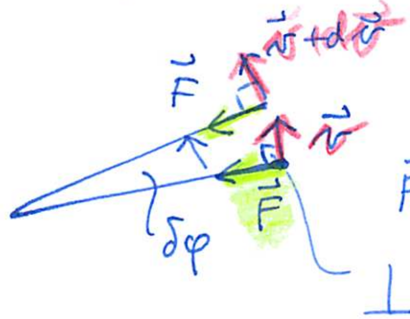
Vrteča osno simetrična vrtavka

Če je vrtavka nesimetrična, ~~se~~ ni rešljivo.
Znamo torej $J_1 = J_2 \neq J_3$.
Kaj vemo od fizike 1. letnika?

a) kroženje

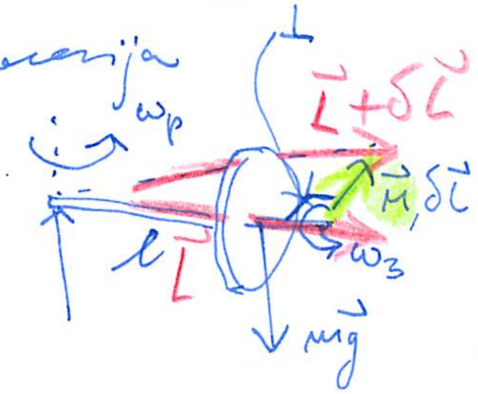


$dv = v\,d\varphi$

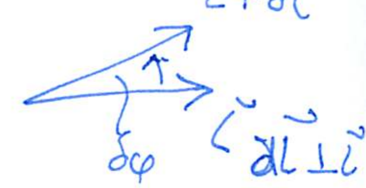
$\vec F \propto d\vec v \perp v$

$$\vec p = m\vec v$$
$$\dot{\vec p} = m\vec a = \vec F$$
$$dp = m\underbrace{v\,d\varphi}_{dv} = F\,dt \Rightarrow \frac{d\varphi}{dt} = \frac{F}{mv} = \frac{F}{p} = \omega$$

b) precesija



$$\vec L = J\vec\omega_3$$
$$\dot{\vec L} = J\dot{\vec\omega}_3$$

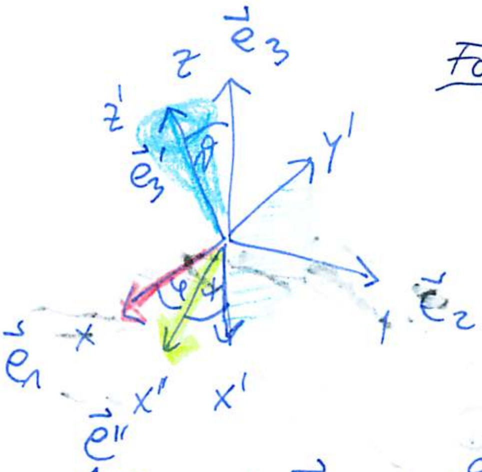


$$dL = J\,d\omega = J\omega\,d\varphi = M\,dt$$
$$\frac{d\varphi}{dt} = \omega_p = \frac{M}{L} = \frac{mgl}{J\omega_3}$$

V obeh primerih je sprememba $\perp$ na količino in velikost konstantna

$$\vec F \parallel \delta\vec v \perp v \quad \text{in} \quad \vec M \parallel \delta\vec\omega_3 \perp \vec\omega_3$$
$$\vec F = \vec F_0 \qquad M = mgl$$

Rezultat: kroženje $\vec p$ oz. $\vec L$, velikost konstantna



Formalno splošno reševanje

$J_1 = J_2$

$$\vec\omega = \dot\varphi\,\vec e_3 + \dot\vartheta\,\vec e_1'' + \dot\psi\,\vec e_3'$$

Zapišimo vse vektorje v sistem ('):

$$\vec e_3 = (\sin\vartheta\sin\psi,\ \sin\vartheta\cos\psi,\ \cos\vartheta)$$

$$\vec e_1'' = (\cos\psi,\ -\sin\psi,\ 0)$$

$$\vec e_3' = (0,\ 0,\ 1)$$

torej

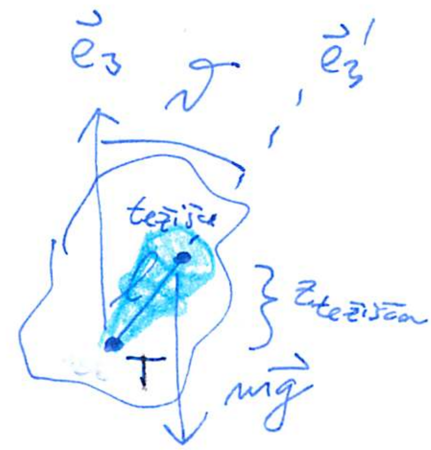
$$\vec\omega = \begin{pmatrix} \sin\vartheta\sin\psi\,\dot\varphi + \cos\psi\,\dot\vartheta \\ \sin\vartheta\cos\psi\,\dot\varphi - \sin\psi\,\dot\vartheta \\ \cos\vartheta\,\dot\varphi + \dot\psi \end{pmatrix}$$

ne pozabimo: v sistem ('), t.j. lastnem.

V lastnem sistemu je torej

$$T = \frac{J_1}{2}(\omega_1^2 + \omega_2^2) + \frac{J_3}{2}\omega_3^2$$

$$V = mgz_T = mgl\cos\vartheta$$



$$L = \frac{1}{2}J_1\Big(\sin^2\vartheta\underbrace{(\sin^2\psi + \cos^2\psi)}_{1}\dot\varphi^2 + \underbrace{(\cos^2\psi + \sin^2\psi)}_{1}\dot\vartheta^2 +$$

$$+ \sin\vartheta\underbrace{(\sin\psi\cos\psi - \sin\psi\cos\psi)}_{0}\dot\varphi\dot\vartheta\Big) + \frac{1}{2}J_3(\dot\psi + \cos\vartheta\,\dot\varphi)^2 -$$

$$- mgl\cos\vartheta$$

Očitno standarden 1-dimenzionalen problem, ki je rešljiv z enim integriranjem. Pravzaprav splošno sicer ne gre, da se pa veliko razumeti:

Velja torej

$$\dot{\vartheta}^2 = \frac{2}{J_1}\left(\tilde{E} - \tilde{V}(\vartheta)\right)$$

$$\frac{d\vartheta}{dt} = \sqrt{\frac{2}{J_1}\left(\tilde{E} - \tilde{V}\right)} \quad \text{in}$$

$$t = \sqrt{\frac{J_1}{2}} \int_{\vartheta(0)}^{\vartheta(t)} \frac{d\vartheta}{\sqrt{\tilde{E} - \tilde{V}(\vartheta)}}\,,$$ kot že prej rečeno.

Preden gremo računat, si oglejmo obračalne točke. Vemo namreč, da mora veljati $\tilde{E} \geq \tilde{V}$, ker $T \geq 0$. V ta namen vpeljemo novo spremenljivko

$u = \cos\vartheta$, $du = -\sin\vartheta\, d\vartheta$, da se znebimo ulomka pod korenom in dobimo polinom v u,

$$t = \int_{u(0)}^{u(t)} \frac{du}{\sqrt{f(u)}}\,,$$

$$f(u) = \underbrace{(1-u^2)}^{\sin^2\vartheta}(\alpha - \beta u) - (b - au)^2\,,$$ [da je fizikalno ≥ 0]

$$\boxed{\alpha = \frac{2\tilde{E}}{J_1}} \quad \text{in} \quad \boxed{\beta = \frac{2mgl}{J_1}}\;; \quad \dot{u}^2 = f(u),$$

ker je $\frac{dt}{du} = \frac{1}{\sqrt{f}} = \frac{1}{|\dot{u}|}$

torej

$$L = \frac{1}{2} J_1 (\sin^2\vartheta\, \dot\varphi^2 + \dot\vartheta^2) + \frac{1}{2} J_3 (\underbrace{\dot\psi + \cos\vartheta\, \dot\varphi}_{\omega_3})^2 - mgl\cos\vartheta .$$

(*)

Očitno sta ψ in φ ciklični koordinati, zato sta ustrezna momenta ohranjena:

I. $$p_\psi = \frac{\partial L}{\partial \dot\psi} = J_3 (\underbrace{\dot\psi + \cos\vartheta\, \dot\varphi}_{\omega_3}) = \boxed{\underset{\text{od prej}}{J_3 \omega_3} \equiv \underset{\text{def. } a}{J_1 a}} = \text{konst.}$$

II. $$p_\varphi = \frac{\partial L}{\partial \dot\varphi} = J_1 \sin^2\vartheta\, \dot\varphi + J_3 \cos\vartheta (\dot\psi + \cos\vartheta\, \dot\varphi) =$$
$$= (J_1 \sin^2\vartheta + J_3 \cos^2\vartheta) \dot\varphi + J_3 \cos\vartheta\, \dot\psi \equiv \underset{\text{def. } b}{\boxed{J_1 b}} = \text{konst.}$$

III. ohranjena je seveda celotna energija,

$E = T + V$ gl. (*) z +

Sedaj iz I. in II. izrazimo $\dot\psi$ in $\dot\varphi$,

$$J_3 \dot\psi = J_1 a - J_3 \cos\vartheta\, \dot\varphi$$

in to vstavimo v II,

$$(J_1 \sin^2\vartheta + J_3 \cos^2\vartheta) \dot\varphi + \cos\vartheta (J_1 a - J_3 \cos\vartheta\, \dot\varphi) = J_1 b$$

0

$$\boxed{\dot\varphi = \frac{b - a\cos\vartheta}{\sin^2\vartheta}} \Rightarrow \varphi(t) \qquad \frac{J_1 a}{J_3} = \dot\psi + u\dot\varphi$$

$$\& \quad \boxed{\dot\psi = \frac{J_1 a}{J_3} - \frac{b - a\cos\vartheta}{\sin^2\vartheta}}$$

In še III.:

$$\text{konst.} = E = \frac{1}{2} J_1 \left(\frac{(b - a\cos\vartheta)^2}{\sin^2\vartheta} + \dot\vartheta^2 \right) + mgl\cos\vartheta + \underbrace{\frac{1}{2} J_3 \omega_3^2}_{= \text{konst}}$$
$$= \tilde{E} + \frac{1}{2} J_3 \omega_3^2$$

torej

$$\boxed{\begin{aligned} \tilde{E} &= \frac{1}{2} J_1 \dot\vartheta^2 + \tilde{V}(\vartheta), \text{ kjer je} \\ \tilde{V}(\vartheta) &= \frac{1}{2} \frac{(b - a\cos\vartheta)^2}{\sin^2\vartheta} + mgl\cos\vartheta . \end{aligned}}$$

1D osnovna enačba vrtavke

$\Rightarrow \vartheta(t)$

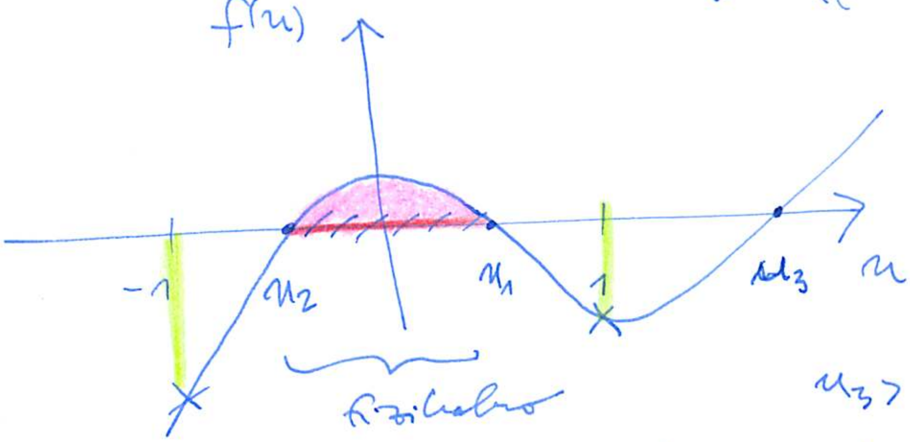
Obračalne točke ($\tilde{E} = \tilde{V}$) so pri $f = 0$. Fizikalno dovoljen režim je $f > 0$. Zanimajo nas različni primeri α, β, a, b. $f(u)$ je polinom 3. stopnje in integral ni analitično rešljiv splošno.

Limite $|u| \to \infty$

$f(u) \to \beta u^3$, torej $\pm$ za $u \pm$

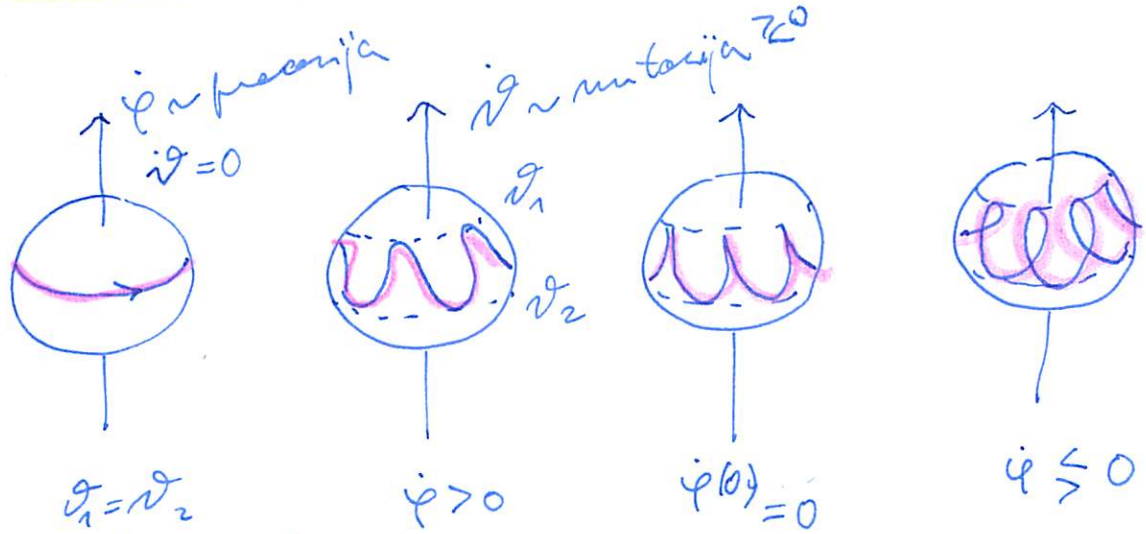
in $f(\pm 1) = -(b \mp a)^2 < 0$

(če $a = b \Rightarrow f(1) = 0$.)



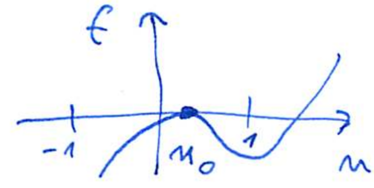
$u_3 > 1 \Rightarrow$ ne $\exists\, \vartheta_3$.

Vrtavka se torej giblje, da je $u_2 \le u \le u_1$, $\boxed{\vartheta_1 \le \vartheta \le \vartheta_2}$.



običajna
precesija
brez nutacije

V splošnem integrala ne moremo izraziti z osnovnimi funkcijami. Posebni primeri ilustrativni:



- Enakomerna precesija

$\dot\vartheta = 0$ in $\dot\varphi = \text{konst.}$, $f(u)$ ima eno samo ničlo,

$$f(u_0) = (1-u_0^2)(\alpha-\beta u_0) - (b-au_0)^2 = 0$$

in

$$\frac{df}{du_0} = 0 = -2u_0(\alpha-\beta u_0) - \beta(1-u_0^2) + 2a(b-au_0)$$

vemo že $\dot\varphi = \frac{b-au}{1-u^2}$, $\dot\varphi^2 = \frac{(b-au)^2}{(1-u^2)^2}$, zato

$u \to u_0$

$$0 = f(u_0) = \cancel{(1-u_0^2)}(\alpha-\beta u_0) - (1-u_0^2)^{\cancel{2}}\dot\varphi^2 = 0 \;; \quad \text{če } u^2 \neq 1$$

$$\alpha - \beta u_0 = \dot\varphi^2(1-u_0^2)$$

$$0 = \frac{df}{du_0} = -2u_0\dot\varphi^2(1-u_0^2) - \beta(1-u_0^2) + 2a(1-u_0^2)\dot\varphi$$

$$(1-u_0^2)\beta = 2a(1-u_0^2)\dot\varphi - 2u_0\dot\varphi^2(1-u_0^2)$$

$$\frac{\beta}{2} = a\dot\varphi - \dot\varphi^2 u_0$$

$$\frac{mgl}{J_1} = \dot\varphi\left(\frac{J_3\omega_3}{J_1} - \dot\varphi\cos\vartheta_0\right)$$

Možni 2 rešitvi; pogoj, da rešitev obstaja, $x_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$-J_1\cos\vartheta_0\,\dot\varphi^2 + J_3\omega_3\dot\varphi - mgl = 0$$

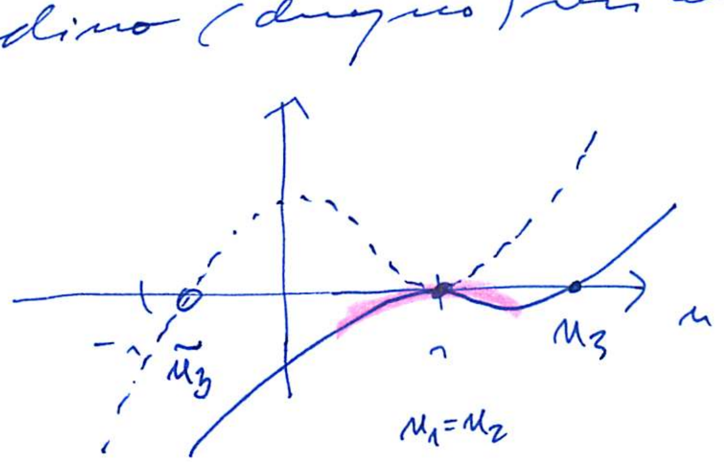
$$\Rightarrow J_3^2\omega_3^2 \geq 4J_1 mgl\cos\vartheta_0$$

$$\boxed{\omega_3 \geq \frac{2}{J_3}\sqrt{J_1 mgl\cos\vartheta_0}\,.}$$

mora se vrteti dovolj hitro!

Speča vrtavka

V bistvu tako, kot prej, samo da imamo edino (dvojno) ničlo pri $u=1$,



Imamo dve možnosti za $u_3 \gtrless 1$. Fizikalna (stabilna) je samo $u_3 > 1$, ker je potem $u_1 = u_2$ edina možna – stabilna.

Torej, $\vartheta = 0$ oz. $u = 1$

$\dot{\vartheta} = 0$ in $f(1) = 0$, in $a = b$, $\alpha = \beta \Rightarrow$

$$f(u) = (1-u^2)(\alpha(1-u)) - a^2(1-u)^2 = \underbrace{(1-u)^2}_{\text{dvojna ničla } u_1=u_2}(\alpha(1+u) - a^2) = 0$$

u_3: $\alpha(1+u_3) = a^2$

$$u_3 = \frac{a^2}{\alpha} - 1 \geq 1 \Rightarrow \frac{a^2}{\alpha} \geq 2 \ ; \ \alpha = \beta .$$

Torej (glej definicije konstant),

$$\left(\frac{J_3\omega_3}{J_1}\right)^2 \frac{J_1}{2mgl} \geq 1 \text{ oz. } \underbrace{\frac{1}{2} J_3 \omega_3^2}_{T} \geq \frac{J_1}{J_3} \underbrace{mgl}_{V}$$

$$\boxed{T \geq \frac{J_1}{J_3} V}$$

(pogoj, da bo $u_3 > 1$, to je, da ni druge rešitve, kot $u_1 = u_2 = 1 \equiv$ speča; ne velja za „cigaro", t.j. $J_3 \to 0$).