... **Spuščena metoda**: iščemo $\vartheta(t)$, $\varphi(t)$ ob ← pogoju

ob $t=0$:

$\dot\varphi(0)=0$ & $\dot\vartheta(0)=0$

1. rešitev ✓

torej

$$0=\dot\varphi=\frac{b-a\cos\vartheta_1}{\sin^2\vartheta} \Rightarrow \boxed{b=a u_1} \quad \mathrm{I}$$

ϑ_1 najvišja točka

$\vartheta(0)=\vartheta_1$

najvišji u

iz energije; $\dot\vartheta=0$, $\dot\varphi=0$

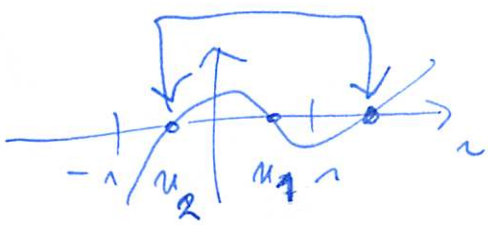
$$\tilde E=\tfrac{1}{2}J_1\underbrace{\dot\vartheta^2}_{=0}+\tilde V(\vartheta) \Rightarrow \tilde E = mgl\cos\vartheta_1$$

$$\tilde E - mgl\,u_1 = 0 \qquad \frac{2}{J_1}$$

$$\frac{2\tilde E}{J_1}-\frac{2mgl}{J_1}u_1=0$$

$$\boxed{\alpha-\beta u_1=0} \quad \mathrm{II}$$

Iščemo še 2. rešitev $u_2=\cos\vartheta_2$ (spodnja meja za u)



$$f(u)=(1-u^2)(\underbrace{\alpha}_{=\boxed{\beta u_1}\ \mathrm{II}}-\beta u)-(\underbrace{b}_{=\boxed{a u_1}\ \mathrm{I}}-au)^2=$$

$$=\beta(1-u^2)(u_1-u)-a^2(u_1-u)^2=$$

$$=(u_1-u)\left(\beta(1-u^2)-a^2(u_1-u)\right)=0 \qquad \left(\text{to je še točno}\right)$$

če $u_1\neq u$: $\beta(1-u_2^2)=a^2(u_1-u_2)$

$$\beta u_2^2 - a^2 u_2 + \underbrace{a^2 u_1}_{ab} - \beta = 0$$

$$u_{2(1,2)} = \frac{a^2 \pm \sqrt{a^4 - 4\beta(ab-\beta)}}{2\beta} \quad \text{itd....}$$

vzamemo rešitev $u_2 < u_1 = \frac{b}{a}$.

Hitra ~~rotacija~~ (spuščana)

Razpon nutacije je $u_2 < u < u_1$; $\Delta u \to 0$.

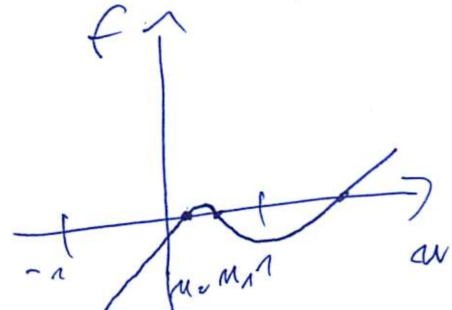
$$x = u_1 - u_2 = \frac{\beta}{a^2}(1 - u_2^2) \quad > 0$$

če je $x \to 0$ vemo od prej, da velja $\frac{a^2}{\beta} \gg 1$

$\frac{a^2}{\beta} > 2$, zelo hitra pa velja

torej

$$x = \underbrace{\frac{\beta}{a^2}}_{\ll 1} \underbrace{(1 - u_2^2)}_{<1} \ll 1 \qquad x \propto \frac{1}{\omega_3^2}$$



Če to velja, velja tudi

$$f(u) = (u_1 - u)\Big(\underbrace{\overset{>0}{\beta(1-u^2)}}_{\text{malo}} - \underbrace{a^2 \overset{>0}{(u_1 - u)}}_{\text{veliko}}\Big) \approx$$

$\dot{u}^2 \sim \ldots u^2$ → nutacije

ah · DN

$$\approx a^2 (u_1 - u)(u - u_2)$$

sicer ne poznamo nutacija.

vaje

$$\int \frac{du}{\sqrt{f(u)}} \Rightarrow \boxed{u = \frac{u_1 + u_2}{2} + \frac{x_1}{2}\cos at}$$

Nutacija in precesija pri konstantna

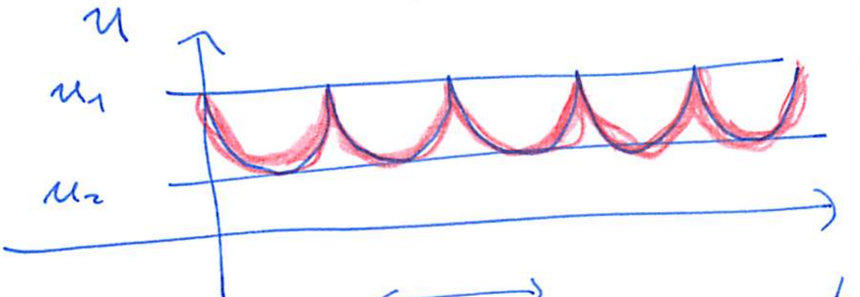
$$\dot{\varphi} = \frac{b - au}{\sin^2\vartheta} = \frac{a(u_1 - u)}{\sin^2\vartheta} = \frac{a}{\sin^2\vartheta}\frac{x_1}{2}(1 - \cos at)$$

Precesija (srednja):

$$\omega_p = \overline{\dot{\varphi}} = \frac{a x_1}{2\sin^2\vartheta} = \frac{\beta}{2a} = \frac{mgl}{J_3\omega_3} = \frac{M}{L_3}$$

kot v osnovni fiziki

cikloida



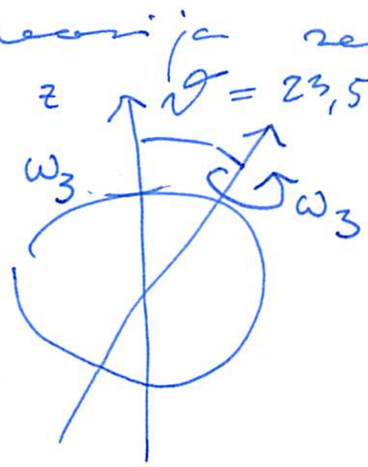
$\} \propto \frac{1}{\omega_3^2} \to 0$

frekvenca $\propto a \to \infty$

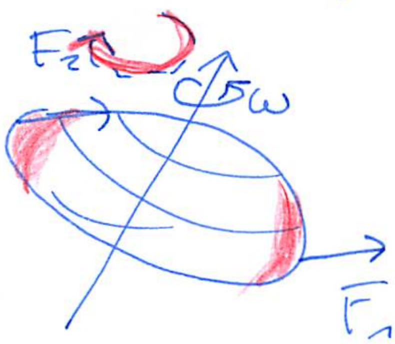
hitrost precesije $\sim \omega_p \propto \frac{1}{\omega_3}$

Primeri vrtavk – Zemlja

• Precesija zemljine osi



Os z je definirana glede na ekliptiko. Zaradi ω_3 Zemlja ni okrogla, ampak geoid (~ elipsoid).



navor Sonca „hoče Zemljo poravnati", zato precesira nazaj.

$F_1 > F_2$ (malo)

Sonce deluje z navorom, rezultat je precesija osi, vrti se nazaj glede na $\vec{\omega}$; „retrogradno".

1 obrat = 25700 let.

•• $\vartheta \sim 22{,}1° \leftrightarrow 24{,}5°$ zaradi drugih planetov, perioda je 41 000 let

∴ drugi planeti spreminjajo ekscentričnost ε. perioda je 105 000 let. Milankovićevi cikli

Stabilnost vrtavke

Naj velja

$$J_1 \neq J_2 \neq J_3 \neq J_1 .$$

in telo se vrti okoli ene od lastnih osi ($\vec{M}=0$); Eulerjeve enačbe so OK:

$$\omega_1 = \Omega \;;\quad \omega_2 = \omega_3 = 0 .$$

Poglejmo majhno odstopanje od tega:

$$\omega_1 = \Omega + \eta_1 \;;\; \omega_2 = \eta_2 \;;\; \omega_3 = \eta_3$$

η_i majhno

Vstavimo v E.e.,

najnižji red:

$$J_1 \dot\eta_1 = 0$$

$$J_2 \dot\eta_2 = \Omega \eta_3 (J_3 - J_1)$$

$$J_3 \dot\eta_3 = \Omega \eta_2 (J_1 - J_2)$$

$$\boxed{\begin{aligned} J_1\dot\omega_1 &= (J_2 - J_3)\,\underbrace{\omega_2\omega_3}_{\eta_2\eta_3 \to 0} \\ J_2\dot\omega_2 &= (J_3 - J_1)\,\underbrace{\omega_3\omega_1}_{\eta_3(\Omega+\eta_1)} \\ J_3\dot\omega_3 &= (J_1 - J_2)\,\underbrace{\omega_1\omega_2}_{(\eta_1+\Omega)\eta_2} \end{aligned}}$$

$$J_2 \ddot\eta_2 = \frac{\Omega^2}{J_3}(J_3 - J_1)(J_1 - J_2)\,\eta_2 = A\,\eta_2$$

$$\ddot\eta_2 - \frac{A}{J_2}\eta_2 = 0 \qquad \sim \frac{\sin}{\cos}\sqrt{\frac{A}{J_2}}\,t$$

očitno: če $A < 0 \Rightarrow$ nihanje η_2

$A > 0 \Rightarrow \eta_2 \sim e^{\pm\sqrt{\frac{A}{J_2}}\,t}$.

$\Rightarrow$ stabilno/nestabilno (če se vrti okoli največje/najmanjše vrednosti). nestabilno: $J_2 < J_1 < J_3$ ali $J_3 < J_1 < J_2$ }.

Hamiltonov formalizem

namesto hitrosti $\dot q_i$ delamo z momenti p_i,

$$p_i = \frac{\partial L}{\partial \dot q_i}$$

$$\frac{dp_i}{dt} = \dot p_i = \frac{\partial L}{\partial q_i} \qquad (L\text{-enačbe})$$

Spomnimo se energije. Enako definiramo novo funkcijo, izraženo s p_i, če $T = \sum_{kl} m_{kl} \dot q_k \dot q_l$

$$H \stackrel{\text{def.}}{=} \sum_i p_i \dot q_i - L \left(\text{oz.} = \sum_i \frac{\partial L}{\partial \dot q_i} \dot q_i - L\right) = T + V \quad (= E)$$

Velja

$$dH = \sum_i (\dot q_i\, dp_i + p_i\, d\dot q_i) - \sum_i \left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot q_i} d\dot q_i\right) - \frac{\partial L}{\partial t} dt$$

$$= \sum_i (\dot q_i\, dp_i - \dot p_i\, dq_i) - \frac{\partial L}{\partial t} dt$$

oz.

$$dH = \sum_i \left(\frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\right) + \frac{\partial H}{\partial t} dt$$

$$\Rightarrow \boxed{\dot q_i = \frac{\partial H}{\partial p_i}, \quad \dot p_i = -\frac{\partial H}{\partial q_i}, \quad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}}$$

velja tudi:

$$\frac{dH}{dt} = \underbrace{\sum_i \left(\frac{\partial H}{\partial q_i} \underbrace{\dot q_i}_{=\frac{\partial H}{\partial p_i}} + \frac{\partial H}{\partial p_i} \underbrace{\dot p_i}_{=-\frac{\partial H}{\partial q_i}}\right)}_{=0} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}.$$

Vemo že od prej: če q_i ciklična $\Rightarrow$ p_i = konst.

<u>Poissonovi oklepaji</u> (pišemo zavite!) ogleti v QM

Vzemimo poljubno funkcijo koordinat in momentov in časa,

$$f(q, p, t) = f(q(t), p(t), t).$$

Poglejmo časovni odvod,

$$\frac{df}{dt} = \sum_i \left(\frac{\partial f}{\partial q_i} \dot{q}_i + \frac{\partial f}{\partial p_i} \dot{p}_i \right) + \frac{\partial f}{\partial t} =$$

$$= \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial H}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial H}{\partial q_i} \right) + \frac{\partial f}{\partial t} =$$

$$\stackrel{\text{def.}}{=} \{f, H\} + \frac{\partial f}{\partial t}.$$

<u>Definirali</u> smo P. oklepaj,

$$\boxed{\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)}.$$

Konstanta gibanja,

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t} \nearrow = \{f, H\}}$$

če $f \neq f(t)$

če $\frac{df}{dt} = 0 \Rightarrow \{f, H\} = 0.$

npr. $\{H, H\} = 0$, če ni f. t.

Lastnosti P. oklepajev

1. linearnost,

$$\{f, \lambda g + \mu h\} = \lambda\{f, g\} + \mu\{f, h\}$$

2. antisimetričnost,

$$\{f, g\} = -\{g, f\}$$

3. produkt,

$$\{f, g \cdot h\} = \sum_i \left(\frac{\partial f}{\partial q_i} \frac{\partial gh}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial gh}{\partial q_i} \right) =$$

$$= \sum_i \left[\frac{\partial f}{\partial q_i} \left(\frac{\partial g}{\partial p_i} h + g \frac{\partial h}{\partial p_i} \right) - \frac{\partial f}{\partial p_i} \left(\frac{\partial g}{\partial q_i} h + g \frac{\partial h}{\partial q_i} \right) \right] =$$

$$= \{f, g\} h + g \{f, h\}.$$

$$\{f, gh\} = g\{f, h\} + \{f, g\} h.$$

4. Jacobijeva zveza (ciklična),

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Primer: Poissonovih oklepajev

(a) $$\{q_i, q_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial q_j}{\partial p_l}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_l}}_{0} \underbrace{\frac{\partial q_j}{\partial q_l}}_{\delta_{jl}} \Big) = 0$$

enako →
$\{q_i, q_j\} = 0$ in $\{p_i, p_j\} = 0$

vendar,
$$\{q_i, p_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial p_j}{\partial p_l}}_{\delta_{jl}} - \underbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}_{0} \Big) = \delta_{il}\delta_{jl} = \delta_{ij}$$

$$\boxed{\{q_i, p_j\} = \delta_{ij}.}$$ Osnovna zveza!

„kanonični spremenljivki"
koordinati

(b) vrtilna količina

$$l_x = y p_z - z p_y$$

$$l_y = z p_x - x p_z$$

$$l_z = x p_y - y p_x$$

Računajmo kar z golo silo:

$$\{l_x, l_y\} = \frac{\partial l_x}{\partial x}\frac{\partial l_y}{p_x} + \frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y} + \frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} -$$

$$- \frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x} - \frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_y}{\partial z} =$$

$$= 0 + p_z 0 + (-p_y)(-x) -$$

$$- 0 - p_z 0 - (+y) p_x =$$

$$= x p_y - y p_x = l_z$$

$$\boxed{\{l_i, l_j\} = \varepsilon_{ijk} l_k} \qquad i,j = x, y, z$$

ker $i = j \rightarrow \varepsilon_{iik} = 0$

$i \leftrightarrow j \rightarrow \varepsilon_{ijk} = -\varepsilon_{jik}$ itd.

Zakaj bi to bilo dobro?

- elegantnost izpeljav
- prehod in QM; podobna (ne enaka) struktura.