Primeri Lagrangeovih enačb

• prost (brez mej) delec v potencialu $V(\vec{r})$

$$L = T - V = \frac{1}{2} m v^2 - V(\vec{r}) =$$

$$= \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \; ; \quad q_{1,2,3} = x, y, z$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = \frac{d}{dt} m\dot{x} + \frac{\partial V}{\partial x} = 0 \Rightarrow m\ddot{\vec{r}} = -\nabla V \checkmark$$

$$\phantom{\frac{d}{dt}} \qquad y \qquad y$$

$$\phantom{\frac{d}{dt}} \qquad z \qquad z$$

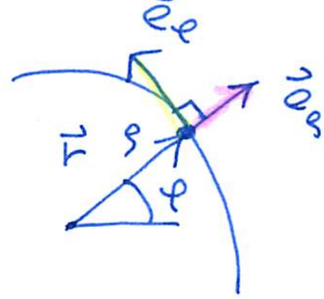
•• gibanje v ravnini (polarne koordinate)

$$\vec{r} = \rho(\cos\varphi, \sin\varphi) = \rho\,\hat{e}_\rho \qquad ; \; q_1 = \rho, \; q_2 = \varphi$$

$$\dot{\vec{r}} = \dot{\rho}(\cos\varphi, \sin\varphi) + \rho(-\sin\varphi, \cos\varphi)\dot{\varphi} =$$

$$= \dot{\rho}\,\hat{e}_\rho + \rho\dot{\varphi}\,\hat{e}_\varphi$$

radialni tangentni
enotni vektor



$$\frac{\partial \dot{\vec{r}}}{\partial \dot{\rho}} = \frac{\partial \vec{r}}{\partial \rho} = \hat{e}_\rho$$

in

$$\frac{\partial \dot{\vec{r}}}{\partial \dot{\varphi}} = \frac{\partial \vec{r}}{\partial \varphi}$$, kot vemo od prej

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial \rho} = \frac{d}{dt}\hat{e}_\rho = \dot{\varphi}\,\hat{e}_\varphi = \frac{\partial \dot{\vec{r}}}{\partial \rho}$$

$$\frac{d}{dt}\frac{\partial \vec{r}}{\partial \varphi} = \frac{d}{dt}\rho\,\hat{e}_\varphi = \dot{\rho}\,\hat{e}_\varphi - \rho\dot{\varphi}\,\hat{e}_\rho = \frac{\partial \dot{\vec{r}}}{\partial \varphi}$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m(\dot{\rho}^2 + \rho^2\dot{\varphi}^2) \; , \text{ ker } \hat{e}_\rho \cdot \hat{e}_\varphi = 0$$

torej

$$L = \frac{1}{2} m (\dot{\rho}^2 + \rho^2 \dot{\varphi}^2) - V(\rho, \varphi)$$

j=1; ρ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\rho}} - \frac{\partial L}{\partial \rho} = 0$$

$$\frac{d}{dt} m\dot{\rho} - \left(m\rho\dot{\varphi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$$

$$\boxed{m\ddot{\rho} = m\rho\dot{\varphi}^2 - \frac{\partial V}{\partial \rho}}$$

j=2; φ:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = 0$$

$$\frac{d}{dt} m\rho^2\dot{\varphi} - \left(-\frac{\partial V}{\partial \varphi}\right) = 0$$

$$\boxed{\underbrace{2m\rho\dot{\rho}\dot{\varphi} + m\rho^2\ddot{\varphi}}_{\frac{d}{dt}(m\rho^2\dot{\varphi})} = -\frac{\partial V}{\partial \varphi}}$$

$$\frac{d}{dt}(m\rho^2\dot{\varphi}) = -\frac{\partial V}{\partial \varphi}$$

če V ni odvisen od φ, $V = V(r)$, t.j. centralni,

$$\frac{\partial V}{\partial \varphi} = 0 \Rightarrow m\rho^2\dot{\varphi} = \text{konst.} = l_0$$

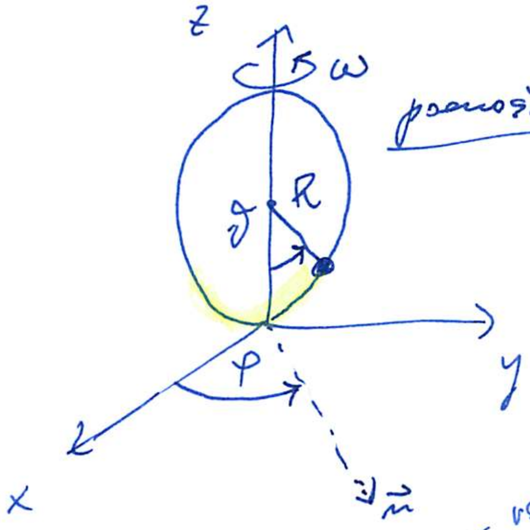
ohranjena količina seveda.

$$\Rightarrow m\ddot{\rho} = m\rho\frac{l_0^2}{m^2\rho^4} - \frac{\partial V}{\partial \rho} \Rightarrow$$

$$\boxed{m\ddot{\rho} = -\frac{\partial}{\partial \rho}\left(V(\rho) + V_c\right); \quad V_c = \frac{l_0^2}{2m\rho^2}}$$

„centrifugalni" potencial

: gibanje mase na vrtljivem obroču:



$q_1=\varphi;\ q_2=\vartheta$

poenostavitveni pogoj $\varphi=\omega t;\ \omega=\text{konst.}$

$x = R\sin\vartheta\cos\omega t;\ \dot{x}=\dots$

$y = R\sin\vartheta\sin\omega t;\ \dot{y}=\dots$

$z = R - R\cos\vartheta;\ \dot{z}=\dots$

vsota kvadratov hitrosti

$$T=\frac{1}{2}m\dot{\vec{r}}^{\,2}=\frac{1}{2}mR^2\left(\dot{\vartheta}^2+\omega^2\sin^2\vartheta\right)$$

$\dot{\varphi}^2$

$$V=mgz=-mgR\cos\vartheta$$

$$L=T-V=\frac{1}{2}mR^2\dot{\vartheta}^2-V_{ef}$$

$$V_{ef}=-mgR\cos\vartheta-\frac{1}{2}mR^2\omega^2\sin^2\vartheta$$

~ centrifugalni

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\vartheta}}-\frac{\partial L}{\partial\vartheta}=0$$

$$\frac{d}{dt}\left(mR^2\dot{\vartheta}\right)+\frac{\partial V_{ef}}{\partial\vartheta}=0$$

$$\boxed{mR^2\ddot{\vartheta}=-\frac{\partial V_{ef}}{\partial\vartheta}}$$

če φ spustimo:

$$\frac{d}{dt}\frac{\partial L}{\partial\dot{\varphi}}-\frac{\partial L}{\partial\varphi}=0=\frac{d}{dt}\left(mR^2\sin^2\vartheta\,\dot{\varphi}\right)$$

$$\Rightarrow \boxed{\left(mR^2\sin^2\vartheta\right)\dot{\varphi}=\text{konst.}}$$

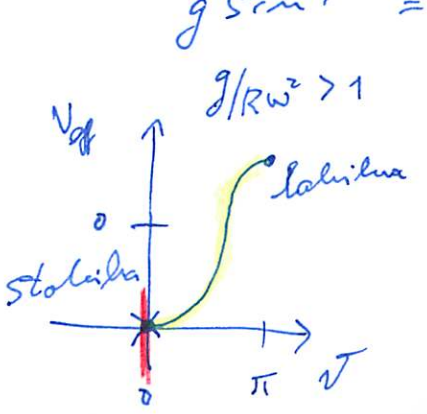
vrtilna količina

stacionarna rešitev: $\dot{\vartheta}=0\ \&\ \ddot{\vartheta}=0 \Rightarrow \frac{\partial V_{ef}}{\partial\vartheta}=0$

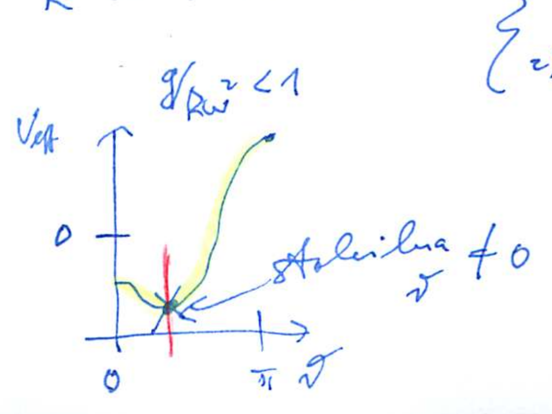
(Σ sil na telo v vrtečem sistemu $=0$).

$$g\sin\vartheta = R\omega^2\sin\vartheta\cos\vartheta \Rightarrow \begin{cases} 1)\ \sin\vartheta=0 \quad \text{ali} \\ 2)\ \cos\vartheta=\frac{g}{R\omega^2}\end{cases}$$

$g/R\omega^2>1$



$g/R\omega^2<1$



Nekatere lastnosti L

- Lagrangeova funkcija je nedoločena do časovnega odvoda funkcije koordinat, če dodamo $\frac{d}{dt}F(q_1, q_2, \dots q_n, t)$ k L, bo nastala funkcija spet Lagrangeova f.

Torej,

$$L' = L + \frac{d}{dt}F(q_1, q_2, \dots q_n, t)$$

$$\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} + \underbrace{\underbrace{\frac{d}{dt}\underbrace{\frac{\partial}{\partial \dot{q}_i}\underbrace{\left(\sum_j \frac{\partial F}{\partial q_j}\dot{q}_j + \frac{\partial F}{\partial t}\right)}_{=\frac{d}{dt}F}}_{=\frac{\partial F}{\partial q_i}}} - \frac{\partial}{\partial q_i}\frac{d}{dt}F}_{=0}$$

torej sledi končno

$$\boxed{\frac{d}{dt}\frac{\partial L'}{\partial \dot{q}_i} - \frac{\partial L'}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i}.}$$

•• mehanska podobnost

Če $L' = cL$ in $c \in \mathbb{R}$, je L' Lagrangeva funkcija, če je to tudi L. (očitno)

Zanimiv primer: potencial je homogena funkcija;

$$U(\alpha\vec{r}_1, \alpha\vec{r}_2, \dots \alpha\vec{r}_N) = \alpha^k U(\vec{r}_1, \vec{r}_2, \dots \vec{r}_N)$$

← ni odvisno od časa

velja torej npr.:

$\vec{r}_i \to \alpha \vec{r}_i$ (spreminjanje dolžinske enote) časovne

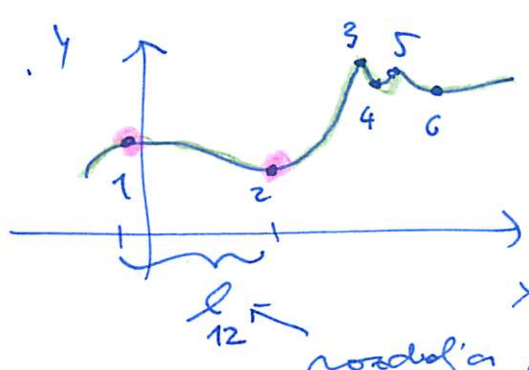
$t \to \beta t$

$\vec{v}_i \to \frac{\alpha}{\beta}\vec{v}_i$

$$\left.\begin{aligned} T &\to \left(\frac{\alpha}{\beta}\right)^2 T \\ U &\to \alpha^k U \end{aligned}\right\}$$ če velja $\left(\frac{\alpha}{\beta}\right)^2 = \alpha^k$ oz. $\beta = \alpha^{1-\frac{1}{2}k}$

$\Rightarrow L \to \alpha^k L$; trajektorije so torej geometrijsko podobne poti

npr.



← razdalja med dogodkoma 1 in 2

delec to prepotuje v času t_{12}

skaliramo enote in tako najdemo:

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k}; \quad \frac{v'}{v} = \left(\frac{l'}{l}\right)^{\frac{1}{2}k}; \quad \frac{E'}{E} = \left(\frac{l'}{l}\right)^k.$$

primeri mehanske podobnosti:

1) $k=2$ (harmonska sila); $V(x) \propto x^2$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{1-\frac{1}{2}k} = 1$$

npr.: nihajni čas nihala neodvisen od amplitude

2) $k=1$ (homogeno polje); $V(x) \propto x$

$$\frac{t'}{t} = \sqrt{l'/l}$$ npr.: prosti pad - $t = \sqrt{2gh}$

3) $k=-1$ (gravitacija, Coulomb); $V(x) \propto \frac{1}{x}$

$$\frac{t'}{t} = \left(\frac{l'}{l}\right)^{3/2}$$ 3. Keplerjev zakon

4) virialni teorem (brez dokaza, naj je): gibanje delcev naj bo omejeno in k

$$\Rightarrow \boxed{2\langle T\rangle = k\langle U\rangle}; \quad \lim_{t_0\to\infty} \frac{1}{t_0}\int_0^{t_0} f(t)\,dt = \langle f\rangle$$

povprečje po času

- nihalo ($k=2$): $\langle T\rangle = \langle U\rangle$

- $k=-1$: $2\langle T\rangle = -\langle U\rangle$

$\Rightarrow \langle U\rangle < 0$ za omejena gibanja

(ne velja za npr. hiperbolične tire; delc prileti od daleč s pozitivno energijo - ni omejeno - in odleti v ∞)

Konstante gibanja (ohranjene količine) (ponovno)

$F(\underline{q}, \underline{\dot{q}}, t)$ je konstanta gibanja, če

$$\frac{dF}{dt} = \sum_i \frac{\partial F}{\partial q_i} \dot{q}_i + \sum_i \frac{\partial F}{\partial \dot{q}_i} \ddot{q}_i + \frac{\partial F}{\partial t} = 0$$

vzdolž trajektorij (q, ki zadošča Lagrangeovim enačbam).

- Naj Lagrangeva funkcija ne odvisna od nekega q_j („ciklična" koordinata) spregledljiva

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial q_j} = 0 \Rightarrow p_j \overset{\text{def.}}{=} \frac{\partial L}{\partial \dot{q}_j} = \text{konst.}$$

npr. gibalna količina, če $V = \text{konst.}$

ali vrtilna količina, če $V(\vec{r}) = V(|\vec{r}|)$ itd.

- • Naj L. funkcija ne odvisna od časa t:

$$\frac{\partial L}{\partial t} = 0 \Rightarrow H = \sum_j \dot{q}_j \frac{\partial L}{\partial \dot{q}_j} - L = \text{konst.}$$

$$= \sum_j p_j \dot{q}_j - L$$

Hamiltonova funkcija

dokaz:

$$\boxed{\frac{dH}{dt}=\sum_j\overbrace{\Big(\ddot q_j\frac{\partial L}{\partial \dot q_j}+\dot q_j\underbrace{\frac{d}{dt}\frac{\partial L}{\partial \dot q_j}-\frac{\partial L}{\partial q_j}}_{=0}\dot q_j-\frac{\partial L}{\partial \dot q_j}\ddot q_j\Big)}^{=0}-\frac{\partial L}{\partial t}=}$$

$$\boxed{=-\frac{\partial L}{\partial t}=0}\ \checkmark$$

Energija:

$$T=\sum_i \frac{1}{2}m_i\dot{\vec r}_i^{\,2}\ ,\quad V\neq V(\dot{\underline q},t)\ !$$

$$V=V(q_1,q_2,\dots q_n)$$

$$\vec r_i=\vec r_i(q_1,q_2,\dots q_n)$$

$$T=\sum_i\frac{1}{2}m_i\sum_{jk}\frac{\partial \vec r_i}{\partial q_j}\dot q_j\cdot\frac{\partial \vec r_i}{\partial q_k}\dot q_k=$$

$$=\sum_{jk}\frac{1}{2}m_{jk}(q_1,q_2,\dots q_n)\,\dot q_j\dot q_k$$ najsplošnejša oblika T

$$\text{in}\quad m_{jk}=\sum_i m_i\frac{\partial \vec r_i}{\partial q_j}\cdot\frac{\partial \vec r_i}{\partial q_k}=m_{kj}$$

$$H=\sum_j\frac{\partial L}{\partial \dot q_j}\dot q_j-L=$$

$$=\underset{\textstyle \sum_{kl} m_{kl}\dot q_k\dot q_l}{\underset{\uparrow}{2T}}-T+V=T+V=\underset{\text{energija}}{E}=\text{konst.}$$

∴ izrek Emmy Noether

Imamo enoparametrično preslikavo koordinat,

$$q_i(t) \to Q_i(s,t) \text{ in } q_i(t) = Q_i(0,t); \; s \in \mathbb{R}.$$

Ta preslikava ostreže zvezni simetriji L, če L ni odvisna od s,

$$\frac{\partial L(Q_i(s,t),\, \dot{Q}_i(s,t),\, t)}{\partial s} = 0.$$

Naj to torej velja za L,

$$0 = \frac{\partial}{\partial s} L = \sum_i \frac{\partial L}{\partial Q_i}\frac{\partial Q_i}{\partial s} + \sum_i \frac{\partial L}{\partial \dot{Q}_i}\frac{\partial \dot{Q}_i}{\partial s} =$$

uzamemo: $s \to 0$

$$= \sum_i \frac{\partial L}{\partial q_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0} =$$

Lagrange

$$= \sum_i \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial Q_i}{\partial s}\Big|_{s=0} + \sum_i \frac{\partial L}{\partial \dot{q}_i}\underbrace{\frac{\partial \dot{Q}_i}{\partial s}\Big|_{s=0}}_{=\frac{d}{dt}\frac{\partial Q_i}{\partial s}} =$$

$$= \frac{d}{dt}\sum_i \underbrace{\frac{\partial L}{\partial \dot{q}_i}}_{=p_i}\frac{\partial Q_i}{\partial s}\Big|_{s=0}.$$

torej,

$$\boxed{\sum_i p_i \frac{\partial Q_i}{\partial s}\Big|_{s=0} = \text{konst.}}$$

Vsaki zvezni simetriji Lagrangeove funkcije torej ustreza ena ohranjena količina primeri (ponovno isto, malo drugače):

1) homogenost prostora

$$\vec{r}_i \to \vec{Q}_i(s,t) = \vec{r}_i + s\vec{n} \quad ; \quad s \in R$$

$\vec{n}$ konstanten vektor

naj velja $L(\vec{r}_i, \dot{\vec{r}}_i, t) = L(\vec{Q}_i, \dot{\vec{Q}}_i, t)$,
to pomeni, da je prostor homogen, se pravi, da translacija sistema ne spremeni enačb gibanja ($\equiv$ in zunanjih sil),

$$\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot \vec{n} = \left(\sum_i \vec{p}_i\right) \cdot \vec{n} \quad \text{za } \forall \vec{n}$$

$$\Rightarrow \sum_i \vec{p}_i = \text{konst.}$$

2) izotropnost prostora

$$\vec{Q}_i = \vec{r}_i + \varphi \vec{n} \times \vec{r}_i$$

naj velja

$L(\vec{r}_i, \dot{\vec{r}}_i, t) = L(\vec{Q}_i, \dot{\vec{Q}}_i, t)$, se pravi,
da ostanejo enačbe gibanja enake v rotiranem sistemu,

$$\sum_i \frac{\partial L}{\partial \dot{\vec{r}}_i} \cdot (\vec{n} \times \vec{r}) = \sum_i \vec{n} \cdot (\vec{r}_i \times \vec{p}_i) = \vec{n} \cdot \vec{L}$$

za $\forall n \Rightarrow \vec{L} = \text{konst.}$

vrtilna količina

3) homogenost časa: L ni odvisna od časa, $\frac{\partial L}{\partial t} = 0$.

$$t \to t + t_0 \Rightarrow H = \text{konst.}$$