Klasična mehanika

Osnovni pojmi

- Newton (1686 Principia); 1 in 2 telesi; N
  - koordinatni sistem, ki miruje ali se giblje s konstantno hitrostjo ($a=0$).
  - dobro definirana lega (in hitrost, pospešek)
  - definirana masa (= „množina snovi"); precej nenatančno; koncept točkastega telesa
  - sila: meri se jo preko učinka (dogovor); smer in velikost; vektor 3D. Npr. standardna vzmet (ne mnogo linearna).
  - „zakoni"

    I. (Galilei) točkasto telo se giblje enakomerno če nanj ne deluje nobena sila

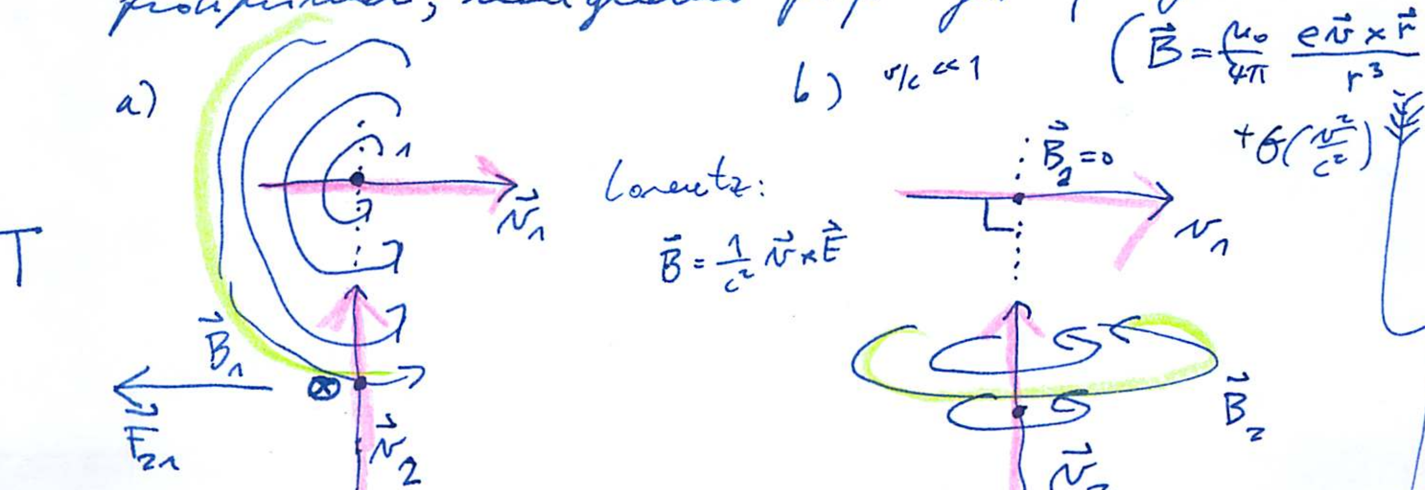
    II. $\vec{F} = m\vec{a}$; $m_a = m_g$ (kasneje)

    III. $\vec{F}_{12} = -\vec{F}_{21}$

    ↑ 1. telo čuti silo 2. telesa

* III. velja za električno in gravitacijsko, vendar ne za magnetno.

protiprimer; magnetno polje gibajočega se naboja $\left(\vec{B} = \frac{\mu_0}{4\pi}\frac{e\vec{v}\times\vec{r}}{r^3}\right)$ $+ \mathcal{O}\left(\frac{v^2}{c^2}\right)$



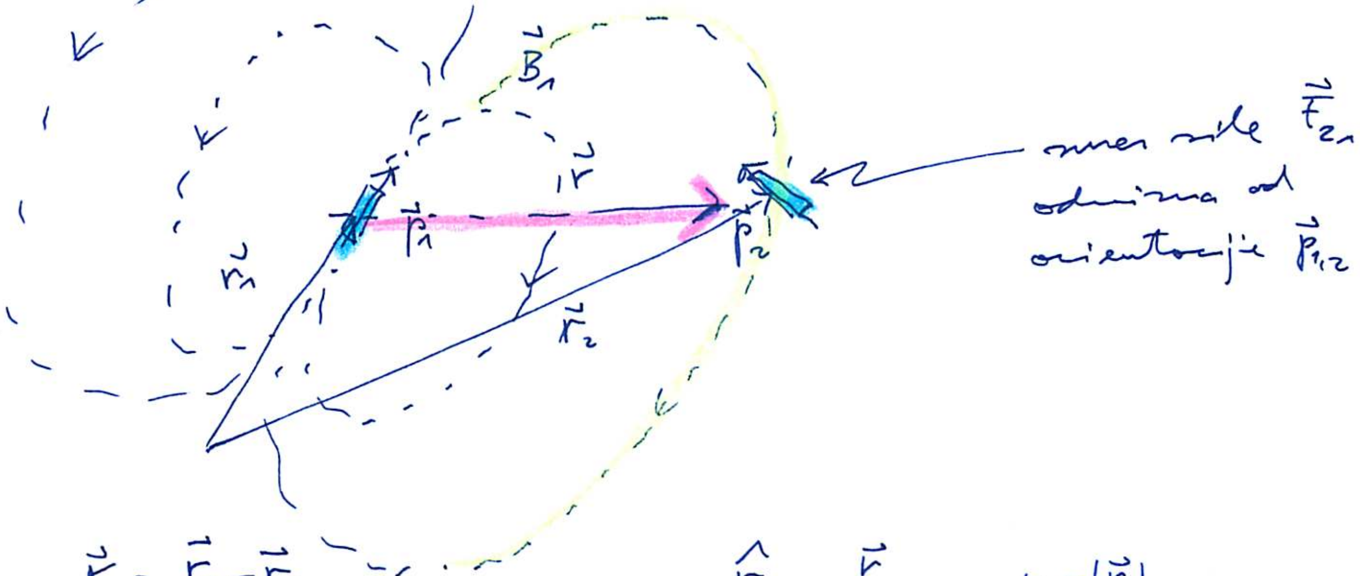
$\vec{F}_{21} = e_2\vec{v}_2\times\vec{B}_1 \neq 0$

$= e_1\vec{v}_1\times\vec{B}_2$

?

$\vec{F}_{12} = 0$, ker ni polja B_2 v smeri gibanja

c) sila med dvema dipolnima magnetoma (točkastima)



$$\vec r = \vec r_2 - \vec r_1 \qquad \hat r = \frac{\vec r}{r}\ ; \qquad r = |\vec r|$$

$$\vec F_{21} = -\nabla_2 \left(-\vec B_1 \cdot \vec p_2 \right)$$

$$\vec B_1 = -\frac{\mu_0}{4\pi} \nabla_2 \frac{\vec p_1 \cdot \vec r}{r^3} \qquad \nabla_2 = \nabla_{\vec r} = \nabla$$

$$\vec F_{21} = -\frac{\mu_0}{4\pi} \nabla \left(\nabla \frac{\vec p_1 \cdot \vec r}{r^3} \cdot p_2 \right) = \ldots$$

$$\ldots = \frac{3\mu_0}{4\pi r^4} \Big((\hat r \times \vec p_1) \times \vec p_2 + (\hat r \times \vec p_2) \times \vec p_1 - 2\hat r (\vec p_1 \cdot \vec p_2) + 5\, \hat r \big((\hat r \times \vec p_1) \cdot (\hat r \cdot \vec p_2) \big) \Big) = -\vec F_{12} .$$

t.j. $(\vec r \to -\vec r)$

torej:

$$\vec F_{21} = -\vec F_{12} \quad \text{in} \quad \vec F_{21} \neq \vec r \frac{|\vec F_{21}|}{|\vec r|}$$

d) v splošnem 3. NZ ne velja za dinamične procese (relativistični popravki polja); če $c \to \infty$, ok.

.. Opis gibanja 1 delca:
za vsak delec 3 koordinate; npr.

$$\vec r(t) = (x(t), y(t), z(t))$$
$$\vec v(t) = \frac{d\vec r}{dt} = \dot{\vec r} = (\dot x, \dot y, \dot z)$$
$$\vec a = \frac{d\vec v}{dt} = \dot{\vec v} = \frac{d^2\vec r}{dt^2} = \ddot{\vec r}$$

deterministična teorija; Descartes: „svet je delo mogočnega urnega mehanizma".

:. $\vec F = m\vec a$; $\vec F(\vec r, \dot{\vec r}) = \dot{\vec p}$; $\vec p = m\dot{\vec r}$
opažanje
$\Leftarrow$ če $\dot m = 0$ (raketa ...)

Ta enačba velja v inercialnem sistemu, ki je definiran z zahtevo, da delec s konstantno maso, brez sil, potuje po premici,

$$\vec r = \vec r_0 + \vec v_0 t$$

N. 1. zakon v bistvu garantira take sisteme.

:: Obstaja 10 neodvisnih transformacij med inercialnimi sistemi:

- 3 rotacije; $\vec r' = \underline{\underline{O}}\,\vec r$; $\underline{\underline{O}}$ je 3×3 ortogonalna matrika
- 3 translacije: $\vec r' = \vec r + \vec c$ ← konstanta
- 3 premiki hitrosti; $\vec r' = \vec r + \vec u t$ ← konstanta
- 1 premik časa $t' = t + t_0$

Te operacije predstavljajo grupo Galilejevih transformacij. Newtonove enačbe so invariantne na te transformacije,

$$\frac{d^2\vec r'}{dt^2} = \frac{d^2\vec r}{dt^2} = m\vec F. \quad (1)$$

t.j., enačba ima enako obliko.

Galilejeva grupa je limitni primer Poincaréjeve grupe (Lorentzove), če $c \to \infty$.

Pomembne količine (1 delec)

• Gibalna količina ($\vec p$; ne $\vec G$):
$\vec{\mathrm{p}}$

$$\vec p = m\vec v$$

$$\frac{d\vec p}{dt} = \dot{\vec p} = \vec F = \sum_{i=1}^{M} \vec F_i$$ vektorska vsota M sil

če $\vec F = 0 \Rightarrow \vec p = \vec p_0 =$ konst. ($\vec v = \vec v_0$). že vemo.

•• Vrtilna količina ($\vec l$ ali $\vec L$; ne $\vec \Gamma$):

$$\vec l = \vec r \times \vec p = m\vec r \times \vec v$$

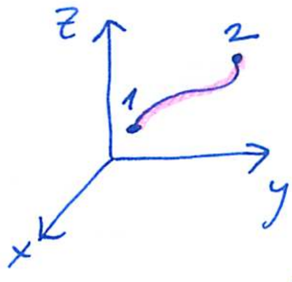
$$\frac{d\vec l}{dt} = m\dot{\vec r}\times\vec v + m\vec r\times\vec a = \cancel{m}\vec r\times\vec F = \vec M$$ navor

če $\vec M = 0 \Rightarrow \vec l = \vec l_0 =$ konst.

∴ Energija

a) kinetična (T; ne W_k):

$$T = \frac{1}{2} m|\vec v|^2 = \frac{1}{2} m v^2 = \frac{1}{2} m \vec v\cdot\vec v$$



pot (tir) - $\vec r(t)$
- $y(x)$ & $z(x)$
- $z(y)$ & $x(y)$
- odvisno od 1 parametra

b) delo sile; $\vec F = \vec F(\vec r)$:

$$A = \int_{pot} \vec F\cdot d\vec r = \int_{t_1}^{t_2} \underbrace{\vec F\cdot\vec v}\, dt = ;$$ $d\vec r = \vec v\, dt$

$$= \int_{t_1}^{t_2} P(t)\,dt$$ $P = \vec F\cdot\vec v$ moč

$$A = \int m\vec a\cdot\vec v\, dt = \int m\dot{\vec v}\cdot\vec v\, dt =$$

$$= \int m\frac{1}{2}(\vec v\cdot\vec v)\,dt = \frac{1}{2} m(v_2^2 - v_1^2) =$$

$$= T(\vec r_2) - T(\vec r_1)$$

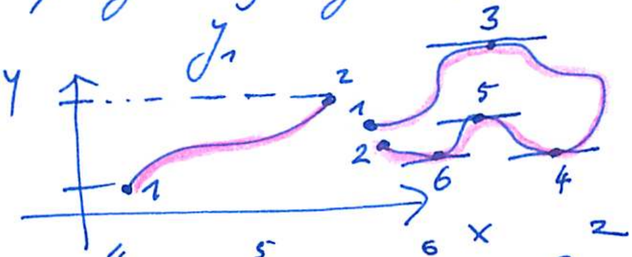
ali: t_2 t_1

konkreten izračun dela

$$A = \int_{\vec r_1\,(pot)}^{\vec r_2} \vec F\cdot d\vec r = \int_{x_1}^{x_2} F_x\,dx + \int_{x_1}^{y_2} F_y\,dy + \int_{z_1}^{z_2} F_z\,dz$$

pot!

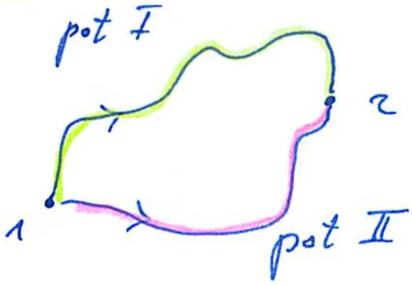
* npr.: $\int_{y_1}^{y_2} F_y\,dy = \int_{y_1}^{y_2} F_y(x(y), y, z(y))\,dy$



$$\int_1^3 + \int_3^4 + \int_4^5 + \int_5^6 + \int_6^2$$

c) potencialna energija (U; ne W_p):

1)



$$A_I = \int_{pot\ I} \vec F\cdot d\vec r, \quad A_{II} = \int_{pot\ II} \vec F\cdot d\vec r$$

če $A_I = A_{II} \Rightarrow \oint \vec F\cdot d\vec r = 0 = A_{II} - A_I$

tudi

2) naj velja

$$\vec F = -\nabla U = -\left(\frac{\partial U}{\partial x}, \frac{\partial U}{\partial y}, \frac{\partial U}{\partial z}\right) = -\frac{\partial U}{\partial \vec r}$$

potem velja, ~~Stokesov izrek~~,

$$A = -\int_{pot} \nabla U(\vec r)\cdot d\vec r = -(U(\vec r_2) - U(\vec r_1))$$

(−)!

d) celotna mehanska energija

$W = T + U \Rightarrow W(\vec r_2) = W(\vec r_1)$ = konst.,

ker velja $0 = \Delta T - A = \Delta T + \Delta U = \Delta W$.

:: Konzervativne sile (pogoji uporabnosti)

če $\exists\, U$:

1) $D=1$ $\qquad U(x) = -\int_{x_0}^{x} F(s)\,ds \;;\; -\frac{dU}{dx} = F(x).$

2) $D=2$ naj bo

$$\vec{F}(x,y) = F_1(x,y)\,\vec{e}_1 + F_2(x,y)\,\vec{e}_2 \qquad \begin{pmatrix} 1=x \\ 2=y \end{pmatrix}$$

in $\vec{F} = -\nabla U = -\frac{\partial U}{\partial x}\vec{e}_1 - \frac{\partial U}{\partial y}\vec{e}_2$

$$-\frac{\partial U}{\partial x} = F_1(x,y) \quad \text{in} \quad -\frac{\partial U}{\partial y} = F_2(x,y)$$

$$-\frac{\partial^2 U}{\partial x \partial y} = \frac{\partial F_1}{\partial y} \qquad -\frac{\partial^2 U}{\partial y \partial x} = \frac{\partial F_2}{\partial x}$$

Če je $\partial^2 U$ zvezen

$$\Rightarrow \frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$$ potreben pogoj za konzervativnost.

3) $D=3$

analogno

:: Nekonzervativne sile ($\oint \vec{A}\cdot d\vec{r} \neq 0$)

a) Linearni zračni upor

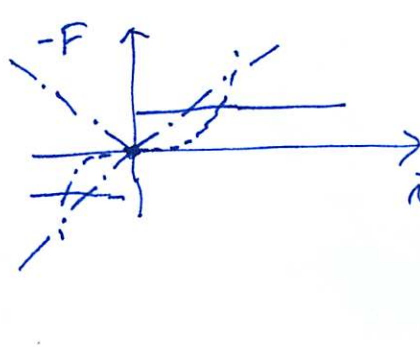
$\vec{F} = -k\vec{v} \qquad k \geq 0$

$A = \int \vec{F}\cdot d\vec{r} = -k\int \vec{v}\cdot\vec{v}\,dt = -k\int v^2\,dt \leq 0$

b) trenje

$(\vec{F})_x = -k F_N\, \mathrm{sign}(\vec{v})_x \qquad 1D$

oz. $\vec{F} = -k F_N \frac{\vec{v}}{v}$

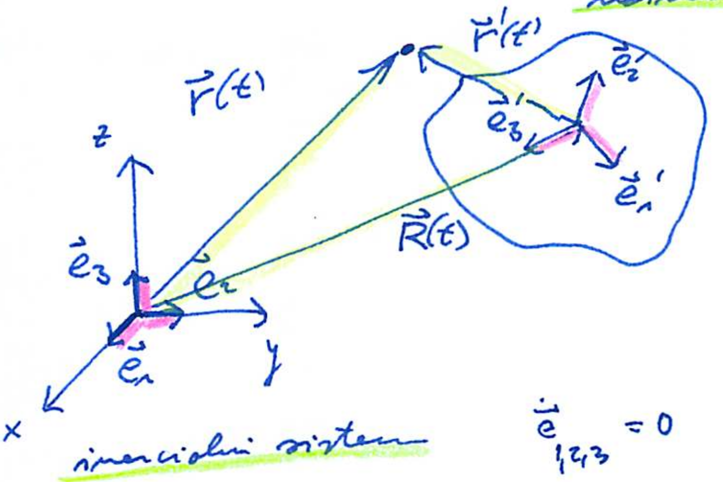


c) kvadratni zračni upor

$\vec{F} = -k v^2 \frac{\vec{v}}{v}$

itd.

Neinercialni koordinatni sistem



$\vec e\,'_{1,2,3}(t)$ ortogonalna baza

$\dot{\vec e}_{1,2,3} = 0$

$$\vec r = \vec R + \vec r\,'$$

$$\vec r = \sum_{\alpha=1}^{3} x_\alpha \vec e_\alpha\ , \quad \vec R = \sum_{\alpha=1}^{3} X_\alpha \vec e_\alpha\ , \quad \vec r\,' = \sum_{\alpha=1}^{3} x'_\alpha \vec e\,'_\alpha$$

a) Kako se izraža hitrost?

$$\vec v = \frac{d\vec r}{dt} = \frac{d\vec R}{dt} + \frac{d\vec r\,'}{dt} \quad \left(= \sum_\alpha \dot x_\alpha \vec e_\alpha\right)$$

$$\vec e_\alpha \cdot \vec e_\beta = \delta_{\alpha\beta}$$

raz koordinat:

$$\frac{d\vec e\,'_\alpha}{dt} = \sum_{\beta=1}^{3} a_{\alpha\beta}\, \vec e\,'_\beta \Rightarrow \Big/ \cdot \vec e\,'_\gamma \ ; \quad \underbrace{\vec e\,'_\alpha \cdot \vec e\,'_\beta = \delta_{\alpha\beta}}_{\check{c}e}$$

$$\frac{d\vec e\,'_\alpha}{dt} \cdot \vec e\,'_\gamma = a_{\alpha\gamma}$$

matrika $a_{\alpha\gamma}$ je antisimetrična. dokaz:

$$a_{\alpha\gamma} = \frac{d\vec e\,'_\alpha}{dt} \cdot \vec e\,'_\gamma = \frac{d}{dt}(\underbrace{\vec e\,'_\alpha \cdot \vec e\,'_\gamma}_{\delta_{\alpha\gamma} = \text{konst.}}) - \vec e\,'_\alpha \cdot \frac{d}{dt}\vec e\,'_\gamma = -a_{\gamma\alpha}$$

torej $\underline{a}$ lahko zapišemo takole:

$$\underline{\underline{a}} = \begin{pmatrix} 0 & a_{12} & a_{13} \\ +a_{21} & 0 & a_{23} \\ +a_{31} & a_{32} & 0 \end{pmatrix} \overset{\text{def.}}{=} \begin{pmatrix} 0 & \omega_3' & -\omega_2' \\ -\omega_3' & 0 & \omega_1' \\ \omega_2' & -\omega_1' & 0 \end{pmatrix},$$

kjer so ω_α komponente rotacije

$$\vec\omega\,' = \sum_{\alpha=1}^{3} \omega'_\alpha \vec e\,'_\alpha \quad \text{oz.} \quad \vec\omega\,' = (\omega_1', \omega_2', \omega_3')$$

v sistemu $(\vec e\,'_\alpha)$.

torej velja za $\alpha=1$ (npr.):

$$\frac{d\vec{e}_1'}{dt} = \sum_{\beta=1}^{3} a_{1\beta}\,\vec{e}_\beta' = a_{12}\,\vec{e}_2' + a_{13}\,\vec{e}_3' = \omega_3'\vec{e}_2' - \omega_2'\vec{e}_3' =$$

$= \vec{\omega}'\times\vec{e}_1$; ciklično lahko menjamo komponente $1\to2\to3$

in dobimo   npr.: $\dot{\vec{e}}_2\cdot\vec{e}_3$

$$\boxed{\frac{d\vec{e}_\alpha'}{dt} = \vec{\omega}'\times\vec{e}_\alpha'} \; ; \quad \vec{\omega} = (a_{23}, -a_{13}, a_{12})$$

To pomeni, da obstaja v vsakem trenutku nek vektor $\vec{\omega}'(t)$ ki predstavlja os rotacije za ortogonalno trojico $\vec{e}_\alpha'$; ker so ti trije vektorji ortonormirani, je tako. Niso neodvisni $\Rightarrow$ njihova časovna odvisnost je lahko samo rotacija. Netrivialna izjava.

Sedaj lahko izrazimo $\vec{v}(t)$. Najprej

$$\frac{d\vec{r}'}{dt} = \frac{d}{dt}\sum_\alpha x_\alpha'\vec{e}_\alpha' = \sum_\alpha \frac{dx_\alpha'}{dt}\vec{e}_\alpha' + \sum_\alpha x_\alpha'\frac{d\vec{e}_\alpha'}{dt} =$$

$$= \left(\frac{d\vec{r}'}{dt}\right)_{\text{v neinerc.}} + \sum_\alpha x_\alpha'\,\vec{\omega}'\times\vec{e}_\alpha' =$$

$$= \left(\frac{d\vec{r}'}{dt}\right)_{\text{nein.}} + \vec{\omega}'\times\vec{r}'.$$

torej

$$\boxed{\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\vec{R}}{dt} + \frac{d\vec{r}'}{dt} = \frac{d\vec{R}}{dt} + \left(\frac{d\vec{r}'}{dt}\right)_{\text{nein.}} + \vec{\omega}'\times\vec{r}'.}$$

zanimivost:

$$\vec{\omega}' = \sum_\alpha \omega_\alpha'\vec{e}_\alpha'$$

$$\boxed{\frac{d\vec{\omega}'}{dt} = \sum_\alpha \dot{\omega}_\alpha'\vec{e}_\alpha' + \underbrace{\vec{\omega}'\times\vec{\omega}'}_{0} = \left(\frac{d\omega'}{dt}\right)_{\text{nein.}}} \text{ je enako!}$$

b) Kako se izraža pospešek?

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{R}}{dt^2} + \frac{d^2\vec{r}\,'}{dt^2} \quad \left(= \sum_\alpha \ddot{x}_\alpha \vec{e}_\alpha\right)$$

postopamo povsem enako še enkrat,

$$\frac{d^2\vec{r}\,'}{dt^2} = \frac{d}{dt}\left(\left(\frac{d\vec{r}\,'}{dt}\right)_{neim.} + \vec{\omega}\,' \times \vec{r}\,' \right) =$$

$$= \frac{d}{dt}\left(\sum_\alpha \dot{x}_\alpha' \vec{e}_\alpha\,' + \vec{\omega}\,' \times \sum_\alpha x_\alpha' \vec{e}_\alpha\,' \right) =$$

$$= \sum_\alpha \ddot{x}_\alpha' \vec{e}_\alpha\,' + \sum_\alpha \dot{x}_\alpha' \underbrace{\dot{\vec{e}}_\alpha\,'}_{\vec{\omega}\,' \times \vec{e}_\alpha\,'} +$$

$$+ \dot{\vec{\omega}}\,' \times \vec{r}\,' + \vec{\omega}\,' \times \sum_\alpha \dot{x}_\alpha' \vec{e}_\alpha\,' + \vec{\omega}\,' \times \underbrace{\sum_\alpha x_\alpha' \underbrace{\dot{\vec{e}}_\alpha\,'}_{\vec{\omega} \times \vec{e}_\alpha\,'}}_{\vec{r}\,'} =$$

$$= \boxed{\left(\frac{d^2\vec{r}\,'}{dt^2}\right)_{neim.}} + \boxed{2\vec{\omega}\,' \times \left(\frac{d\vec{r}\,'}{dt}\right)_{neim.}} + \vec{\omega}\,' \times (\vec{\omega}\,' \times \vec{r}\,') + \dot{\vec{\omega}}\,' \times \vec{r}\,' .$$

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$$

$$\vec{\omega} \times (\vec{\omega} \times \vec{r}) = \vec{\omega}(\vec{\omega} \cdot \vec{r}) - \omega^2 \vec{r}$$

$\left(\dfrac{d\vec{r}\,'}{dt}\right)_{neim.} = \vec{v}\,'_{neim.}$ hitrost v neinercialnem sistemu

$\left(\dfrac{d^2\vec{r}\,'}{dt^2}\right)_{neim.} = \vec{a}\,'_{neim.}$ pospešek —"—

(npr. „glede na Zemljo")

$$\left.\begin{array}{l} \vec{v}\,' = \left(\dfrac{dx'}{dt}, \dfrac{dy'}{dt}, \dfrac{dz'}{dt}\right) \\ \vec{a}\,' = (a_x', a_y', a_z') \end{array}\right\} \text{merjeno npr. glede na Zemljo}$$

Zakaj je to sploh dobro?

c) → Sistemske sile

$$\vec F = m\frac{d^2\vec r}{dt^2}$$

Kako to izgleda v nein. sistemu?

$$\vec F = m\frac{d^2\vec R}{dt^2} + m\frac{d^2\vec r\,'}{dt^2}$$

zapišemo v naslednji obliki:

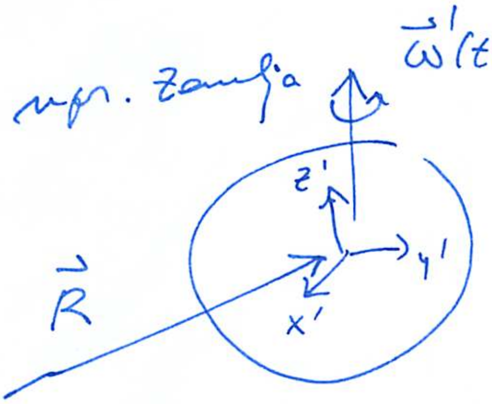
$$\vec F + \vec F_s \stackrel{\text{def.}}{\equiv} m\left(\frac{d^2\vec r\,'}{dt^2}\right)_{nein.}$$

↖ sistemska sila

$$\vec F_s = -m\frac{d^2\vec R}{dt^2} \underbrace{-\,2m\vec\omega\,'\times\vec v\,'_{nein.}}_{\text{Coriolisova}} \underbrace{-\,m\vec\omega\,'\times(\vec\omega\,'\times\vec r\,')}_{\text{centrifugalna}} - m\left(\frac{d\vec\omega\,'}{dt}\right)_{nein.}\times\vec r\,'$$

centrifugalna $= +m\omega'^2\vec r - m(\vec\omega\,'\cdot\vec r)\vec\omega\,'$

$$\left(\frac{d\vec\omega\,'}{dt}\right)_{nein.} = \frac{d\omega'}{dt}$$ vemo od prej

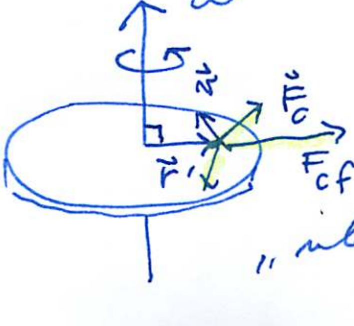


$$\vec r\,' = (x', y', z')$$

$$\boxed{m\left(\frac{d^2\vec r\,'}{dt^2}\right)_{nein.} = \vec F'\left(\vec r\,', \left(\frac{d\vec r\,'}{dt}\right)_{nein.}, t\right).}$$

na vrtiljaku „čutiš ne delež"

primer: delec na vrteči se plošči



$$\vec F_s = -2m\vec\omega\,'\times\vec v\,'_{nein} + m\omega'^2\vec r\,' - m\dot{\vec\omega}\,'\times\vec r\,'.$$

„vleče v desno"   „vleče navzven"