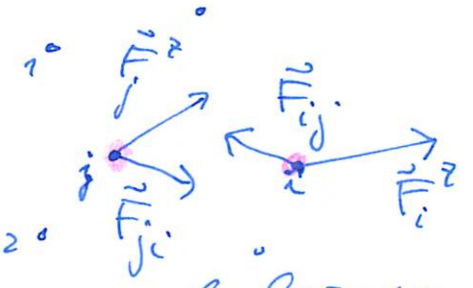
Sistem N točkastih teles

N točkastih teles, za vsakega velja Newtonov zakon, za vsak par $\vec F_{ij} = -\vec F_{ji}$.

$\vec F_i^{z} + \sum_{j\neq i} \vec F_{ij}$ sila na i



- Gibalna količina sistema

$$m_1\vec a_1 = \vec F_1^z + \vec F_{12} + \sum_{j>2} \vec F_{1j}$$
$$m_2\vec a_2 = \vec F_2^z + \vec F_{21} + \sum_{j>3} \vec F_{2j}$$
$$\vdots$$

$$\sum_{i=1}^{N} m_i \vec a_i = \underbrace{\sum_i \vec F_i^z}_{\vec F^z} + \underbrace{\vec F_{12} + \vec F_{21}}_{0} + \underbrace{\sum_{i}\sum_{j\neq i} \vec F_{ij}}_{\sum_i \sum_{j<i} \underbrace{\vec F_{ij} + \vec F_{ji}}_{0}}$$

$$\vec p \overset{\text{def.}}{=} \sum_i \vec p_i$$
$$\dot{\vec p} = \vec F^z$$

Če $\vec F^z = 0 \Rightarrow \vec p = \vec p_0 =$ konst.

Težišče

$$M \vec r_T = \sum_i m_i \vec r_i \; ; \quad M = \sum_i m_i$$

$$\vec p = \sum_i m_i \vec v_i = M \dot{\vec r}_T = M \vec v_T \quad \text{hitrost težišča}$$

Izgleda kot Newtonov zakon za vse skupaj;

$$M \vec a_T = M \dot{\vec v}_T = M \ddot{\vec r}_T = \dot{\vec p} = \vec F^z.$$

pozor: uporaba tega s premislekom.

npr.: zakaj kolesa poganjajo avto?
sila trenja; efektivne sile podsistemov

Vrtilna količina sistema

$$\vec L = \sum_i \vec l_i = \sum_i \vec r_i \times \vec p_i$$

$$\dot{\vec L} = \sum_i \frac{d}{dt}(\vec r_i \times \vec p_i) = \sum_i \underbrace{\dot{\vec r}_i \times m\vec v_i}_{0} + \sum_i \vec r_i \times \underbrace{m\vec a_i}_{\dot{\vec p}_i}$$

$$\dot{\vec L} = \sum_i \vec r_i \times \vec F_i^z + \sum_{\substack{i \\ j\neq i}} \vec r_i \times \vec F_{ij} = \vec M^z + \vec M^n = \vec M$$

ponovno simetriziramo vsoto

$$\sum_{\substack{i \\ j\neq i}} \vec r_i \times \vec F_{ij} = \sum_{\substack{i \\ j<i}} (\vec r_i \times \vec F_{ij} + \vec r_j \times \vec F_{ji}) =$$

$$= \sum_{\substack{i \\ j<i}} (\vec r_i - \vec r_j) \times \vec F_{ij} = \vec M^n$$

če so sile centralne, $\vec F_{ij} = F_{ij}\dfrac{\vec r_i - \vec r_j}{|\vec r_i - \vec r_j|}$, je

$$\vec M^n = 0.$$

To ne velja za magnetno silo; npr. za silo med magnetnimi dipoli.

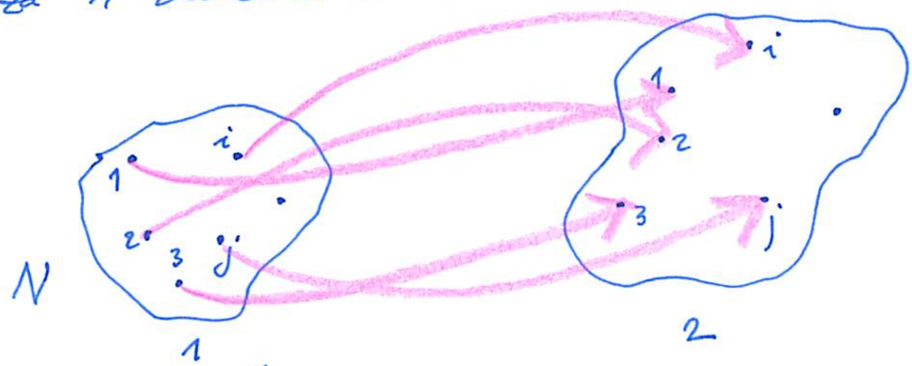
Velja torej

$$\dot{\vec L} = \vec M^z, \quad \vec M^z = \sum_i \vec r_i \times \vec F_i^z$$

če $\vec M^z = 0 \Rightarrow \vec L = \vec L_0 = \text{konst.}$

∴ Energija sistema N teles

v bistvu ponovimo isto kar smo naredili za 1 delec ...



– delo sile

$$A = \sum_i \int_{1,\ \text{pot } i}^{2} \vec{F}_i \cdot d\vec{r}_i = A^z + A^n$$

$$A^z = \sum_i \int_{1,\ \text{pot } i}^{2} \vec{F}_i^{\,z} \cdot d\vec{r}_i \qquad A^n = \sum_{\substack{i \\ j \neq i}} \int_{1,\ \text{pot } i}^{2} \vec{F}_{ij} \cdot d\vec{r}_i$$

enako kot prej

$$A = \sum_i \int_{t_1}^{t_2} m_i \dot{\vec{v}}_i \cdot \underbrace{\vec{v}_i\, dt}_{d\vec{r}_i} =$$

$$= \int_{t_1}^{t_2} \left(\frac{d}{dt} \sum_i \frac{1}{2} m_i v_i^2 \right) dt =$$

$$= T(t_2) - T(t_1) = \Delta T$$

– kinetična energija sistema

$$T = \frac{1}{2} \sum_i m_i v_i^2$$

$$\vec{v}_i = \vec{v}_T + \vec{v}_i{}'$$

$$T = \frac{1}{2} \sum_i m_i (\vec{v}_T + \vec{v}_i{}') \cdot (\vec{v}_T + \vec{v}_i{}') =$$

$$= \frac{1}{2} \sum_i m_i (v_T^2 + \vec{v}_i{}'^2 + 2 \vec{v}_i{}' \cdot \vec{v}_T) =$$

$$= \underbrace{\frac{1}{2} M \vec{v}_T^{\,2}}_{\text{translacija težišča}} + \underbrace{\frac{1}{2} \sum_i m_i \vec{v}_i{}'^2}_{\text{glede na težišče = rotacija}} + \underbrace{\vec{v}_T \cdot \sum_i m_i \vec{v}_i{}'}_{0}$$

$$\vec{v}_i{}' = \vec{v}_i - \vec{v}_T, \qquad \sum_i m_i \vec{v}_i{}' = M\vec{v}_T - M\vec{v}_T = 0.$$

– konzervativne (potencialne) sile; ponovno velja

$$\vec F_i^{\,z} = -\nabla_i U_i^z(\vec r_i)$$

$$\int_{\vec r_{i1}}^{\vec r_{i2}} \vec F_i^{\,z}\cdot d\vec r_i = -\int_{\vec r_{i1}}^{\vec r_{i2}} \nabla_i U_i^z(\vec r_i)\cdot d\vec r_i = -U_i^z(\vec r_{i2}) + U_i^z(\vec r_{i1})$$

$$A^z = \sum_i \int_1^2 \vec F_i^{\,z}\, d\vec r_i = -U^z(t_2) + U^z(t_1) = -\Delta U^z$$

$$U^z = \sum_i U_i^z(\vec r_i)\Big|_{t=\dots}$$

Za „notranje" sile predpostavimo, da so tudi potencialne (gravitacijske, električne) in centralne

$$V_{ij} = V_{ij}(\vec r_{ij}) = V_{ij}(|\vec r_{ij}|); \quad \vec r_{ij} = \vec r_i - \vec r_j$$

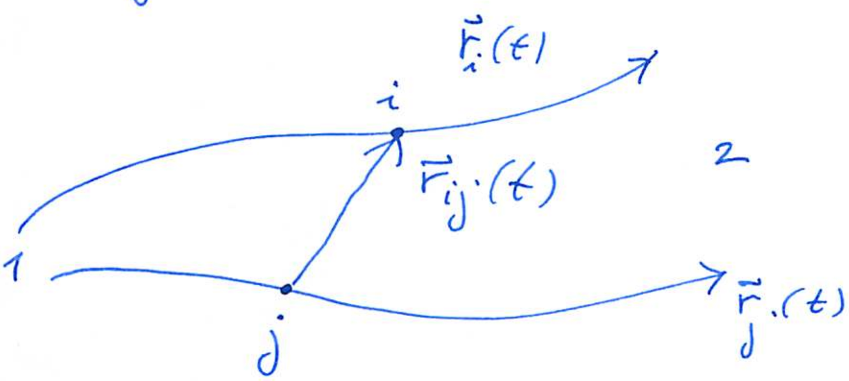
$$= V_{ji}$$

$$A^n = \sum_{\substack{i\\ j<i}} \left(\int_1^2 \vec F_{ij}\cdot d\vec r_i + \int_1^2 \vec F_{ji}\cdot d\vec r_j \right) =$$

$$= \sum_{\substack{i\\ j<i}} \int_1^2 \vec F_{ij}\cdot d\vec r_{ij}$$

; kdaj je $A^n = ?$

- $F \perp dr$
- $F = 0$
- $dr = 0$



$$\vec F_{ij} = -\nabla_{ij} V_{ij}(r_{ij}) = -\left(\frac{\partial}{\partial x_i} V_{ij}(r_{ij}), \frac{\partial}{\partial y_i}\dots, \frac{\partial}{\partial z_i}\dots\right)$$

gradient centralnega potenciala

$$V = V(|\vec{r}|) = V(r) \quad ; \quad r = \sqrt{x^2+y^2+z^2}$$

$$\nabla V(r) = \left(\frac{\partial}{\partial x} V, \frac{\partial}{\partial y} V, \frac{\partial}{\partial z} V\right) =$$

$$= \left(\frac{\partial V(r)}{\partial r}\frac{\partial r}{\partial x}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial y}, \frac{\partial V}{\partial r}\frac{\partial r}{\partial z}\right) = \frac{\partial V}{\partial r}\nabla r$$

$$\nabla r = \left(\frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2}, \frac{\partial}{\partial y}\sqrt{}, \frac{\partial}{\partial z}\sqrt{}\right) =$$

$$= \left(\frac{1 \cdot 2x}{2\sqrt{x^2+y^2+z^2}}, \frac{2y}{2r}, \frac{z}{r}\right) = \frac{\vec{r}}{r}$$

$$\nabla V(r) = \frac{\partial V(r)}{\partial r}\frac{\vec{r}}{r}.$$

torej,

$$\vec{F}_{ij} = -\frac{\partial V_{ij}(r_{ij})}{\partial r_{ij}}\frac{\vec{r}_{ij}}{r_{ij}}$$

delo notranjih sil je torej

$$A^n = -\sum_{\substack{i\\ j<i}}\int_1^2 \nabla_{ij} V_{ij}(r_{ij}) \cdot d\vec{r}_{ij} = -\sum_{\substack{i\\ j<i}}\int_1^2\left(\frac{\partial}{\partial x_i}V_{ij}\,dx_i + \dots\right) =$$

$$= -U^n(t_2) + U^n(t_1), \quad = -\Delta U^n$$

$$U^n = \sum_{\substack{i\\ j<i}} V_{ij}(r_{ij}) = \frac{1}{2}\sum_{ij} V_{ij}(r_{ij})$$

celotna mehanska energija $W = T + \underbrace{U^z + U^n}_{U}$

$$A = \underbrace{A^z + A^n}_{-\Delta U} = \Delta T \implies \Delta W = 0 = \Delta T + \Delta U$$

$$W = W_0 = \text{konst.}$$

:: Virialni teorem (Clausius; vis, vinis = moč, sila)

Začnemo z N. zakonom za N teles; $i=1,\dots N$:

$$m_i \ddot{\vec{r}}_i = \sum_{\substack{j \\ j\neq i}} \vec{F}_{ij} + \vec{F}_i^{z} \qquad / \cdot \vec{r}_i \, ; \sum_i$$

$$* \quad \sum_{\substack{ij \\ j\neq i}} m_i \ddot{\vec{r}}_i \cdot \vec{r}_i = \sum_{\substack{ij \\ j\neq i}} \vec{F}_{ij} \cdot \vec{r}_i + \sum_i \vec{F}_i^{z} \cdot \vec{r}_i$$

pomožni računi:

a) $$\frac{d^2}{dt^2} \sum_i m_i \vec{r}_i^{\,2} = \frac{d}{dt}\left(2 \sum_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i\right) = 2 \cdot 2 \underbrace{\sum_i \frac{1}{2} m_i \dot{r}_i^{\,2}}_{2T} + 2 \underbrace{\sum_i m_i \ddot{\vec{r}}_i \cdot \vec{r}_i}_{*}$$

b) če $\vec{F}_{ij} = -\vec{F}_{ji}$, velja

$$\sum_{\substack{ij \\ j\neq i}} \vec{F}_{ij} \cdot \vec{r}_i = \frac{1}{2} \sum_{\substack{ij \\ j\neq i}} (\vec{F}_{ij} + \underbrace{\vec{F}_{ij}}_{-\vec{F}_{ji}}) \cdot \vec{r}_i = \frac{1}{2} \sum_{\substack{ij \\ j\neq i}} \vec{F}_{ij} \cdot (\vec{r}_i - \vec{r}_j) =$$

$$i \leftrightarrow j: \; -\vec{F}_{ij} \cdot \vec{r}_j$$

$$= \frac{1}{2} \sum_{\substack{ij \\ j\neq j}} \vec{F}_{ij} \cdot \vec{r}_{ij} \qquad \vec{r}_{ij} \cdots \vec{r}_i - \vec{r}_j$$

za naprej: $\sum_{\substack{ij \\ j\neq i}} \equiv \sum_{j\neq i}$

in $\sum_{\substack{ij \\ j<i}} \equiv \sum_{j<i}$

$$\Bigg\} \quad \frac{1}{2} \sum_{j\neq i} = \sum_{j<i}$$

e) $\vec{F}_{ij}(\vec{r}_{ij}) = -\nabla_{ij} V_{ij}(\vec{r}_{ij}) = -\frac{\partial V_{ij}}{\partial r_{ij}} \frac{\vec{r}_{ij}}{r_{ij}}$

Sedaj združimo a, b in c in d.

$$\frac{d^2}{dt^2} \sum_i \frac{m_i \vec{r}_i^{\,2}}{2} = 2T - \frac{1}{2} \sum_{j \neq i} \frac{\partial V_{ij}}{\partial r_{ij}} \frac{\vec{r}_{ij}}{r_{ij}} \cdot \vec{r}_{ij} + \sum_i \vec{F}_i^{z} \vec{r}_i$$

Povprečimo sedaj po času,

$$\overline{f} = \frac{1}{T} \int_0^T f(t) dt,$$

$$\frac{1}{T} \int_0^T \frac{d^2}{dt^2} \sum_i \frac{m_i r_i^2}{2} dt = \frac{1}{T} \frac{d}{dt} \sum_i m_i \frac{r_i^2}{2} \Big|_0^T = \frac{1}{T} \sum_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i \Big|_0^T$$

$\lim_{T \to \infty} \frac{1}{T} \sum_i m_i \vec{r}_i \cdot \dot{\vec{r}}_i = 0$, če $\vec{r}_i \cdot \dot{\vec{r}}_i \Big|_0^T$ končno. (omejeno gibanje).

Tako torej

$$\boxed{2\overline{T} = \frac{1}{2} \sum_{j \neq i} \overline{\frac{\partial V_{ij}}{\partial r_{ij}} r_{ij}} - \overline{\sum_i \vec{F}_i^{z} \cdot \vec{r}_i}}$$

to je Virialni teorem

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primer: $V(\vec{r}) = \alpha r^n$ in delci potovanje omejeni

$$2\overline{T} = \underbrace{\overline{\frac{1}{2} \sum_{j \neq i} n \alpha r_{ij}^{n-1} r_{ij}}}_{= \sum_{j<i} n \alpha \underbrace{r_{ij}^n}_{V(r_{ij})}} - \underbrace{\overline{\sum_i \vec{F}_i^{z} \cdot \vec{r}_i}}_{=0}$$

$=0$, če npr. delci (plin) v škatli in na steno v povprečju $=0$ (steno-stena) ali pa, če $\vec{F}_i^{z} \equiv 0$.

torej $\boxed{2\overline{T} = n\overline{V}}$ $\qquad V = \sum_{j<i} V_{ij}$.