Primeri: Poissonovih oklepajev

(a)
$$\{q_i, q_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial q_j}{\partial p_l}}_{0} - \underbrace{\frac{\partial q_i}{\partial p_l}}_{0} \underbrace{\frac{\partial q_j}{\partial q_l}}_{\delta_{jl}} \Big) =$$
$$= 0$$

enako →
$\{q_i, q_j\} = 0$ in $\{p_i, p_j\} = 0$

vendar,
$$\{q_i, p_j\} = \sum_l \Big(\underbrace{\frac{\partial q_i}{\partial q_l}}_{\delta_{il}} \underbrace{\frac{\partial p_j}{\partial p_l}}_{\delta_{jl}} - \underbrace{\frac{\partial q_i}{\partial p_l} \frac{\partial p_j}{\partial q_l}}_{0} \Big) = \delta_{il}\delta_{jl} = \delta_{ij}$$

$$\boxed{\{q_i, p_j\} = \delta_{ij}}$$ Osnovna zveza!

„kanonični spremenljivki" koordinati

- $\dot{q}_i = \{q_i, H\}$
- $\dot{p}_i = \{p_i, H\}$

(b) vrtilna količina

$$l_x = y p_z - z p_y$$

$$l_y = z p_x - x p_z$$

$$l_z = x p_y - y p_x$$

Računajmo kar zgornjo zvezo:

$$\{l_x, l_y\} = \frac{\partial l_x}{\partial x}\frac{\partial l_y}{p_x} + \frac{\partial l_x}{\partial y}\frac{\partial l_y}{\partial p_y} + \frac{\partial l_x}{\partial z}\frac{\partial l_y}{\partial p_z} -$$

$$- \frac{\partial l_x}{\partial p_x}\frac{\partial l_y}{\partial x} - \frac{\partial l_x}{\partial p_y}\frac{\partial l_y}{\partial y} - \frac{\partial l_x}{\partial p_z}\frac{\partial l_y}{\partial z} =$$

$$= 0 + p_z 0 + (-p_y)(-x) -$$

$$- 0 - p_z 0 - (+y) p_x =$$

$$= x p_y - y p_x = l_z$$

$$\boxed{\{l_i, l_j\} = \varepsilon_{ijk} l_k}$$

$i,j = x, y, z$

za $i = j \to \varepsilon_{iik} = 0$

$i \leftrightarrow j \to \varepsilon_{ijk} = -\varepsilon_{jik}$ itd.

Zakaj bi to bilo dobro?

- elegantnost izpeljav
- prehod in QM; podobna (ne enaka) struktura.

ali na npr.,

$$\{l^2, l_x\} = \{l_x^2 + l_y^2 + l_z^2, l_x\} =$$

$$= 0 + \{l_y^2, l_x\} + \{l_z^2, l_x\} =$$

$$= 2 l_y \underbrace{\{l_y, l_x\}}_{-l_z} + 2 l_z \underbrace{\{l_z, l_x\}}_{l_y} = 0.$$

Torej

$$\boxed{\{l^2, l_i\} = 0.}$$

c) Laplace – Runge Lenz

$$\vec{A} = \vec{p} \times \vec{L} - km\frac{\vec{r}}{r}$$

direkten račun pokaže

$$\{l_i, A_j\} = \varepsilon_{ijk} A_k .$$

Vemo že od prej, da če

$$H = \frac{p^2}{2m} - \frac{km}{r}$$

$$\Rightarrow \{\vec{A}, H\} = 0 , \quad \{\vec{l}, H\} = 0 ,$$

$$\{l^2, H\} = 0$$

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d) $\{l_i, q_j\} = \varepsilon_{ijk} q_k$

$\{l_i, p_j\} = \varepsilon_{ijk} p_k$

e) $\{\vec{p}, \vec{n}\cdot\vec{l}\} = \vec{n} \times \vec{p}$

Nabit delec v magnetnem polju: Lagrangeev in Hamiltonov pristop

Že od prej vemo, da je potencial lahko odvisen od hitrosti in je sila podana z

$$F_i = -\frac{\partial U(q_i,\dot q_i)}{\partial q_i} + \frac{d}{dt}\frac{\partial U}{\partial \dot q_i} \quad \text{(posplošena sila)} \quad (*)$$

Ker je sila na nabit delec v magnetnem polju odvisna od hitrosti, bomo imeli tak primer. Poiščimo $U(\vec r, \dot{\vec r})$!

Velja Lorentzova sila

$$\boxed{\vec F = e(\vec E + \vec v \times \vec B)}.$$

Običajno se nadaljuje z vpeljavo skalarnega in vektorskega potenciala, $(\phi, \vec A)$,

$$\vec B = \nabla \times \vec A \quad \text{in} \quad \vec E = -\nabla\phi - \frac{\partial \vec A}{\partial t},$$

kar očitno zadošča Maxwellovim enačbam

$$\nabla \cdot \vec B = 0 \quad \text{in}$$

$$\nabla \times \vec E = -\underbrace{\nabla \times \nabla\phi}_{0} - \nabla \times \frac{\partial \vec A}{\partial t} = -\frac{\partial}{\partial t}\nabla \times \vec A = -\frac{\partial \vec B}{\partial t}, \checkmark$$

Silo torej lahko izrazimo takole

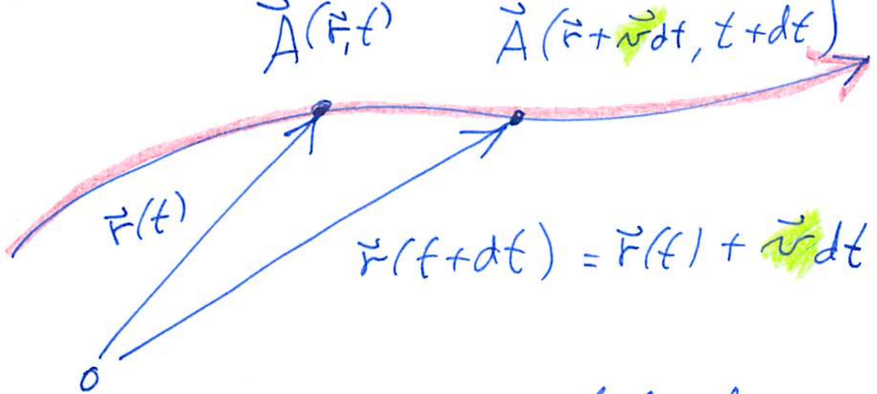
$$\boxed{\vec F = e\left[-\nabla\phi - \frac{\partial \vec A}{\partial t} + \vec v \times (\nabla \times \vec A)\right]},$$

kar bomo sedaj predelali v obliko $(*)$;

(člen $\frac{d}{dt}\frac{\partial U}{\partial \dot q_i}$ je časovni odvod gradienta glede na hitrost)

$$\frac{d}{dt}\nabla_{\vec v} U; \quad \boxed{\vec F = -\frac{\partial}{\partial \vec r}U + \frac{d}{dt}\frac{\partial U}{\partial \dot{\vec r}} \quad (*)}$$

Spomnimo se „substancialnega" odvoda.
Imamo delec, ki se giblje po trajektoriji $\vec{r}(t)$,



$\vec{r}(t+dt) = \vec{r}(t) + \vec{v}dt \quad ; \quad dt \to 0.$

Delec se giblje v vektorskem polju $\vec{A}(\vec{r},t)$ in v časovnem intervalu dt se na mestu delca polje spremeni za $d\vec{A}$,

$$dA_x = A_x(\vec{r}+\vec{v}dt, t+dt) - A_x(\vec{r},t) =$$

$$= \frac{\partial A_x}{\partial x}\underbrace{dx}_{=v_x dt} + \frac{\partial A_x}{\partial y}\underbrace{dy}_{=v_y dt} + \frac{\partial A_x}{\partial z}\underbrace{dz}_{=v_z dt} + \frac{\partial A_x}{\partial t}dt =$$

$$= \left[\left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x + \frac{\partial A_x}{\partial t}\right]dt =$$

$$= \left(\vec{v}\cdot\nabla A_x + \frac{\partial A_x}{\partial t}\right)dt$$

Enako velja za y in z, torej

$$\boxed{\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + \underbrace{(\vec{v}\cdot\nabla)\vec{A}}}$$

vektor: $\begin{pmatrix} \vec{v}\cdot\nabla A_x \\ \vec{v}\cdot\nabla A_y \\ \vec{v}\cdot\nabla A_z \end{pmatrix}$ ~ „odvod v smeri $\frac{\vec{v}}{v}$" $\cdot v$

Velja tudi zveza

$$\vec{v}\times(\nabla\times\vec{A}) = \nabla(\vec{v}\cdot\vec{A}) - (\vec{v}\cdot\nabla)\vec{A},$$

kjer je $\vec{v}$ hitrost delca. Dokažemo z grobo silo po komponentah (ali verjamemo...):

Leva stran (komponenta x):

$$v_y(\nabla\times\vec{A})_z - v_z(\nabla\times\vec{A})_y =$$

$$= v_y\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - v_z\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)$$

Desna stran:

$$\frac{\partial}{\partial x}(v_x A_x + v_y A_y + v_z A_z) - \left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)A_x =$$

$$= \underbrace{v_x\frac{\partial A_x}{\partial x} - v_x\frac{\partial A_x}{\partial x}}_{=0} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} - v_y\frac{\partial A_x}{\partial y} - v_z\frac{\partial A_x}{\partial z}.$$

✓ res enako

torej velja,

$$\boxed{\frac{d\vec{A}}{dt} = \frac{\partial\vec{A}}{\partial t} + \nabla(\vec{v}\cdot\vec{A}) - \vec{v}\times(\nabla\times\vec{A})}$$

oziroma

$$\vec{F} = e\left[-\nabla_{\vec{r}}(\phi - \vec{v}\cdot\vec{A}) - \frac{d\vec{A}}{dt}\right] =$$

$$= e\left[-\nabla_{\vec{r}}(\phi - \vec{v}\cdot A) + \frac{d}{dt}\left(-\nabla_{\vec{v}}(\vec{v}\cdot\vec{A})\right)\right] =$$

$$= -\nabla_{\vec{r}}U + \frac{d}{dt}\nabla_{\vec{v}}U,$$

če

$$\boxed{U = e\phi - e\vec{v}\cdot\vec{A}}.$$

Lagrangeova funkcija je torej

$$L = \frac{1}{2} m v^2 - e\phi + e\vec{v}\cdot\vec{A}$$ in

ustrezne enačbe gibanja so izraz za Lorentzovo silo in posplošeni moment je

$$\vec{p} = \nabla_{\vec{v}} L = m\vec{v} + e\vec{A}.$$!

Hamiltonova funkcija je definirana

$$H = \vec{p}\cdot\vec{v} - L =$$

$$= m v^2 + e\vec{v}\cdot\vec{A} - \frac{1}{2} m v^2 + e\phi - e\vec{v}\cdot\vec{A} =$$

$$= \frac{1}{2} m v^2 + e\phi = \leftarrow$$ še ni izražena z impulzi

$$= \frac{|\vec{p} - e\vec{A}|^2}{2m} + e\phi.$$

Komentar: Lorentzova sila je $\perp$ na $\vec{v}$, $e\vec{v}\times\vec{B}$, zato je $P = \vec{F}\cdot\vec{v} = 0$, torej polje ne deluje z močjo (ne opravlja mehanskega dela). Torej ne spremeni energije in zato ne nastopi v izrazu

$$\frac{1}{2} m v^2 + e\phi.$$

Vpliva pa, seveda, na obliko tira.

Izpeljava Hamiltonovih enačb iz „Hamiltonovega" principa (minimalna akcija)

Delamo kot pri L, samo da imamo kot spremenljivke q_i in p_i (neodvisne),

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i, t)\,dt \, .$$

L. enačbe dobimo pri pogoju $\delta q_i(t_1) = \delta q_i(t_2) = 0$ in $\delta S = 0$.

Zapišimo podobne to z q_i, p_i,

$$S = \int_{t_1}^{t_2} \Big(\sum_i p_i \dot{q}_i - H\Big)\,dt \quad ; \quad \dot{q}_i(q_i, p_i) = \dot{q}_i \, ,$$

$$\delta S = \int_{t_1}^{t_2} \sum_i \Big(\delta p_i \dot{q}_i + p_i \delta\dot{q}_i - \frac{\partial H}{\partial q_i}\delta q_i - \frac{\partial H}{\partial p_i}\delta p_i\Big)\,dt =$$

$$= \int_{t_1}^{t_2} \sum_i \Big[\Big(\dot{q}_i - \frac{\partial H}{\partial p_i}\Big)\delta p_i + \Big(-\dot{p}_i - \frac{\partial H}{\partial q_i}\Big)\delta q_i\Big]\,dt + \sum_i p_i \delta q_i \Big|_{t_1}^{t_2}$$

per partes

za $\forall\, \delta p_i, \delta q_i$ je $\delta S = 0$

$$\Rightarrow \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \, .$$

Uporabili smo $\delta q_i \Big|_{t_1, t_2} = 0$, za δp_i ni pogoja

! (q_i in p_i ~~nista~~ povsem simetrična).

Kanonične transformacije

Recimo, da zamenjamo spremenljivke,

$$q_i \to Q_i(\underline{q}, \underline{p}, t) \text{ in}$$

$$p_i \to P_i(\underline{q}, \underline{p}, t).$$

Transformacija med koordinatami je **kanonična** (def.), če ohranja Poissonove oklepaje,

$$\{f, g\}_{QP} = \{f, g\}_{qp}$$

$$f(\underline{q}(\underline{Q},\underline{P}), \underline{p}(\underline{Q},\underline{P}), t) \equiv \tilde{f}(\underline{Q}, \underline{P}, t) \qquad f(\underline{q}, \underline{p}, t)$$

Če je tako, potem so H. enačbe invariantne na transformacijo v smislu

$$\boxed{\dot{Q}_i = \{Q_i, H\}_{qp} + \frac{\partial Q_i}{\partial t}} = \{Q_i, H\}_{QP} + \frac{\partial Q_i}{\partial t} = \boxed{\frac{\partial H}{\partial P_i}} + \frac{\partial Q_i}{\partial t}$$

$$\boxed{\dot{P}_i = \{P_i, H\}_{qp} + \frac{\partial P_i}{\partial t} = \ldots = -\frac{\partial H}{\partial Q_i}} + \frac{\partial P_i}{\partial t}$$

Če $\frac{\partial Q_i}{\partial t} = 0$ in $\frac{\partial P_i}{\partial t} = 0$, so enačbe enake. 0?

Kako se poišče nove koordinate?

Nekateri def. kanoničnost s pogojem $\{Q_i, P_j\} = \delta_{ij}$

Recimo, da imamo zvezo

$$Q_i = Q_i(\underline{q}, t) \quad \text{in} \quad q_i = q_i(\underline{Q}, t)$$

$$\tilde{L}(\underline{Q}_i, \dot{\underline{Q}}_i, t) = L(\underline{q}(\underline{Q}, t), \dot{\underline{q}}(Q, \dot{Q}, t), t).$$

Kako najdemo P_i?

Vemo, da lahko L-u prištejemo popolni odvod,

$$\tilde{L} = L - \frac{dF}{dt}.$$

Želimo $(\underline{Q}, \underline{P})$, $K(\underline{Q}, \underline{P}, t)$ in

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} \quad \text{in} \quad \dot{P}_i = -\frac{\partial K}{\partial Q_i},$$

torej

$$\sum_i \dot{Q}_i P_i - K = \sum_i \dot{q}_i p_i - H - \frac{dF}{dt}.$$

- Izberimo posebno funkcijo, $F = F_1(q, Q, t)$ in

$$\frac{dF_1}{dt} = \sum_i \left(\frac{\partial F_1}{\partial q_i} \dot{q}_i + \frac{\partial F_1}{\partial Q_i} \dot{Q}_i \right) + \frac{\partial F_1}{\partial t}$$

in vstavimo v gornjo enačbo. Dobimo

zveze

$$\boxed{P_i = -\frac{\partial F_1}{\partial Q_i}}, \quad \boxed{p_i = +\frac{\partial F_1}{\partial q_i}}, \quad \boxed{K = H + \frac{\partial F_1}{\partial t}}.$$

Tako smo „našli" P_i. Opomba: p_i so izraženi s q_i in Q_i.

v informacijo:

V splošnem imamo 4 možnosti za F:

1) $F = F_1(q, Q, t)$: $p_i = \frac{\partial F_1}{\partial q_i}$, $P_i = -\frac{\partial F_1}{\partial Q_i}$, $K = H + \frac{\partial F_1}{\partial t}$

2) $F_2(q, P, t) = F + QP$: $p_i = \frac{\partial F_2}{\partial q_i}$, $Q_i = \frac{\partial F_2}{\partial P_i}$, $K = H + \frac{\partial F_2}{\partial t}$

3) $F_3(p, Q, t) = F - qp$: $p_i = -\frac{\partial F_3}{\partial p_i}$, $P_i = -\frac{\partial F_3}{\partial Q_i}$, $K = H + \frac{\partial F_3}{\partial t}$

4) $F_4(p, P, t) = F_2 - qp$: $q_i = -\frac{\partial F_4}{\partial p_i}$, $Q_i = \frac{\partial F_4}{\partial P_i}$, $K = H + \frac{\partial F_4}{\partial t}$

Legendrove transformacije