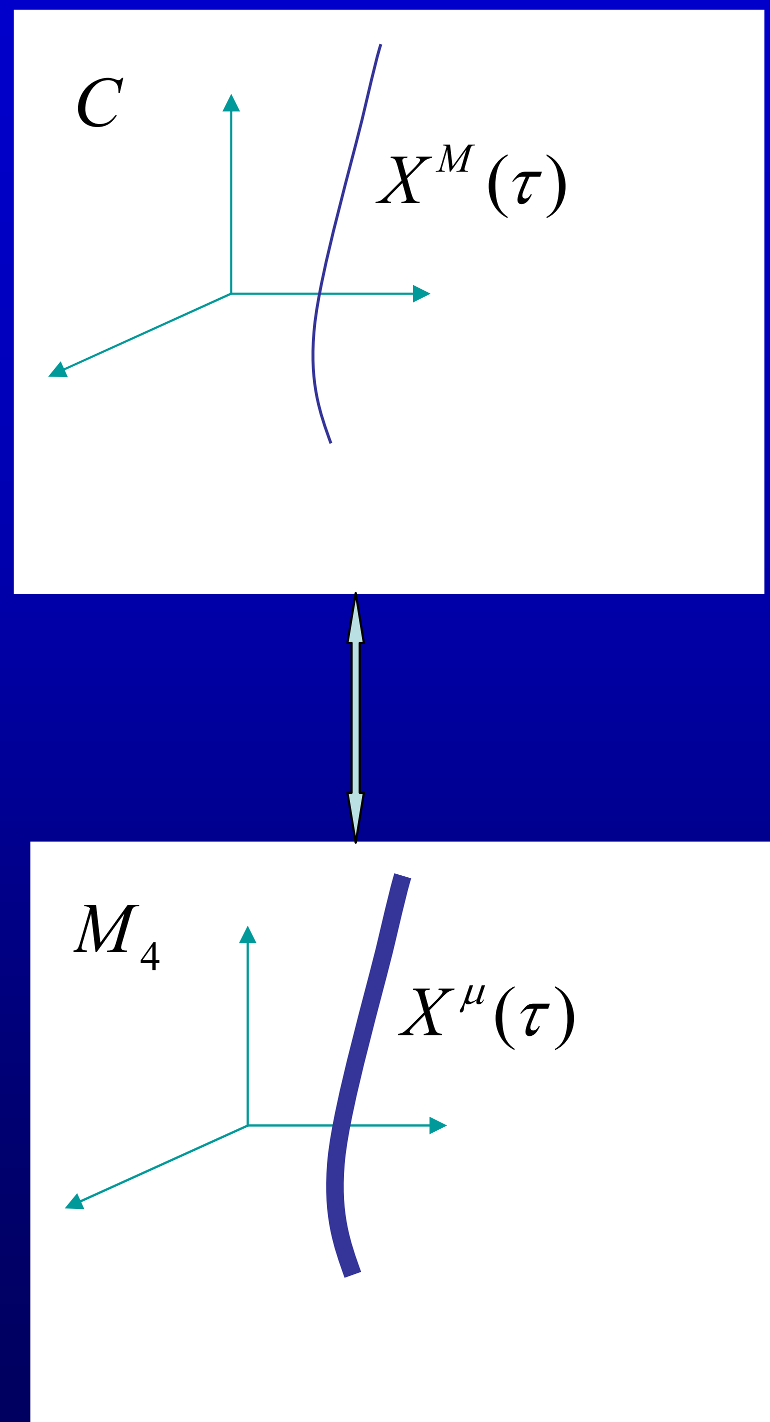
$$+ + + + + + + + - - - - - - - -$$

$(8,8)$

The above dynamics holds for tensionless branes. For the branes with tension one has to introduce curved Clifford space.

Thick point particles and strings

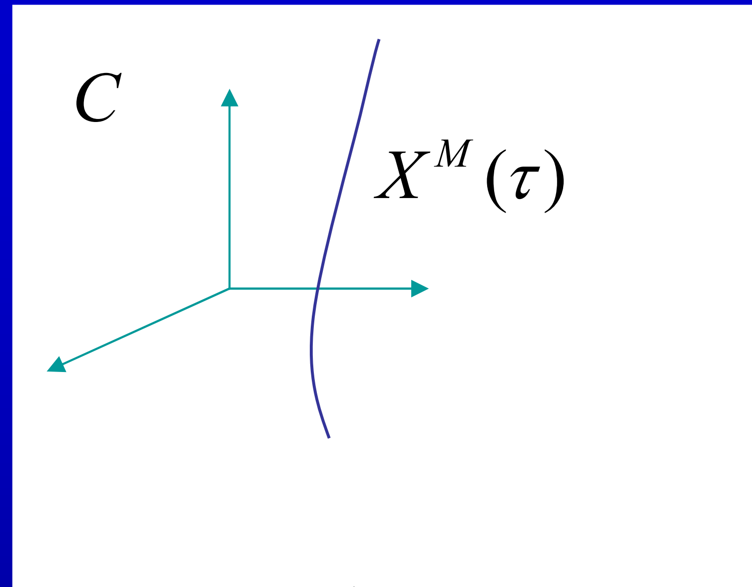
A world line in C represents the evolution of a `thick' particle in spacetime M_4

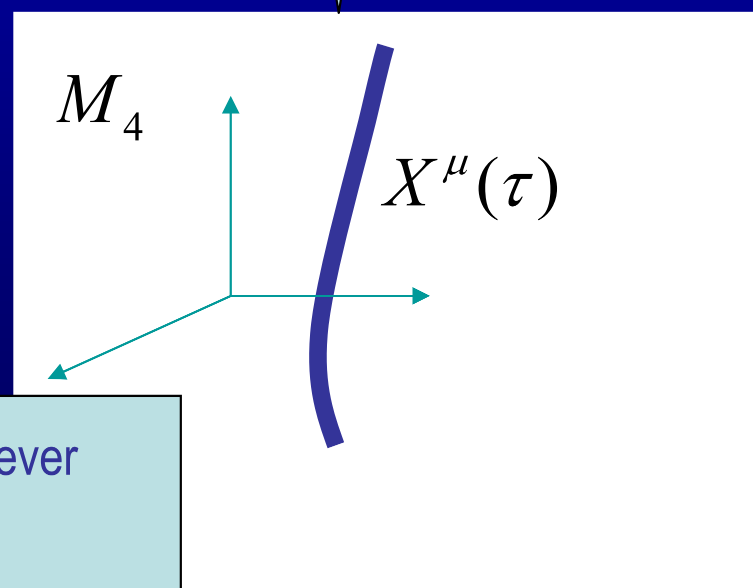


Thick particle can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.

A world line in *C* represents the evolution of a `thick' particle in spacetime M_4



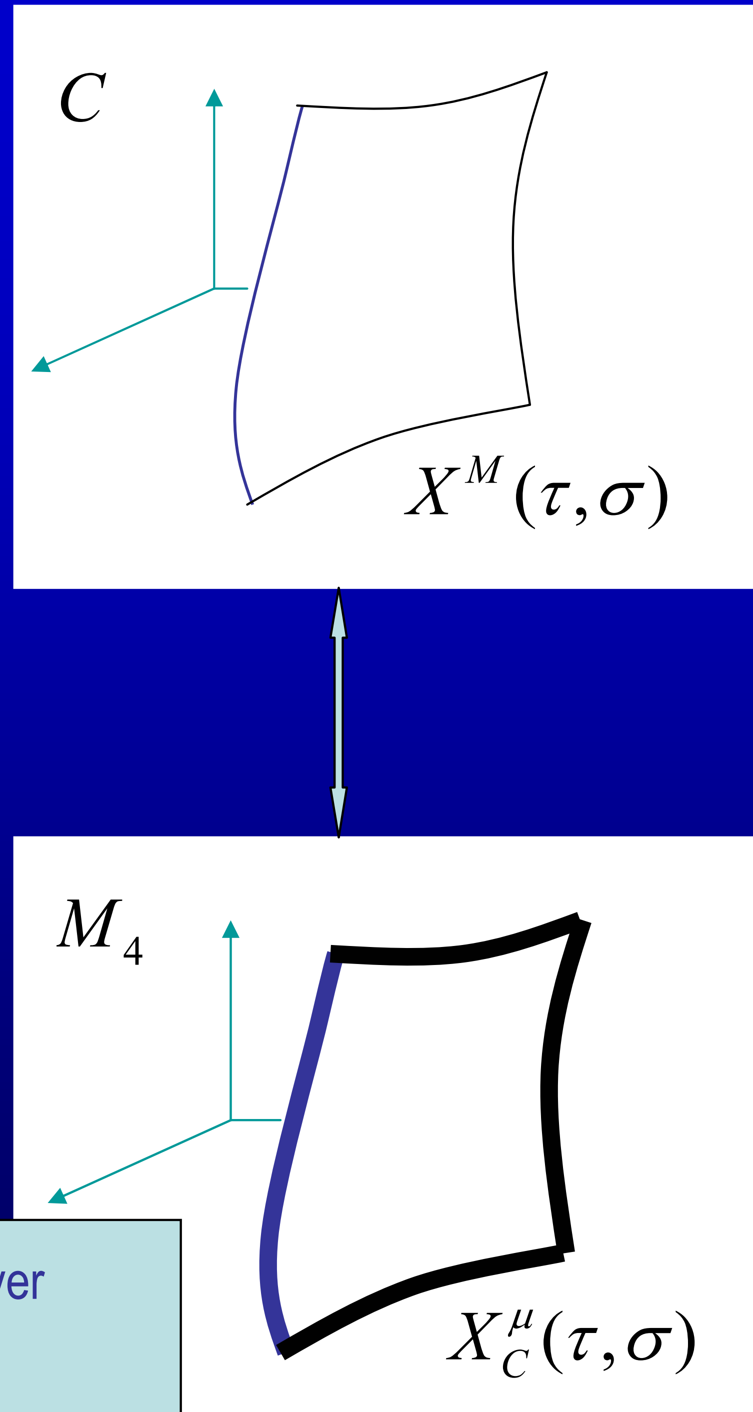


Thick particle can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.

Thick particle may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1\mu_2\ldots\mu_r}$

A world sheet in *C* represents the evolution of a `thick' string in spacetime M_4



Thick string can be an aggregate *p*-branes for various *p*=0,1,2,…

But such interpretation is not obligatory.

Thick string may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}$

$X^\mu(\tau)$

Usual strings are infinitely thin object. Although called `extended objects', they are not fully extended.

Instead of infinitely thin strings we thus consider thick strings. Their thickness is encoded in polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}$.

Infinitely thin strings are singular objects

String action

$$I = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}\sigma \, (\dot{X}^M \dot{X}^N - X'^M X'^N) \, G_{MN}$$

Conformal gauge

No extra dimensions of the spacetime are required

The necessary extra dimensions for consistency of string theory are in 16-dimensional Clifford space.

Jackiw-Kim-Noz definition of vacuum

No central terms in the Virasoro algebra, if the space in which the string lives has signature $(+ + + \; ... \; - - -)$

The space in which out string lives is Clifford space.
Its dimension is 16, and signature (8,8).

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Conformal gauge

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are in 16-dimensiona

Jackiw-Kim-Noz defi

No central terms in t
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The space in wh
Its dimension is

$$X^M = (x, x^\mu, x^{\mu\nu}, ...)$$

$$\gamma^M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu,}, ...)$$

$$X^M \gamma_M$$

Polyvector

(It contains spinors)

Summary

- We have considered a theory in which spacetime is replaced by a larger space, namely the configuration space associated with a system under consideration.
- The ordinary special and general relativity are recovered for a particular metric of the configuration space.
- Since configuration space has extra dimensions, its metric provides description of additional interactions, beside the 4-dimensional gravity, just as in Kaluza-Klein theories
- In this theory there is no need for extra dimensions of spacetime. The latter space is a subspace of the configuration space.

All dimensions of the configuration space C are physical. Therefore there is no need for a compactification of the extra dimensions of C.

Some more related material can be found in a book

M. Pavšič: The Landscape of Theoretical Physics: A Global view;
From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle
(Kluwer Academic, 2001)

where the description with a metric tensor has been surpassed.

Very promising is the description in terms of the Clifford algebra equivalent of the tetrad field which simplifies calculations significantly.

Some other related publications:

Class.Quant.Grav.20:2697-2714,2003, gr-qc/0111092

Kaluza-Klein theory without extra dimensions: Curved Clifford space.
Phys.Lett.B614:85-95,2005, hep-th/0412255

Clifford space as a generalization of spacetime: Prospects for QFT of point particles and strings, Found.Phys.35:1617-1642,2005, hep-th/0501222

Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory in 4-dimensional spacetime, Int.J.Mod.Phys.A21:5905-5956,2006, gr-qc/0507053

A novel view on the physical origin of E8, J. Phys. A41: 332001, 2008,
0806.4365 [hep-th]

Phase space action

$$I\left[X^M, P_M, \Lambda\right] = \int d\tau \left(P_M \dot{X}^M - H\right)$$

$$H = \frac{\Lambda}{2M}\left(P_M P_N G^{MN} - M^2\right)$$

Splitting $X^M = (X^\mu, X^{\bar{M}})$

$$I[X^\mu, X^{\bar{M}}, p_\mu, P_{\bar{M}}, \Lambda] = \int d\tau \left[p_\mu \dot{X}^\mu + P_{\bar{M}} \dot{X}^{\bar{M}} - H \right]$$

$$H = \frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu - A_\mu{}^{\bar{J}} P_{\bar{J}}\right)\left(p_\nu - A_\nu{}^{\bar{K}} P_{\bar{K}}\right) + \phi^{\bar{M}\bar{N}} P_{\bar{M}} P_{\bar{N}} - M^2 \right]$$

Hamiltonian

We assume that the extra (or 'internal') space admits isometries given by Killing vector fields $k_\alpha{}^{\bar{J}}$

Projection of momentum onto Killing vector $k_\alpha{}^{\bar{J}} P_{\bar{J}} \equiv p_\alpha$ Charge

$$A_\mu{}^{\bar{J}} = k_\alpha{}^{\bar{J}} A_\mu{}^\alpha \qquad \phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha{}^{\bar{M}} k_\beta{}^{\bar{N}}$$

$$H = \frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu - A_\mu{}^\alpha p_\alpha\right)\left(p_\nu - A_\nu{}^\beta p_\beta\right) + \varphi^{\alpha\beta} p_\alpha p_\beta - M^2 \right]$$

$$H=\frac{\Lambda}{2M}\left[g^{\mu\nu}\left(p_\mu-A_\mu{}^\alpha p_\alpha\right)\left(p_\nu-A_\nu{}^\beta p_\beta\right)+\varphi^{\alpha\beta}p_\alpha p_\beta-M^2\right]$$

$$k_\alpha{}^{\bar{J}}P_{\bar{J}}\equiv p_\alpha$$

$$\dot{p}_\alpha=\{p_\alpha,H\}$$

$$\{p_\alpha,p_\beta\}=\frac{\partial p_\alpha}{\partial X^J}\frac{\partial p_\beta}{\partial P_J}-\frac{\partial p_\beta}{\partial X^J}\frac{\partial p_\alpha}{\partial P_J}=\left(k_{\alpha,J}{}^M k_\beta{}^J-k_{\beta,J}{}^M k_\alpha{}^J\right)p_M=-C_{\alpha\beta}{}^\gamma p_\gamma$$

$$\left(k_{\alpha,J}{}^M k_\beta{}^J-k_{\beta,J}{}^M k_\alpha{}^J\right)=-C_{\alpha\beta}{}^\gamma k_\gamma{}^M$$

$$p_\mu-A_\mu{}^{\bar{J}}P_{\bar{J}}\equiv\pi_\mu,\qquad g^{\mu\nu}\pi_\nu=\frac{M}{\Lambda}\dot{X}^\mu$$

$$\dot{p}_\alpha=C_{\alpha\beta}{}^\gamma\,p_\gamma\,A_\mu{}^\beta\,\dot{X}^\mu-\frac{\Lambda}{2M}\varphi^{\alpha'\beta'}{}_{,\bar{J}}\,p_{\alpha'}p_{\beta'}\,k_\alpha{}^{\bar{J}}$$

Wong equation

One can choose a frame in which

$$k_\alpha{}^M=\left(k_\alpha{}^\mu,k_\alpha{}^{\bar{M}}\right),\quad k_\alpha{}^\mu=0,\quad k_\alpha{}^{\bar{M}}\neq0$$

$$\dot{p}_\mu = \left\{p_\mu, H\right\} = -\frac{\partial H}{\partial X^\mu}$$

$$F_{\mu\nu}{}^{\alpha} = \partial_\mu A_\nu{}^{\alpha} - \partial_\nu A_\mu{}^{\alpha} + C_{\alpha'\beta'}{}^{\alpha} A_\mu{}^{\alpha'} A_\nu{}^{\beta'}$$

Yang-Mills field strength

$$g^{\mu\nu}\pi_\nu = \frac{M}{\Lambda}\dot{X}^\mu, \qquad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu}\dot{X}^\nu$$

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu}\pi^\rho\pi^\sigma + F_{\mu\nu}{}^{\alpha} p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}}\, k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta = 0$$

Wong equation
(Equation of geodesic
+ Yang-Mills)

Extra contribution
due to `scalar' fields

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}}$$

Four dimensional mass m is given by the higher dimensional mass M and the contribution due to the extra components of momentum $p_{\bar{M}}$

From the perspective of 4-dimensioal spacetime, m has the role of inertial mass. This can be seen if we rewrite the equation of motion

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu} \pi^\rho \pi^\sigma + F_{\mu\nu}{}^\alpha p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta = 0$$

$$g^{\mu\nu}\pi_\nu = \frac{M}{\Lambda}\dot{X}^\mu, \qquad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu} \dot{X}^\nu$$

$$\Lambda^2 = \dot{X}^M \dot{X}^N G_{MN}, \qquad \lambda^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

$$\frac{m}{M} = \frac{\lambda}{\Lambda}$$

$$\frac{1}{\lambda}\frac{d}{d\tau}\left(\frac{\dot{X}^\mu}{\lambda}\right) + {}^{(4)}\Gamma^\mu{}_{\rho\sigma} \frac{\dot{X}^\rho \dot{X}^\sigma}{\lambda^2} + \frac{p_\alpha}{m} F_{\mu\nu}{}^\alpha \frac{\dot{X}^\nu}{\lambda}$$

$$+ \frac{1}{2m^2}\left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'}\right) p_\alpha p_\beta + \frac{1}{\lambda m}\frac{dm}{d\tau} = 0$$

Example of internal space S_2

Solution to the Killing equation $k^{\alpha}{}_{\bar{M};\bar{N}} + k^{\alpha}{}_{\bar{N};\bar{M}} = 0$ gives:

$$k_1{}^{\vartheta} = \sin\varphi\,, \qquad k_1{}^{\varphi} = \cot\vartheta\cos\varphi\,,$$
$$k_2{}^{\vartheta} = \cos\varphi\,, \qquad k_1{}^{\varphi} = -\cot\vartheta\sin\varphi\,,$$
$$k_3 = 0\,, \qquad k_3{}^{\varphi} = 1$$

$$p_\alpha \equiv k_\alpha{}^{\bar{M}} P_{\bar{M}}$$

$$p_{\bar{M}} = (p_\theta, p_\varphi)$$

$$p_1 = \sin\varphi\, p_\vartheta + \cot\vartheta\, \cos\varphi\, p_\varphi$$
$$p_2 = \cos\varphi\, p_\vartheta - \cot\vartheta\, \sin\varphi\, p_\varphi$$
$$p_3 = p_\varphi$$

$$\varphi^{\alpha\beta} = \delta^{\alpha\beta}\frac{1}{r^2}$$

$$\phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha{}^{\bar{M}} k_\beta{}^{\bar{N}}$$

$$\phi^{\bar{M}\bar{N}} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2\sin^2\vartheta} \end{pmatrix}$$

$$\phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = \varphi^{\alpha\beta} p_\alpha p_\beta$$

Example: the Dirac membrane

$$X^{\mu}(\xi^{a}) = (X^{0},\, r\sin\vartheta\cos\varphi,\, r\sin\vartheta\sin\varphi,\, r\cos\vartheta)$$

$$\xi^{a} = (\tau, \vartheta, \varphi)\,, \qquad X^{0} = X_{0}$$

$$\gamma_{ab} = \begin{pmatrix} \dot{X}_{0}^{2} - \dot{r}^{2} & 0 & 0 \\ 0 & -r^{2} & 0 \\ 0 & 0 & -r^{2}\sin^{2}\vartheta \end{pmatrix}$$

$$\sqrt{|\det\gamma|} \equiv \sqrt{|\gamma|} = \sqrt{\dot{X}_{0}^{2} - \dot{r}^{2}}\; r^{2}\sin\vartheta$$

X^{0}, r functions of τ

$$I = \int \mathrm{d}\tau\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\sqrt{|\gamma|} = \int \mathrm{d}\tau\, 4\pi r^{2}\sqrt{\dot{X}_{0}^{2} - \dot{r}^{2}}$$

Variation with respect to r and X^{0}

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{r}}{\sqrt{\dot{X}_{0}^{2} - \dot{r}^{2}}}\right) + \frac{2\dot{X}_{0}^{2}}{r\sqrt{\dot{X}_{0}^{2} - \dot{r}^{2}}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r^{2}\dot{X}_{0}}{\sqrt{\dot{X}_{0}^{2} - \dot{r}^{2}}}\right) = 0$$

Example: the Dirac membrane

$$X^{\mu}(\xi^{a}) = (X^{0},\ r\sin\vartheta\cos\varphi,\ r\sin\vartheta\sin\varphi,\ r\cos\vartheta)$$

$$\xi^{a} = (\tau, \vartheta, \varphi)\ , \qquad X^{0} = X_{0}$$

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$$\sqrt{|\det\gamma|} \equiv \sqrt{|\gamma|} = \sqrt{\dot{X}_0^2 - \dot{r}^2}\ r^2\sin\vartheta$$

$$I = \int \mathrm{d}\tau\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\sqrt{|\gamma|} = \int \mathrm{d}\tau\, 4\pi r^2\sqrt{\dot{X}_0^2 \ldots}$$

Va

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{r}}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) + \frac{2\dot{X}_0^2}{r\sqrt{\dot{X}_0^2 - \dot{r}^2}} = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{r^2\dot{X}_0}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}\right) = 0$$

$$I = \int \mathrm{d}S, \qquad \mathrm{d}S = \mathrm{d}\tau\, 4\pi r^2\sqrt{\dot{X}_0^2 - \dot{r}^2}$$

$$X^{123} = \frac{1}{3!}\int \mathrm{d}r\,\mathrm{d}\vartheta\,\mathrm{d}\varphi\ \partial_{[a}X^1\partial_b X^2\partial_{c]}X^3 = \frac{4\pi r^3}{3}$$

$$\dot{X}^{123} = 4\pi r^2\dot{r}$$

$$\frac{dX^{123}}{dS} = \frac{\dot{X}^{123}}{4\pi r^2\sqrt{\dot{X}_0^2 - \dot{r}^2}} = \frac{\dot{r}}{\sqrt{\dot{X}_0^2 - \dot{r}^2}}$$

$$\frac{\mathrm{d}^2X^{123}}{\mathrm{d}S^2} + \frac{2}{3X^{123}}\left(1 + \left(\frac{\mathrm{d}X^{123}}{\mathrm{d}S}\right)^2\right) = 0$$

Equation in new variables

Action in C-space

$$I[X^M] = \int \mathrm{d}S = \int d\tau \, (G_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

δX^M

$$\frac{1}{\sqrt{\dot{X}^2}} \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\dot{X}^M}{\sqrt{\dot{X}^2}} \right) + \Gamma^M_{JK} \frac{\dot{X}^J \dot{X}^K}{\dot{X}^2} = 0$$

Let us consider a subspace $X^M = (X^0, X^{123})$

with the metric $G_{MN} = \begin{pmatrix} C\tilde{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix}$

$\tilde{X} \equiv X^{123}$

$$\frac{\mathrm{d}^2 X^{123}}{\mathrm{d}S^2} + \frac{2}{3X^{123}} \left(1 + \left(\frac{\mathrm{d}X^{123}}{\mathrm{d}S} \right)^2 \right) = 0$$

The same equation as obtained directly for the Dirac membrane

Action in C-space

$$I[X^M] = \int \mathrm{d}S = \int d\tau (G_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Let us consider a subspace $X^M = (X^0, X^{123})$

with the metric

$$G_{MN} = \begin{pmatrix} C\tilde{X}^{4/3} & 0 \\ 0 & -1 \end{pmatrix}$$

$\tilde{X} \equiv X^{123}$

$$\mathrm{d}S^2 = G_{00}(\mathrm{d}X^0)^2 + G_{\tilde{X}\tilde{X}}\mathrm{d}\tilde{X}^2$$

$$\mathrm{d}S^2 = C\tilde{X}^{4/3}(\mathrm{d}X^0)^2 - \mathrm{d}\tilde{X}^2$$

$$\tilde{X} = \frac{4\pi r^3}{3}, \qquad \mathrm{d}\tilde{X} = 4\pi r^3 \mathrm{d}r$$

$$\tilde{X}^{4/3} = \left(\frac{4\pi}{3}\right)^{4/3} r^4$$

$$C\left(\frac{4\pi}{3}\right)^{4/3} = (4\pi)^2$$

$$\mathrm{d}S^2 = (4\pi r^2)^2 \left(\mathrm{d}(X^0)^2 - \mathrm{d}r^2\right)$$

$$I = \int \mathrm{d}\tau (4\pi r^2)^2 \sqrt{(\dot{X}^0)^2 - \dot{r}^2}$$

The C-space action for this particular case is equivalent to the action for the Dirac membrane