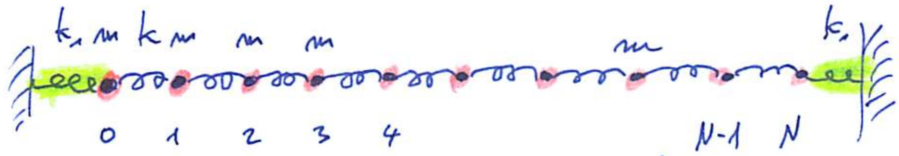
Dinamika zveznega sredstva

Poglejmo si najprej eno vajo: gibalne enačbe za verigo nihal.



stabilne lege

$$L = T - V;\quad q_i = \underbrace{ia}_{q_i^0} + u_i\ ;\quad i = 0, 1, \dots N$$

$$L = \frac{1}{2}\sum_{i=0}^{N} m\dot{u}_i^2 - \frac{1}{2}\sum_{i=0}^{N-1} k(u_{i+1}-u_i)^2 - \frac{1}{2}k_1(u_0^2+u_N^2).$$

„Robni" pogoji npr.: (a) $k_1 \to 0$, veriga se lahko translira, vendar končni k_1 potreben za definiranje ravnovesne lege

(b) $k_1 \to \infty$, togo vpeti robni masi.

(c) linearne kombinacije se jim in na robu itd.

Enačbe gibanja,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{u}_i} - \frac{\partial L}{\partial u_i} = 0:$$

$$m\ddot{u}_i + k(u_{i+1}-u_i)(-1) + k(u_i - u_{i-1}) = 0\ ,\quad \text{če } i \neq 0, N$$

oz.

$$m\ddot{u}_i = k(u_{i-1} - 2u_i + u_{i+1}).$$

Če $N \to \infty$, lahko pri fiksni celotni dolžini X vse skupaj obravnavamo kot palico z vzdolžnim nihanjem,

$$a = \frac{X}{N}\ ,\quad m = \frac{M_{palica}}{N+1}.$$

Spomnimo se recepture za numerično odvajanje:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\boxed{\frac{d^2 f(x)}{dx^2} = \lim_{\Delta x \to 0} \frac{f'(x+\Delta x) - f'(x)}{\Delta x}} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x+2\Delta x) - f(x+\Delta x) - f(x+\Delta x) + f(x)}{(\Delta x)^2} =$$

$$\boxed{\simeq \left[f(x+2\Delta x) - 2f(x+\Delta x) + f(x) \right] / (\Delta x)^2}$$

Pri nas vzamemo $x = ia$ in

$$u(x,t) = u_i(t), \; (N \to \infty)$$

$$m\,\ddot{u}(x,t) = k \overset{\Delta x}{a^2} \frac{\partial^2}{\partial x^2} u(x,t)$$

mora na robu sedaj „robni pogoji".

$$\frac{\partial^2 u}{\partial t^2} = u_{tt}$$

$$\frac{\partial^2 u}{\partial x^2} = u_{xx}$$

$$\boxed{u_{tt} = \underbrace{k\frac{a^2}{m}}_{c^2} u_{xx}}$$ običajna valovna enačba

c^2 hitrost valovanja

Hitrost valovanja povezali z znanimi konstantami. Torej

$$\frac{\Delta l}{l} = \frac{F}{S_0 E_Y} \qquad E_Y \sim \text{Youngov modul}, \quad S_0 \sim \text{presek palice}$$

$$\frac{u_i}{a} = \frac{k u_i}{S_0 E_Y} \Rightarrow k = \frac{S_0 E_Y}{a}$$

$$m = \rho S_0 a, \qquad \boxed{\frac{k a^2}{m} = \frac{S_0 E_Y a^2}{\rho S_0 a a} = \frac{E_Y}{\rho} = c^2}$$

(znan rezultat)

Sedaj poiščimo L, da bo uslovna enačba „enačba gibanja". Imamo že gostoto mase, ρ. Definiramo gostoto energije

$$\frac{T}{S_0 a} = \boxed{\mathcal{T} = \frac{1}{2}\frac{m}{S_0 a} u_t^2 = \frac{1}{2}\rho u_t^2} \quad ; \quad u_t = \frac{\partial u}{\partial t} = \dot{u}(x,t)$$

$$\boxed{\mathcal{V} = \frac{V}{S_0 a} = \frac{1}{2} E_Y u_x^2}.$$

Torej ← Lagrangeova gostota

$$\mathcal{L} = \frac{1}{2}\rho u_t^2 - \frac{1}{2} E_Y u_x^2 \quad \text{in}$$

$$L = S_0 \int_0^X \mathcal{L}(u, u_x, u_t, x, t)\, dx \quad \left(= \int \mathcal{L}\, d^3 r\right)$$

L. funkcija. novo!

To je za 1 dimenzijo.

Euler-Lagrangeove enačbe

Postopamo analogno diskretnemu primeru, in sicer z upeljano akcijo,

$$S = \int_{t_1}^{t_2} L\, dt = S_0 \int_{t_1}^{t_2} \int_0^X \mathcal{L}\, dx\, dt = \min .$$

Poenostavimo $\mathcal{L} = \mathcal{L}(u, u_x, u_t, t)$, ker sicer ni mogoče. Najdemo $u(x,t)$ rešitev in jo variiramo,

$$u(x,t,\alpha) = u(x,t) + \alpha\, \eta(x,t),$$

kjer velja

$$\forall x: \quad \eta(x,t_1) = \eta(x,t_2) = 0 \quad \text{in}$$

$$\forall t: \quad \eta(0,t) = \eta(X,t) = 0.$$

Variiramo:

$$\delta S = S_0 \int_{t_1}^{t_2} \int_0^X \left(\frac{\partial \mathcal{L}}{\partial u} \frac{\partial u}{\partial \alpha} + \frac{\partial \mathcal{L}}{\partial u_t} \frac{\partial u_t}{\partial \alpha} + \underbrace{\frac{\partial \mathcal{L}}{\partial u_x} \frac{\partial u_x}{\partial \alpha}}_{\text{novo!}} \right) \alpha \,(\text{oz. } \delta\alpha)\, dx\, dt =$$

$$= \alpha S_0 \int_{t_1}^{t_2} \int_0^X \left(\frac{\partial \mathcal{L}}{\partial u} \eta + \frac{\partial \mathcal{L}}{\partial u_t} \underset{\frac{\partial \eta}{\partial t}}{\dot{\eta}} + \frac{\partial \mathcal{L}}{\partial u_x} \frac{\partial \eta}{\partial x} \right) dx\, dt .$$

Velja

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial u_t} \frac{\partial \eta}{\partial t}\, dt = - \int_{t_1}^{t_2} \eta \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial u_t} \right) dt + \underbrace{\eta \frac{\partial \mathcal{L}}{\partial u_t} \Big|_{t_1}^{t_2}}_{0} ,$$

$$\int_0^X \frac{\partial \mathcal{L}}{\partial u_x} \frac{\partial u_x}{\partial t}\, dx = - \int_0^X \frac{d}{dx} \left(\frac{\partial \mathcal{L}}{\partial u_x} \right) dx + \eta \frac{\partial \mathcal{L}}{\partial u_x} \Big|_0^X .$$

Torej

$$\delta S = \alpha S_0 \int_{t_1}^{t_2}\int_0^X \left[\frac{\partial \mathcal{L}}{\partial u} - \frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial u_t}\right) - \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right)\right]\eta\,dx\,dt = 0$$

za $\forall\ \eta(x,t) \Rightarrow$

$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0.}$$

E.L. en.

V primeru palice od prej

$$\frac{d}{dt}\rho u_t - \frac{d}{dx}E_Y u_x = \rho u_{tt} - E_Y u_{xx} = 0. \checkmark$$

Kako je z energijo? Poglejmo si še vse skupaj v Hamiltonovem formalizmu. Vpeljemo posplošen impulz,

$$\boxed{\Pi = \frac{\partial \mathcal{L}}{\partial u_t} = \Pi(x,t)}$$

Kanonični spremenljivki sta

torej

$u(x,t)$ in $\Pi(x,t)$ (za palico $= \rho u_t$) gibalna količina, ene mase v [illegible]

$$H = S_0 \int_0^X (\Pi u_t - \mathcal{L})\,dx \equiv S_0 \int_0^X \mathcal{H}\,dx,$$

$$\mathcal{H} = \frac{1}{2}\rho u_t^2 + \frac{1}{2}E_Y u_x^2,$$ po pričakovanjih

Kdaj se ohranja energija ohranja?

Pričakujemo, da takrat, ko se mrežna robu ne sklaplja z okolico. Konkretno:

$$0 \stackrel{?}{=} \frac{dH}{dt} = S_0 \int_0^X \frac{d\mathcal{H}}{dt}\,dx = S_0 \int_0^X (\underbrace{\rho u_t u_{tt}} + E_y u_x u_{xt})\,dx$$

valovna enačba od prej $\rightarrow$ $\rho u_{tt} = E_y u_{xx}$

$u_{xt} = u_{tx}$

torej

$$\frac{dH}{dt} = S_0 E_y \int_0^X (u_t u_{xx} + u_x u_{tx})\,dx =$$

$$= S_0 E_y \int_0^X \frac{d}{dx}(u_t u_x)\,dx = S_0 E_y u_t u_x \Big|_0^X .$$

kdaj je: $u_t(0,t)\,u_x(0,t) \stackrel{?}{=} 0$
$\quad X \qquad X$

Zagotovo velja to za ~~togo vpeto~~ palico (hitrosti in odmiki na robu $= 0$), so še druge možnosti

$\longrightarrow$ mehanika kontinuuma

in zato

$$H = E_0 = \text{konst.}$$