„Problem" dveh teles

se naravno zreducira na eno telo, kot sledi:

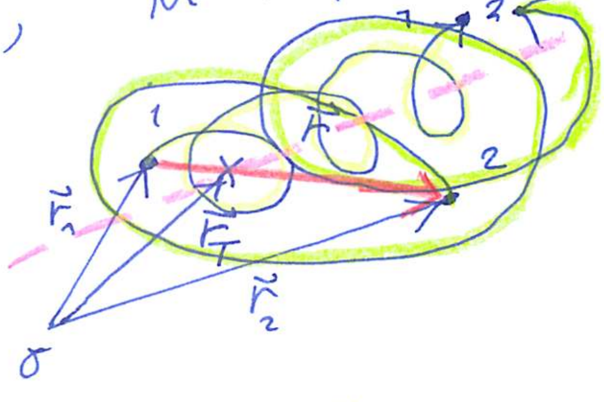
$$T = \frac{1}{2} m_1 |\dot{\vec{r}}_1|^2 + \frac{1}{2} m_2 |\dot{\vec{r}}_2|^2, \quad V = U(\vec{r}, \dot{\vec{r}}),$$

$$\vec{r} = \vec{r}_2 - \vec{r}_1$$ relativna koordinata

6 leg + 6 hitrosti

Težišče,

$$M \vec{r}_T = m_1 \vec{r}_1 + m_2 \vec{r}_2 , \qquad M = m_1 + m_2$$



$$\vec{r}_1 = \vec{r}_T - \frac{m_2}{M} \vec{r}$$
$$\vec{r}_2 = \vec{r}_T + \frac{m_1}{M} \vec{r}$$

$$T = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2 ; \quad m = \frac{m_1 m_2}{M}$$

$$\frac{1}{m} = \frac{1}{m_1} + \frac{1}{m_2}$$

reducirana masa

če m_1 ali $m_2 \to \infty$, je $m = m_2$ ali m_1.

$$L = \frac{1}{2} M |\dot{\vec{r}}_T|^2 + \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}}),$$

- konstante

$\vec{r}_T$ ciklična $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{\vec{r}}_T} = 0 \Rightarrow M \dot{\vec{r}}_T = \text{konst.}$

$$\vec{r}_T(t) = \vec{r}_T(0) + \dot{\vec{r}}_T(0) t$$

začetni pogoji: 3 + 3.

ostanejo 3+3:

$$L = \frac{1}{2} m |\dot{\vec{r}}|^2 - U(\vec{r}, \dot{\vec{r}}).$$

$$E = \frac{1}{2} m |\dot{\vec{r}}|^2 + U = \text{konst.}$$

če je sila centralna, $V(\vec{r}) = V(|\vec{r}|) = V(r)$,

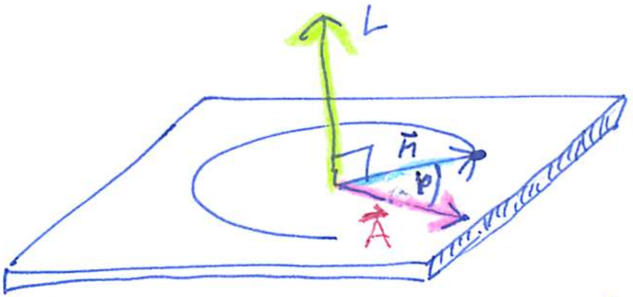
$$\vec{F} = -\nabla V = -\frac{dV(r)}{dr}\frac{\vec{r}}{r},$$

$$\dot{\vec{L}} = \vec{M} = \vec{r} \times \vec{F} = 0, \quad \vec{L} = \text{konst.} \quad (\text{že vemo})$$

Torej lahko definiramo ravnino,

$\vec{L} \cdot \vec{r} = 0$, stalno ravnino (x, y),

$$\vec{L} = L\,\hat{e}_z$$



Vpeljemo že znane polarne koordinate v tej ravnini; r, φ,

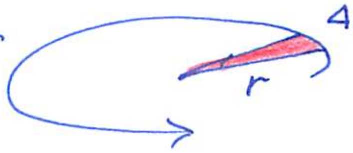
$$L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) - V(r), \quad \varphi \text{ ciklična},$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = l = l_0 = \text{konst.}$$

V času Δt opiše $\vec{r}$ lok oz. ploščino P,

$$\Delta P = \frac{1}{2} r \cdot r\Delta\varphi = \frac{1}{2} r^2 \dot{\varphi}\Delta t \Rightarrow \frac{dP}{dt} = \frac{1}{2} r^2 \dot{\varphi} = \text{konst.}$$

2. Keplerjev zakon



(velja za $\forall\, V(r)$.)

Ali so še kakšne konstante gibanja?

Za primer $V \propto \frac{1}{r}$ se ohranja

Laplace-Runge-Lenzov vektor

neodvisen od teh ga ni odkril (gl. wiki)

simetrija: delec se giblje po 4D hipersferi; QM.

Ponovimo... Torej,

$\vec r_T$ je očitno ciklična koordinata, zato,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\vec r}_T} = 0 \Rightarrow M\dot{\vec r}_T = M\vec v_T = \vec P_T = \text{konst.}$$

gibalna količina.

Torej zadošča, če vzamemo le enodelčni problem z

$$L = \frac{1}{2} m|\dot{\vec r}|^2 - U(\vec r, \dot{\vec r}).$$

Konstante gibanja (razen $\vec P_T$ in celotna L in E).

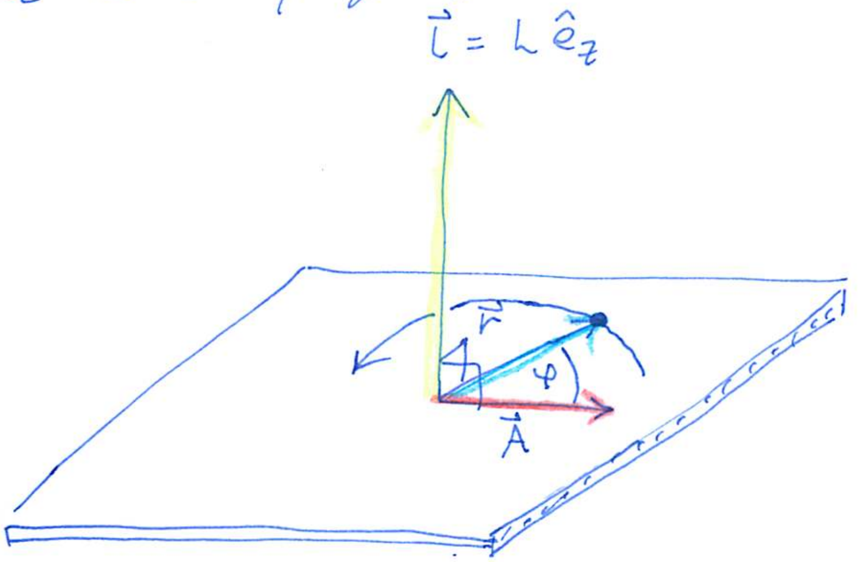
Obravnavamo torej točkast delec v izotropnem danem potencialu $V(|\vec r|)$,

$$\vec F = -\nabla V(r) = -\frac{\partial V}{\partial r}\frac{\vec r}{r},$$

$$\vec M = \vec r \times \vec F = 0 \Rightarrow \dot{\vec L} = \vec M = 0,$$

- $\vec L = \vec r \times \vec p =$ konst.; $\vec r \cdot \vec L = \vec r \cdot (\vec r \times \vec p) = 0$; $\vec p = m\dot{\vec r}$

Delec se torej giblje v ravnini ⊥ $\vec L$.



: Seveda se ohranja tudi celotna energija

Laplace-Runge-Lenzov vektor

naj velja

$$m\ddot{\vec r} = \vec F(\vec r) = f(r)\frac{\vec r}{r} \; ; \quad f(r) = -\frac{\partial V(r)}{\partial r}$$

$$\dot{\vec p} = m\ddot{\vec r}$$

$$\dot{\vec p}\times\vec L = \frac{m f(r)}{r}\left(\vec r\times(\vec r\times\dot{\vec r})\right) =$$

$$= \frac{m f(r)}{r}\left[\vec r\,(\vec r\cdot\dot{\vec r}) - \dot{\vec r}\, r^2\right]$$

kot že večkrat, $\frac{d}{dt} r^2 = \underbrace{2 r\dot r}_{\uparrow} = \frac{d}{dt}\vec r\cdot\vec r = 2\underbrace{\vec r\cdot\dot{\vec r}}_{\uparrow}$

torej

$$\frac{d}{dt}(\vec p\times\vec L) = \dot{\vec p}\times\vec L + \vec p\times\underset{=0}{\dot{\vec L}} =$$

$$= -\frac{m f(r)}{r}\left(\frac{\dot{\vec r}}{r} - \frac{\dot r\,\vec r}{r^2}\right) =$$

$$= -m f(r) r^2 \frac{d}{dt}\left(\frac{\vec r}{r}\right)$$

Če velja (npr. gravitacija): $-f(r)\boxed{r^2} = \underset{\text{konstanta}}{k} \,(= mMG)$,

$\Rightarrow \frac{d}{dt}(\vec p\times\vec L) = \frac{d}{dt}\, mk\frac{\vec r}{r}$ in zato očitno

$$\boxed{\vec A = \vec p\times\vec L - mk\frac{\vec r}{r} = \text{konst.}} \text{ oz. } \dot{\vec A} = 0 .$$

~ masa Zemlje

$k = mMG$ ← gravitacijska konstanta

~ masa Sonca (v limiti $m/M \to 0$).

$\vec A = \vec A_0$ je torej konstanta gibanja pri Keplerjevem problemu, $V \propto \frac{1}{r}$.

Očitno velja

$$\vec{A}\cdot\vec{L} = \underbrace{(\vec{p}\times\vec{L})\cdot\vec{L}}_{=0} - mk\frac{\overbrace{\vec{r}\cdot\vec{L}}^{=0}}{r} \equiv 0 .$$

Torej je $\vec{A}$ v ravnini ⊥ na $\vec{L}$, kot $\vec{r}$.
Konkretna vrednost $\vec{A}$ je odvisna od začetnih pogojev. Imamo torej

$$3(\vec{L}) + 3(\vec{A}) + 1(E) + 3(\vec{P}_T) = 7+3$$

$$= 10 \text{ konstant gibanja}$$

(niso neodvisne).

Konstante gibanja bistveno poenostavijo računanje (pravzaprav – omogočijo).
Tir določimo tako:

$$\vec{A}\cdot\vec{r} = Ar\cos\varphi = \underbrace{\vec{r}\cdot(\vec{p}\times\vec{L})}_{\vec{L}\cdot(\vec{r}\times\vec{p}) = L^2 \equiv l^2 = \text{konst.}} - mkr$$

$$r(A\cos\varphi + mk) = l^2$$

$$\boxed{r = \frac{l^2}{mk(1+\frac{A}{mk}\cos\varphi)} = \frac{r_0}{1+\varepsilon\cos\varphi}}$$

± (glede na def. kota)

$$\varepsilon = \frac{A}{mk}$$

$$r_0 = \frac{l^2}{mk}$$

$\Rightarrow$ stožnica; $A \neq 0$

Če $A=0$, je to prost. pad in je očiten drugačna (trivialno sicer).

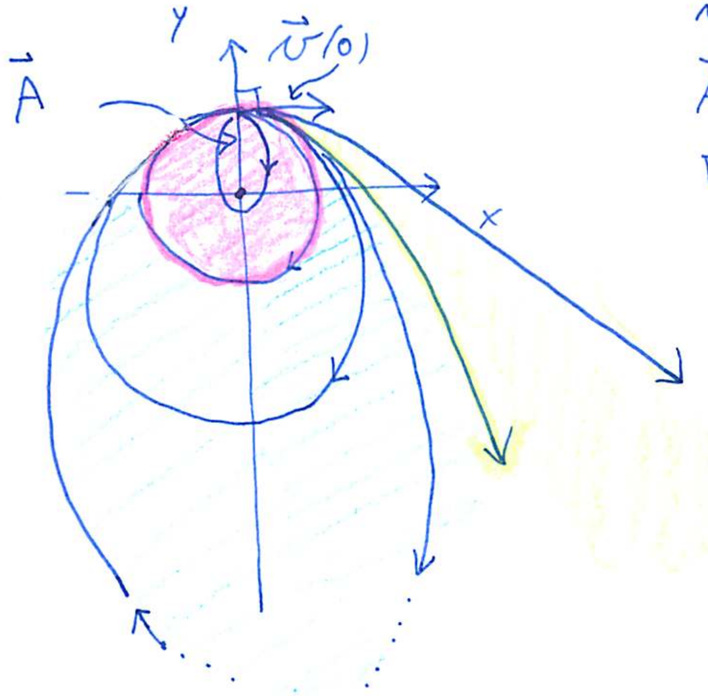
Orbite pri Keplerjevem problemu

Izberemo si začetne pogoje takole:

$\vec{v}(0) = (v, 0, 0)$, $v \neq 0$

$\vec{A} = (0, A, 0)$

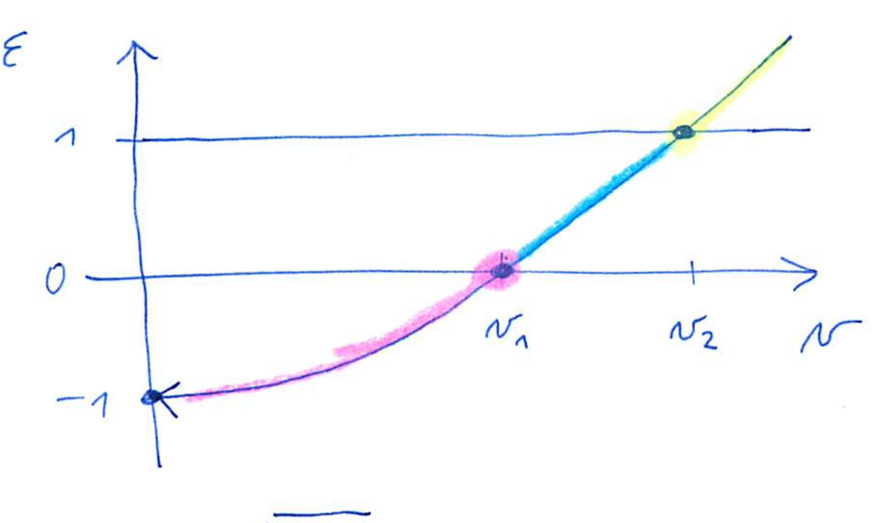
$\vec{r}(0) = (0, R, 0)$



$$\vec{A} = m\vec{v} \times (\vec{r} \times m\vec{v}) - mk\frac{\vec{r}}{r}$$

$$\pm|\vec{A}| = m^2 R v^2 - m^2 MG$$

$$\varepsilon = \frac{A}{mk} = \frac{mRv^2}{mMG} - 1 = \frac{Rv^2}{MG} - 1$$



(a)

$\varepsilon = 0$: $v_1 = \sqrt{\frac{MG}{R}}$ krožnica

Energija,

$$E_1 = \frac{1}{2} m v_1^2 - \frac{mMG}{R} = -\frac{1}{2}\frac{mMG}{R} < 0$$

virialni izrek:

$$\langle T \rangle = \frac{k}{2} \langle V \rangle$$

$$k = -1$$

(b)

$$\varepsilon = 1 : \quad v_2 = \sqrt{2\frac{MG}{R}}$$

$$E_2 = 0$$

hiperbola za večje hitrosti; če $v = v_2$ je parabola („elipsa z enim goriščem → ∞).

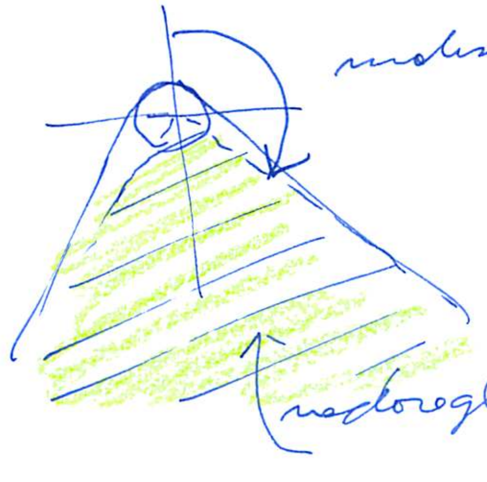
(c) $\varepsilon = 1$ in $v_1 > v_2$,

$E > 0$ ← virialni teorem ne velja za neomejeno gibanje

$$r(\varphi) = \frac{r_0}{1 + \varepsilon\cos\varphi}$$

$$1 \pm \varepsilon\cos\varphi_0 = 0 \Rightarrow \cos\varphi_0 = \frac{1}{\varepsilon} \quad (\text{oz. } \varphi_0 + 2\pi)$$

maksimalni kot φ_0



nedosegljivo ($\cos\varphi > 1$)

opomba: pri $\varepsilon = 1$ se smer vektorja $\vec{A}$ obrne (ε spremeni predznak); vloga gorišč se zamenja.