Majhna nihanja

$$L = T - V = \frac{1}{2}\sum_{ij} m_{ij}(\underline{q})\,\dot{q}_i\dot{q}_j - V(\underline{q})$$ in

$m_{ij} = m_{ji}$.

Ohranja se energija

$$E = \sum_j \frac{\partial L}{\partial \dot{q}_j}\dot{q}_j - L = T + V = H.$$

Najprej poiščemo stabilne lege (če so) in nihanja. Naj bo

$$q_i(t) = q_i^\circ \quad \text{in} \quad \dot{q}_i(t) = 0 \quad \text{stabilna lega}$$

$$\left.\frac{\partial L}{\partial q_i}\right|_{\underline{q}^\circ} = -\left.\frac{\partial V}{\partial q_i}\right|_{\underline{q}^\circ} = 0 \quad ; \quad \underline{q}^\circ = (q_1^\circ, q_2^\circ, \dots, q_n^\circ).$$

(sila na malo telo = 0)

Absolutno stabilno stanje, če

$$V(\underline{q}) > V(\underline{q}^\circ) \quad \text{za} \ \forall \underline{q}.$$

Zadošča lokalna stabilnost, to je, če je $\underline{q}$ v okolici $\underline{q}^\circ$.

Naj bo $V(\underline{q})$ analitična funkcija koordinat,

$$V(\underline{q}) = V(\underline{q}^0) + \underbrace{\sum_i \frac{\partial V}{\partial q_i}\bigg|_{\underline{q}^0} \eta_i}_{0 \text{ (stabilnost)}} + \frac{1}{2} \sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\underline{q}^0} \eta_i \eta_j + \ldots$$

in

$$\eta_i = q_i - q_i^0 \quad \text{za } i = 1, \ldots n.$$

Torej $\frac{1}{2} \sum_{ij} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\underline{q}^0} \eta_i \eta_j > 0$ za $\forall\, \eta_i, \eta_j$

- Matrika $\boxed{V_{ij} \stackrel{\text{def.}}{=} \frac{\partial^2 V}{\partial q_i \partial q_j}\bigg|_{\underline{q}^0}}$ je torej pozitivno definitna

- V $\tilde{L}$ je najnižji red v odvodih še

$$w_{ij}(\underline{q}) = w_{ij}(\underline{q}^0) + \underbrace{\sum_l \frac{\partial w_{ij}}{\partial q_l}\bigg|_{\underline{q}^0} \eta_l + \ldots}_{\text{višji redi}}$$

Torej je vodilni red

$$\boxed{T_{ij} = w_{ij}(\underline{q}^0)} \quad \text{in}$$

$$\tilde{L} \stackrel{\text{def.}}{=} \frac{1}{2} \sum_{ij} T_{ij} \dot{\eta}_i \dot{\eta}_j - V_{ij} \eta_i \eta_j \; ; \qquad L = T - V, \quad \tilde{L} = T - (V - V_0),$$

in $\sum_j T_{ij} \ddot{\eta}_j + \sum_j V_{ij} \eta_j = 0$ za $\forall_i$, kar sledi iz

$$\frac{d}{dt}\frac{\partial \tilde{L}}{\partial \dot{\eta}_i} - \frac{\partial \tilde{L}}{\partial \eta_i} = 0 \quad , \quad \text{za } i = 1, \dots n .$$

Definiramo stolpce,

$$\underline{\eta} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_n \end{pmatrix} \quad \text{in} \quad \underline{\eta}^T = (\eta_1, \eta_2, \dots \eta_n)$$

in

$$T_{ij} \rightarrow \quad \underline{\underline{T}} = \begin{pmatrix} T_{11} & T_{12} & \dots & T_{1n} \\ \vdots & & & \\ T_{n1} & \dots & & T_{nn} \end{pmatrix}$$

in zato

$$\tilde{L} = \frac{1}{2}\left(\dot{\underline{\eta}}^T \underline{\underline{T}}\, \eta - \underline{\eta}^T \underline{\underline{V}}\, \eta \right),$$

$$\text{in} \quad \underline{\underline{V}} = \begin{pmatrix} V_{11} & V_{12} & \dots & V_{1n} \\ V_{21} & \dots & & \\ \vdots & & & \\ V_{n1} & \dots & & V_{nn} \end{pmatrix}.$$

---

opomba: k-ti vektor: $\underline{a}_k = \begin{pmatrix} (a_k)_1 \\ (a_k)_2 \\ \vdots \\ (a_k)_n \end{pmatrix}$ $\quad$ $\underline{a}_\ell^T = \left((a_\ell^T)_1, (a_\ell^T)_2, (a_\ell^T)_3, \dots (a_\ell^T)_n \right)$

$$\left(\underline{\underline{A}}\, \underline{a}_k\right)_i = \sum_j A_{ij} (\underline{a}_k)_j \quad \text{in} \quad \left(\underline{a}_\ell^T \underline{\underline{A}}\right)_i = \sum_j (\underline{a}_\ell^T)_j A_{ji}$$

i-ti člen stolpca (element)

i-ti člen v vrstici

Lastna nihanja

$$\eta_i(t) = \alpha\, a_i\, e^{i\omega t}$$

$$\sum_j V_{ij} a_j - \omega^2 \sum_j T_{ij} a_j = 0$$

oz $\underline{\underline{V}}\,\underline{a} = \omega^2 \underline{\underline{T}}\,\underline{a} = \lambda \underline{\underline{T}}\,\underline{a}$

Posplošena diagonalizacija; simetričnost in pozitivno definitnost $\omega^2 \geq 0$.

modi ν $\lambda = \omega^2 \in \mathbb{R}$

Poseben primer: $\underline{\underline{T}} = T\,\underline{\underline{I}}$ (npr. enake mase)

$$(\underline{\underline{V}} - \lambda \underline{\underline{T}})\underline{a} = 0 \rightarrow (\underline{\underline{V}} - \underbrace{\lambda T}_{\tilde{\lambda}} \underline{\underline{I}})\underline{a} = 0$$

za $\forall k$: $(\underline{\underline{V}} - \tilde{\lambda}_k)\,\underline{a}_k = 0 = \underline{\underline{V}}\,\underline{a}_k - \tilde{\lambda}_k \underline{a}_k$ ← stolpec

in tudi $\underline{a}_l^T(\underline{\underline{V}} - \tilde{\lambda}_l) = 0$

vrstica →

$$\left.\begin{array}{l} \underline{a}_l^T \underline{\underline{V}}\,\underline{a}_k = \tilde{\lambda}_k\, \underline{a}_l^T \underline{a}_k \\ \underline{a}_l^T \underline{\underline{V}}\,\underline{a}_k = \tilde{\lambda}_l\, \underline{a}_l^T \underline{a}_k \end{array}\right\} -$$

$$0 = (\tilde{\lambda}_l - \tilde{\lambda}_k)\, \underline{a}_l^T \underline{a}_k$$

$$\Rightarrow \text{če } \tilde{\lambda}_l \neq \tilde{\lambda}_k \Rightarrow \underline{a}_l^T \underline{a}_k = 0 \quad (\delta_{lk})$$

Lastne vektorje zložimo v matriko,

$$\underline{\underline{A}} = [\underline{a}_1, \underline{a}_2, \ldots \underline{a}_n] = \begin{bmatrix} a_{11} & \ldots & a_{n1} \\ a_{12} & & a_{n2} \\ \vdots & & \vdots \\ a_{1n} & & a_{nn} \end{bmatrix}$$

$\underline{\underline{A}}^T$ je transponirana

(*) $$\underline{\underline{A}}^T \underline{\underline{V}} \, \underline{\underline{A}} - \underline{\underline{A}}^T \underline{\underline{A}} \, \underline{\underline{\tilde{\Lambda}}} = 0 \quad ; \quad \underline{\underline{\tilde{\Lambda}}} = \begin{pmatrix} \tilde{\lambda}_1 & & & \\ & \tilde{\lambda}_2 & & \\ & & \ddots & \\ & & & \tilde{\lambda}_n \end{pmatrix}$$

Ker so vektorji $\underline{a}_i$ ortonormirani, je matrika ortogonalna,

$$\underline{\underline{A}}^T \underline{\underline{A}} = \underline{\underline{A}} \, \underline{\underline{A}}^T = \underline{\underline{I}} \Rightarrow \underline{\underline{A}}^{-1} = \underline{\underline{A}}^T \text{ in}$$

$$\underline{\underline{A}}^T \underline{\underline{V}} \, \underline{\underline{A}} = \underline{\underline{\tilde{\Lambda}}} \quad \text{diagonaliziranje } V\text{-ja.}$$

V praksi imamo bolj zložen primer (kot recimo, npr. različne mase):

$$\underline{\underline{V}} \, \underline{a} = \underline{\underline{T}} \, \underline{a} \qquad \underline{\underline{T}} \text{ ni diagonalna}$$

Podobno, kot prej,

$$\underline{\underline{V}} \, \underline{a}_k = \lambda_k \underline{\underline{T}} \, a_k \quad \text{in} \quad \underline{a}_l^T \underline{\underline{V}} = \lambda_l a_l^T \underline{\underline{T}}$$

a_l^T $\qquad\qquad$ a_k

$$\left.\begin{aligned} \underline{a}_l^T \underline{\underline{V}}\, \underline{a}_k &= \lambda_k \, \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k \\ \underline{a}_l^T \underline{\underline{V}}\, \underline{a}_k &= \lambda_l \, \underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k \end{aligned}\right\} -$$

$$0 = (\lambda_k - \lambda_l)\, \underbrace{\underline{a}_l^T \underline{\underline{T}}\, \underline{a}_k}$$

izberemo : δ_{lk} ($\underline{\underline{T}}$ pozitivno definitna)

Zato

$$\boxed{\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}} = \underline{\underline{I}}}$$

in

$$\boxed{\underline{\underline{A}}^T \underline{\underline{V}}\, \underline{\underline{A}} = \underbrace{\underline{\underline{A}}^T \underline{\underline{T}}\, \underline{\underline{A}}}\, \underline{\underline{\Lambda}} = \underline{\underline{\Lambda}}} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

Normalne koordinate

Imamo torej

$$\tilde{L} = \frac{1}{2} \underline{\dot{\eta}}^T \underline{\underline{T}}\, \underline{\dot{\eta}} - \frac{1}{2} \underline{\eta}^T \underline{\underline{V}}\, \underline{\eta}$$ in

želimo ta izraz diagonalizirati.

Nastavek:

$$\eta_i(t) = \sum_k \alpha_k(t)\, a_{ki} = \sum_k a_{ki}\, \alpha_k(t)$$

oz. $\underline{\eta} = \underline{\underline{A}}\, \underline{\alpha}$ in $\underline{\eta}^T = \underline{\alpha}^T \underline{\underline{A}}^T$.

Torej,

$$\tilde{L} = \frac{1}{2}\,\underline{\dot{\alpha}}^T \underbrace{\underline{\underline{A}}^T \underline{\underline{I}}\,\underline{\underline{A}}}_{\underline{\underline{I}}}\,\underline{\dot{\alpha}} - \frac{1}{2}\,\underline{\alpha}^T \underbrace{\underline{\underline{A}}^T \underline{\underline{V}}\,\underline{\underline{A}}}_{\underline{\underline{\Lambda}}}\,\underline{\alpha} =$$

$$= \frac{1}{2}\,\underline{\dot{\alpha}}^T \underline{\dot{\alpha}} - \frac{1}{2}\,\underline{\alpha}^T \underline{\underline{\Lambda}}\,\underline{\alpha},$$

oz.

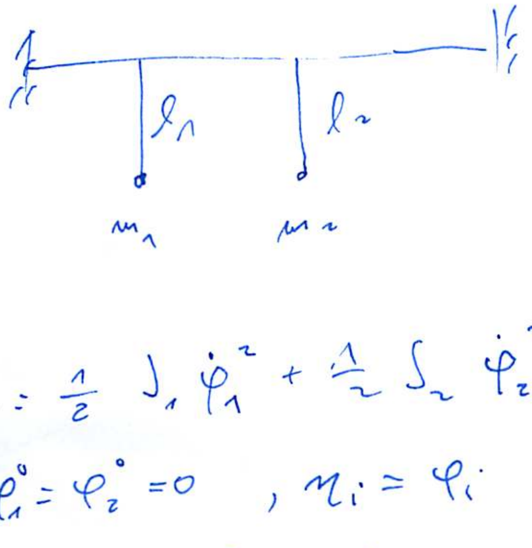
$$\boxed{\begin{array}{l} \tilde{L} = \frac{1}{2}\sum_k \left(\dot{\alpha}_k^2 - \omega_k^2 \alpha_k^2\right) \text{ in} \\ \text{ustrezno } \tilde{H} = \frac{1}{2}\sum_k \left(\dot{\alpha}_k^2 + \omega_k^2 \alpha_k^2\right) \text{ in} \end{array}}$$

enačbe gibanja so

$$\ddot{\alpha}_k + \omega_k^2 \alpha_k = 0 \quad \text{za } \forall k \text{ (neodvisni oscilatorji)}$$

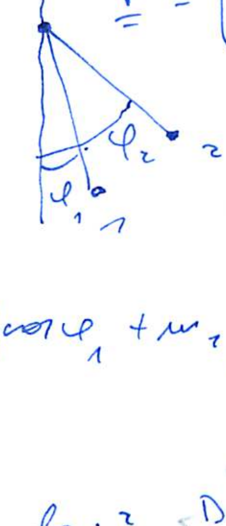
$$\Rightarrow \quad \alpha_k(t) = \alpha_{k_0} \cos(\omega_k t + \delta_k).$$

Primer:



$$\underline{\underline{I}} = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}$$

$$\underline{\underline{V}} = \begin{pmatrix} m_1 g l_1 + D & -2D \\ -2D & m_2 g l_2 + D \end{pmatrix}$$



$$L = \frac{1}{2} J_1 \dot{\varphi}_1^2 + \frac{1}{2} J_2 \dot{\varphi}_2^2 + m_1 g l_1 \cos\varphi_1 + m_2 g l_1 \cos\varphi_2 - \frac{D}{2}(\varphi_1 - \varphi_2)^2$$

$$\varphi_1^0 = \varphi_2^0 = 0 \quad , \eta_i = \varphi_i$$

$$\tilde{L} = \frac{1}{2} J_1 \dot{\eta}_1^2 + \frac{1}{2} J_2 \dot{\eta}_2 - \frac{m_1 g l_1}{2} \eta_1^2 - \frac{m_2 g l_2}{2} \eta_2^2 - \frac{D}{2}(\eta_1 - \eta_2)^2$$