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Classical Dynamics

University of Cambridge Part II Mathematical Tripos

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Recommended Books and Resources

- L. Hand and J. Finch, *Analytical Mechanics*

This very readable book covers everything in the course at the right level. It is similar to Goldstein's book in its approach but with clearer explanations, albeit at the expense of less content.

There are also three classic texts on the subject

- H. Goldstein, C. Poole and J. Safko, *Classical Mechanics*

In previous editions it was known simply as "Goldstein" and has been the canonical choice for generations of students. Although somewhat verbose, it is considered the standard reference on the subject. Goldstein died and the current, third, edition found two extra authors.

- L. Landau an E. Lifshitz, *Mechanics*

This is a gorgeous, concise and elegant summary of the course in 150 content packed pages. Landau is one of the most important physicists of the 20th century and this is the first volume in a series of ten, considered by him to be the "theoretical minimum" amount of knowledge required to embark on research in physics. In 30 years, only 43 people passed Landau's exam!

- V. I. Arnold, *Mathematical Methods of Classical Mechanics*

Arnold presents a more modern mathematical approach to the topics of this course, making connections with the differential geometry of manifolds and forms. It kicks off with "The Universe is an Affine Space" and proceeds from there...

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1. Newton's Laws of Motion

> "So few went to hear him, and fewer understood him, that oftimes he did, for want of hearers, read to the walls. He usually stayed about half an hour; when he had no auditors he commonly returned in a quarter of that time."
>
> *Appraisal of a Cambridge lecturer in classical mechanics, circa 1690*

1.1 Introduction

The fundamental principles of classical mechanics were laid down by Galileo and Newton in the 16^{th} and 17^{th} centuries. In 1686, Newton wrote the *Principia* where he gave us three laws of motion, one law of gravity and pretended he didn't know calculus. Probably the single greatest scientific achievement in history, you might think this pretty much wraps it up for classical mechanics. And, in a sense, it does. Given a collection of particles, acted upon by a collection of forces, you have to draw a nice diagram, with the particles as points and the forces as arrows. The forces are then added up and Newton's famous "$F = ma$" is employed to figure out where the particle's velocities are heading next. All you need is enough patience and a big enough computer and you're done.

From a modern perspective this is a little unsatisfactory on several levels: it's messy and inelegant; it's hard to deal with problems that involve extended objects rather than point particles; it obscures certain features of dynamics so that concepts such as chaos theory took over 200 years to discover; and it's not at all clear what the relationship is between Newton's classical laws and quantum physics.

The purpose of this course is to resolve these issues by presenting new perspectives on Newton's ideas. We shall describe the advances that took place during the 150 years after Newton when the laws of motion were reformulated using more powerful techniques and ideas developed by some of the giants of mathematical physics: people such as Euler, Lagrange, Hamilton and Jacobi. This will give us an immediate practical advantage, allowing us to solve certain complicated problems with relative ease (the strange motion of spinning tops is a good example). But, perhaps more importantly, it will provide an elegant viewpoint from which we'll see the profound basic principles which underlie Newton's familiar laws of motion. We shall prise open "$F = ma$" to reveal the structures and symmetries that lie beneath.

Moreover, the formalisms that we'll develop here are the basis for *all* of fundamental modern physics. Every theory of Nature, from electromagnetism and general relativity, to the standard model of particle physics and more speculative pursuits such as string theory, is best described in the language we shall develop in this course. The new formalisms that we'll see here also provide the bridge between the classical world and the quantum world.

There are phenomena in Nature for which these formalisms are not particularly useful. Systems which are dissipative, for example, are not so well suited to these new techniques. But if you want to understand the dynamics of planets and stars and galaxies as they orbit and spin, or you want to understand what's happening at the LHC where protons are collided at unprecedented energies, or you want to know how electrons meld together in solids to form new states of matter, then the foundations that we'll lay in in this course are a must.

1.2 Newtonian Mechanics: A Single Particle

In the rest of this section, we'll take a flying tour through the basic ideas of classical mechanics handed down to us by Newton. We'll start with a single particle.

A *particle* is defined to be an object of insignificant size. e.g. an electron, a tennis ball or a planet. Obviously the validity of this statement depends on the context: to first approximation, the earth can be treated as a particle when computing its orbit around the sun. But if you want to understand its spin, it must be treated as an extended object.

The motion of a particle of mass m at the position $\mathbf{r}$ is governed by *Newton's Second Law* $\mathbf{F} = m\mathbf{a}$ or, more precisely,

$$\mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}) = \dot{\mathbf{p}} \tag{1.1}$$

where $\mathbf{F}$ is the force which, in general, can depend on both the position $\mathbf{r}$ as well as the velocity $\dot{\mathbf{r}}$ (for example, friction forces depend on $\dot{\mathbf{r}}$) and $\mathbf{p} = m\dot{\mathbf{r}}$ is the momentum. Both $\mathbf{F}$ and $\mathbf{p}$ are 3-vectors which we denote by the bold font. Equation (1.1) reduces to $\mathbf{F} = m\mathbf{a}$ if $\dot{m} = 0$. But if $m = m(t)$ (e.g. in rocket science) then the form with $\dot{\mathbf{p}}$ is correct.

General theorems governing differential equations guarantee that if we are given $\mathbf{r}$ and $\dot{\mathbf{r}}$ at an initial time $t = t_0$, we can integrate equation (1.1) to determine $\mathbf{r}(t)$ for all t (as long as $\mathbf{F}$ remains finite). This is the goal of classical dynamics.

Equation (1.1) is not quite correct as stated: we must add the caveat that it holds only in an *inertial frame*. This is defined to be a frame in which a free particle with $\dot{m} = 0$ travels in a straight line,

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{v}t \tag{1.2}$$

Newtons's first law is the statement that such frames exist.

An inertial frame is not unique. In fact, there are an infinite number of inertial frames. Let S be an inertial frame. Then there are 10 linearly independent transformations $S \to S'$ such that S' is also an inertial frame (i.e. if (1.2) holds in S, then it also holds in S'). These are

- 3 Rotations: $\mathbf{r}' = O\mathbf{r}$ where O is a 3×3 orthogonal matrix.
- 3 Translations: $\mathbf{r}' = \mathbf{r} + \mathbf{c}$ for a constant vector $\mathbf{c}$.
- 3 Boosts: $\mathbf{r}' = \mathbf{r} + \mathbf{u}t$ for a constant velocity $\mathbf{u}$.
- 1 Time Translation: $t' = t + c$ for a constant real number c

If motion is uniform in S, it will also be uniform in S'. These transformations make up the *Galilean Group* under which Newton's laws are invariant. They will be important in section 2.4 where we will see that these symmetries of space and time are the underlying reason for conservation laws. As a parenthetical remark, recall from special relativity that Einstein's laws of motion are invariant under Lorentz transformations which, together with translations, make up the Poincaré group. We can recover the Galilean group from the Poincaré group by taking the speed of light to infinity.

1.2.1 Angular Momentum

We define the *angular momentum* $\mathbf{L}$ of a particle and the *torque* $\boldsymbol{\tau}$ acting upon it as

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \quad , \quad \boldsymbol{\tau} = \mathbf{r} \times \mathbf{F} \tag{1.3}$$

Note that, unlike linear momentum $\mathbf{p}$, both $\mathbf{L}$ and $\boldsymbol{\tau}$ depend on where we take the origin: we measure angular momentum with respect to a particular point. Let us cross both sides of equation (1.1) with $\mathbf{r}$. Using the fact that $\dot{\mathbf{r}}$ is parallel to $\mathbf{p}$, we can write $\frac{d}{dt}(\mathbf{r} \times \mathbf{p}) = \mathbf{r} \times \dot{\mathbf{p}}$. Then we get a version of Newton's second law that holds for angular momentum:

$$\boldsymbol{\tau} = \dot{\mathbf{L}} \tag{1.4}$$

1.2.2 Conservation Laws

From (1.1) and (1.4), two important conservation laws follow immediately.

- If $\mathbf{F} = 0$ then $\mathbf{p}$ is constant throughout the motion
- If $\boldsymbol{\tau} = 0$ then $\mathbf{L}$ is constant throughout the motion

Notice that $\boldsymbol{\tau} = 0$ does not require $\mathbf{F} = 0$, but only $\mathbf{r} \times \mathbf{F} = 0$. This means that $\mathbf{F}$ must be parallel to $\mathbf{r}$. This is the definition of a *central force*. An example is given by the gravitational force between the earth and the sun: the earth's angular momentum about the sun is constant. As written above in terms of forces and torques, these conservation laws appear trivial. In section 2.4, we'll see how they arise as a property of the symmetry of space as encoded in the Galilean group.

1.2.3 Energy

Let's now recall the definitions of energy. We firstly define the *kinetic energy* T as

$$T = \tfrac{1}{2} m\, \dot{\mathbf{r}} \cdot \dot{\mathbf{r}} \tag{1.5}$$

Suppose from now on that the mass is constant. We can compute the change of kinetic energy with time: $\frac{dT}{dt} = \dot{\mathbf{p}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}}$. If the particle travels from position $\mathbf{r}_1$ at time t_1 to position $\mathbf{r}_2$ at time t_2 then this change in kinetic energy is given by

$$T(t_2) - T(t_1) = \int_{t_1}^{t_2} \frac{dT}{dt}\, dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}}\, dt = \int_{\mathbf{r}_1}^{\mathbf{r}_2} \mathbf{F} \cdot d\mathbf{r} \tag{1.6}$$

where the final expression involving the integral of the force over the path is called the *work done* by the force. So we see that the work done is equal to the change in kinetic energy. From now on we will mostly focus on a very special type of force known as a *conservative* force. Such a force depends only on position $\mathbf{r}$ rather than velocity $\dot{\mathbf{r}}$ and is such that the work done is independent of the path taken. In particular, for a closed path, the work done vanishes.

$$\oint \mathbf{F} \cdot d\mathbf{r} = 0 \qquad \Leftrightarrow \qquad \nabla \times \mathbf{F} = 0 \tag{1.7}$$

It is a deep property of flat space $\mathbf{R}^3$ that this property implies we may write the force as

$$\mathbf{F} = -\nabla V(\mathbf{r}) \tag{1.8}$$

for some *potential* $V(\mathbf{r})$. Systems which admit a potential of this form include gravitational, electrostatic and interatomic forces. When we have a conservative force, we necessarily have a conservation law for energy. To see this, return to equation (1.6) which now reads

$$T(t_2) - T(t_1) = -\int_{\mathbf{r}_1}^{\mathbf{r}_2} \nabla V \cdot d\mathbf{r} = -V(t_2) + V(t_1) \tag{1.9}$$

or, rearranging things,

$$T(t_1) + V(t_1) = T(t_2) + V(t_2) \equiv E \tag{1.10}$$

So $E = T + V$ is also a constant of motion. It is the energy. When the energy is considered to be a function of position $\mathbf{r}$ and momentum $\mathbf{p}$ it is referred to as the *Hamiltonian* H. In section 4 we will be seeing much more of the Hamiltonian.

1.2.4 Examples

- Example 1: The Simple Harmonic Oscillator

This is a one-dimensional system with a force proportional to the distance x to the origin: $F(x) = -kx$. This force arises from a potential $V = \frac{1}{2}kx^2$. Since $F \neq 0$, momentum is not conserved (the object oscillates backwards and forwards) and, since the system lives in only one dimension, angular momentum is not defined. But energy $E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$ is conserved.

- Example 2: The Damped Simple Harmonic Oscillator

We now include a friction term so that $F(x, \dot{x}) = -kx - \gamma\dot{x}$. Since F is not conservative, energy is not conserved. This system loses energy until it comes to rest.

- Example 3: Particle Moving Under Gravity

Consider a particle of mass m moving in 3 dimensions under the gravitational pull of a much larger particle of mass M. The force is $\mathbf{F} = -(GMm/r^2)\hat{\mathbf{r}}$ which arises from the potential $V = -GMm/r$. Again, the linear momentum $\mathbf{p}$ of the smaller particle is not conserved, but the force is both central and conservative, ensuring the particle's total energy E and the angular momentum $\mathbf{L}$ are conserved.

1.3 Newtonian Mechanics: Many Particles

It's easy to generalise the above discussion to many particles: we simply add an index to everything in sight! Let particle i have mass m_i and position $\mathbf{r}_i$ where $i = 1, \ldots, N$ is the number of particles. Newton's law now reads

$$\mathbf{F}_i = \dot{\mathbf{p}}_i \tag{1.11}$$

where $\mathbf{F}_i$ is the force on the i^{th} particle. The subtlety is that forces can now be working between particles. In general, we can decompose the force in the following way:

$$\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}} \tag{1.12}$$

where $\mathbf{F}_{ij}$ is the force acting on the i^{th} particle due to the j^{th} particle, while $\mathbf{F}_i^{\text{ext}}$ is the external force on the i^{th} particle. We now sum over all N particles

$$\begin{aligned} \sum_i \mathbf{F}_i &= \sum_{i,j \text{ with } j \neq i} \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{\text{ext}} \\ &= \sum_{i<j} (\mathbf{F}_{ij} + \mathbf{F}_{ji}) + \sum_i \mathbf{F}_i^{\text{ext}} \end{aligned} \tag{1.13}$$

where, in the second line, we've re-written the sum to be over all pairs $i < j$. At this stage we make use of *Newton's third law of motion*: every action has an equal and opposite reaction. Or, in other words, $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$. We see that the first term vanishes and we are left simply with

$$\sum_i \mathbf{F}_i = \mathbf{F}^{\text{ext}} \tag{1.14}$$

where we've defined the total external force to be $\mathbf{F}^{\text{ext}} = \sum_i \mathbf{F}_i^{\text{ext}}$. We now define the total mass of the system $M = \sum_i m_i$ as well as the *centre of mass* $\mathbf{R}$

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M} \tag{1.15}$$

Then using (1.11), and summing over all particles, we arrive at the simple formula,

$$\mathbf{F}^{\text{ext}} = M\ddot{\mathbf{R}} \tag{1.16}$$

which is identical to that of a single particle. This is an important formula. It tells that the centre of mass of a system of particles acts just as if all the mass were concentrated there. In other words, it doesn't matter if you throw a tennis ball or a very lively cat: the center of mass of each traces the same path.

1.3.1 Momentum Revisited

The *total momentum* is defined to be $\mathbf{P} = \sum_i \mathbf{p}_i$ and, from the formulae above, it is simple to derive $\dot{\mathbf{P}} = \mathbf{F}^{\text{ext}}$. So we find the conservation law of total linear momentum for a system of many particles: $\mathbf{P}$ is constant if $\mathbf{F}^{\text{ext}}$ vanishes.

Similarly, we define *total angular momentum* to be $\mathbf{L} = \sum_i \mathbf{L}_i$. Now let's see what happens when we compute the time derivative.

$$\begin{aligned}
\dot{\mathbf{L}} &= \sum_i \mathbf{r}_i \times \dot{\mathbf{p}}_i \\
&= \sum_i \mathbf{r}_i \times \left(\sum_{j \neq i} \mathbf{F}_{ij} + \mathbf{F}_i^{\text{ext}} \right) && (1.17) \\
&= \sum_{i,j \text{ with } i \neq j} \mathbf{r}_i \times \mathbf{F}_{ji} + \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}} && (1.18)
\end{aligned}$$

The last term in this expression is the definition of *total external torque*: $\boldsymbol{\tau}^{\text{ext}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{\text{ext}}$. But what are we going to do with the first term on the right hand side? Ideally we would like it to vanish! Let's look at the circumstances under which this will happen. We can again rewrite it as a sum over pairs $i < j$ to get

$$\sum_{i<j} (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} \tag{1.19}$$

which will vanish if and only if the force $\mathbf{F}_{ij}$ is parallel to the line joining to two particles $(\mathbf{r}_i - \mathbf{r}_j)$. This is the strong form of Newton's third law. If this is true, then we have a statement about the conservation of total angular momentum, namely $\mathbf{L}$ is constant if $\boldsymbol{\tau}^{\text{ext}} = 0$.

Most forces do indeed obey both forms of Newton's third law: $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and $\mathbf{F}_{ij}$ is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$. For example, gravitational and electrostatic forces have this property. And the total momentum and angular momentum are both conserved in these systems. But some forces don't have these properties! The most famous example is the Lorentz force on two moving particles with electric charge Q. This is given by,

$$\mathbf{F}_{ij} = Q\mathbf{v}_i \times \mathbf{B}_j \tag{1.20}$$

where $\mathbf{v}_i$ is the velocity of the i^{th} particle and $\mathbf{B}_j$ is the magnetic field generated by the j^{th} particle. Consider two particles crossing each other in a "T" as shown in the diagram. The force on particle 1 from particle 2 vanishes. Meanwhile, the force on particle 2 from particle 1 is non-zero, and in the direction

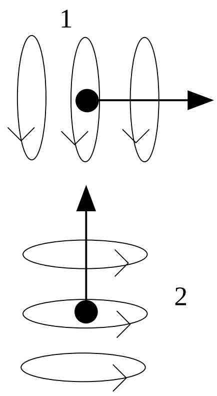

Figure 1: The magnetic field for two particles.

$$\mathbf{F}_{21} \sim \uparrow \times \otimes \sim \leftarrow \tag{1.21}$$

Does this mean that conservation of total linear and angular momentum is violated? Thankfully, no! We need to realise that the electromagnetic field itself carries angular momentum which restores the conservation law. Once we realise this, it becomes a rather cheap counterexample to Newton's third law, little different from an underwater swimmer who can appear to violate Newton's third law if we don't take into account the momentum of the water.

1.3.2 Energy Revisited

The total kinetic energy of a system of many particles is $T = \frac{1}{2}\sum_i m_i \dot{\mathbf{r}}_i^2$. Let us decompose the position vector $\mathbf{r}_i$ as

$$\mathbf{r}_i = \mathbf{R} + \tilde{\mathbf{r}}_i \tag{1.22}$$

where $\tilde{\mathbf{r}}_i$ is the distance from the centre of mass to the particle i. Then we can write the total kinetic energy as

$$T = \tfrac{1}{2}M\dot{\mathbf{R}}^2 + \tfrac{1}{2}\sum_i m_i \dot{\tilde{\mathbf{r}}}_i^2 \tag{1.23}$$

Which shows us that the kinetic energy splits up into the kinetic energy of the centre of mass, together with an *internal energy* describing how the system is moving around its centre of mass. As for a single particle, we may calculate the change in the total kinetic energy,

$$T(t_2) - T(t_1) = \sum_i \int \mathbf{F}_i^{\text{ext}} \cdot d\mathbf{r}_i + \sum_{i \neq j} \int \mathbf{F}_{ij} \cdot d\mathbf{r}_i \tag{1.24}$$

Like before, we need to consider conservative forces to get energy conservation. But now we need both

- Conservative external forces: $\mathbf{F}_i^{\text{ext}} = -\nabla_i V_i(\mathbf{r}_1, \ldots, \mathbf{r}_N)$
- Conservative internal forces: $\mathbf{F}_{ij} = -\nabla_i V_{ij}(\mathbf{r_1}, \ldots, \mathbf{r_N})$

where $\nabla_i \equiv \partial/\partial \mathbf{r}_i$. To get Newton's third law $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ together with the requirement that this is parallel to $(\mathbf{r}_i - \mathbf{r}_j)$, we should take the internal potentials to satisfy $V_{ij} = V_{ji}$ with

$$V_{ij}(\mathbf{r}_1, \ldots \mathbf{r}, N) = V_{ij}(|\mathbf{r}_i - \mathbf{r}_j|) \tag{1.25}$$

so that V_{ij} depends only on the distance between the i^{th} and j^{th} particles. We also insist on a restriction for the external forces, $V_i(\mathbf{r}_1, \ldots, \mathbf{r}_N) = V_i(\mathbf{r}_i)$, so that the force on particle i does not depend on the positions of the other particles. Then, following the steps we took in the single particle case, we can define the *total potential energy* $V = \sum_i V_i + \sum_{i<j} V_{ij}$ and we can show that $H = T + V$ is conserved.

1.3.3 An Example

Let us return to the case of gravitational attraction between two bodies but, unlike in Section 1.2.4, now including both particles. We have $T = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2$. The potential is $V = -Gm_1m_2/|\mathbf{r}_1 - \mathbf{r}_2|$. This system has total linear momentum and total angular mometum conserved, as well as the total energy $H = T + V$.