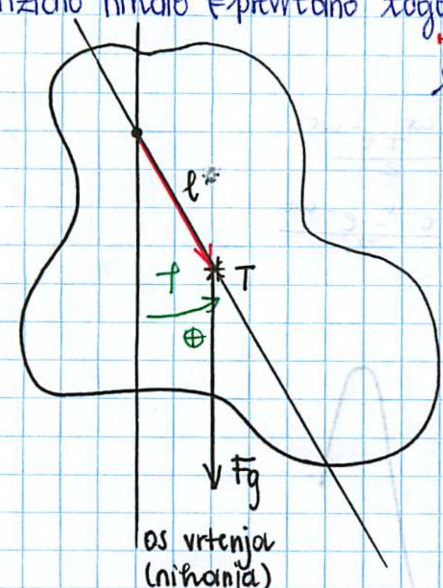
(a) Fizično nihalo (=prevrtano togo telo)



$\longrightarrow$ ročica

l^* - razdalja od težišča do osišča

$J\alpha = M$ (Uporabimo Steinerjev izrek)

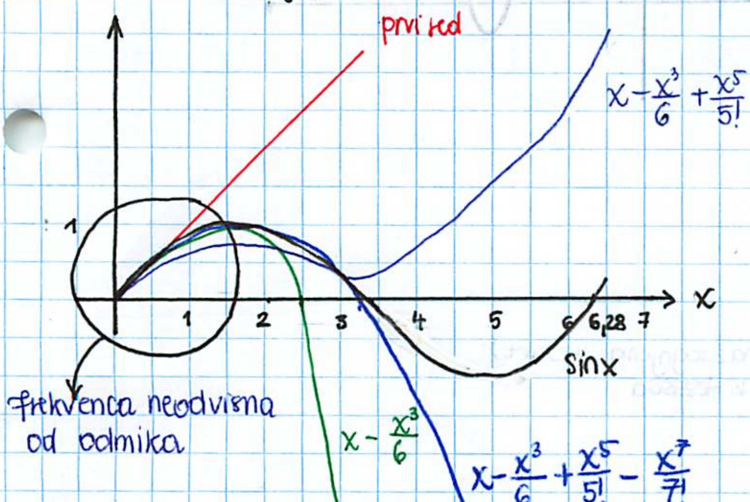
$J\ddot{\varphi} = l^* mg\sin\varphi\,(-1) = -ml^* g\sin\varphi$

zaradi teže je pospešek kota negativen, saj je kot definiran v pozitivno smer

$\ddot{\varphi} + \omega^2 \sin\varphi = 0$; $\omega^2 = \frac{mgl^*}{J}$

$\varphi(t) = ?$

$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \dots \approx \varphi$ ← lineariziramo enačbo



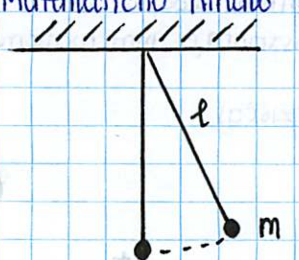
frekvenca neodvisna od odmika

Nihajni čas je odvisen od začetnega odmika. Če lineariziramo pa več ni pomemben!

v ω nimamo odmika!

$\ddot{\varphi} + \omega^2\varphi = 0 \Longrightarrow \varphi(t) = A\cos\omega t + B\sin\omega t$

(b) Matematično nihalo (lahka palica, $m=0$) (točkasto telo)



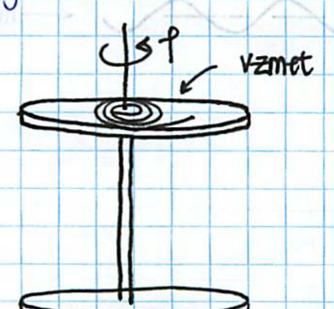
$\omega^2 = \frac{mgl}{ml^2} = \frac{g}{l}$

$N=5$

$t = 8{,}67\,s$; $8{,}68\,s$; $8{,}48\,s$ (večji odmik)

$t_0 = \frac{t}{N} = \frac{8{,}67\,s}{5} = 1{,}734\,s$

(c) Torzijsko nihalo



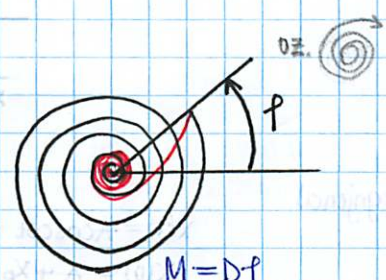
$J\alpha = M = -D\varphi$

$J\ddot{\varphi} + D\varphi = 0$

$\ddot{\varphi} + \omega^2\varphi = 0$

$\omega^2 = \frac{D}{J}$;

$\omega^2 = \frac{k}{m}$



$M = D\varphi$

↑

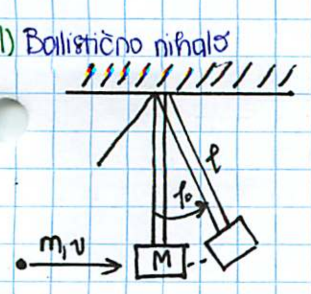
koeficient torzijske vzmeti

$N=2$

$t_1 = 11{,}91\,s$

$t_2 = 11{,}32\,s$ ← večji φ

(d) Balistično nihalo



$t = 0^+$

$\varphi(0) = 0$

$\dot{\varphi}(0) = \omega_1$

$\ddot{\varphi} + \omega^2\varphi = 0$

$\varphi(t) = A\sin\omega t = \frac{\omega_1}{\omega}\sin\omega t$

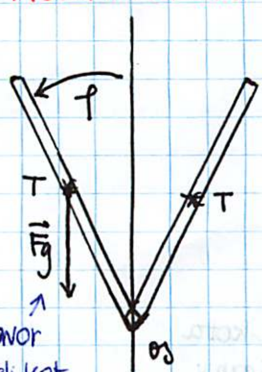
$\dot{\varphi} = \omega A = \omega_1$

$\Delta G_1 = \Delta G_2$

$mv + 0 = (m+M)v_1$

$v = \frac{m+M}{m} v_1 = \frac{m+M}{m} \cdot l\omega_1$

LABILNA LEGA



Navor želi kot pospešit v levo!

$J\alpha = M$

$J\ddot{\varphi} = mgl^* \sin\varphi \ (+1)$

$\ddot{\varphi} - \omega^2\varphi = 0$ → ni periodična rešitev, ampak eksponentno nihanje; $e^{\pm\omega t}$, $ch\,\omega t = \frac{e^{\omega t}+e^{-\omega t}}{2}$

$+\sin\varphi$

$\ddot{\varphi} = \omega^2\varphi$

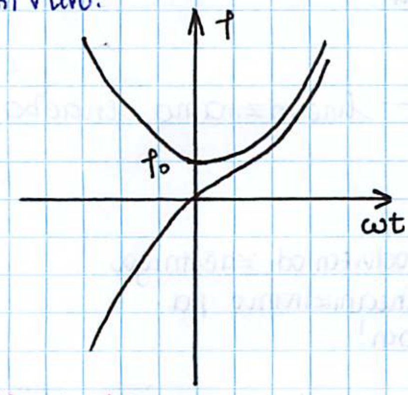
$sh\,\omega t = \frac{e^{\omega t}-e^{-\omega t}}{2}$

$\varphi = A\,ch\,\omega t + B\,sh\,\omega t$

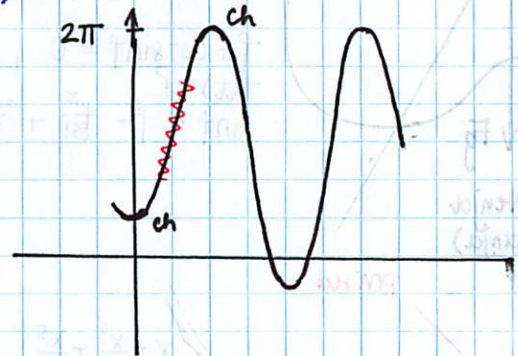
$t=0: \ \varphi(0)=\varphi_0$

$\dot{\varphi}(0) = 0$

$\Longrightarrow \varphi = \varphi_0\, ch\,\omega t$

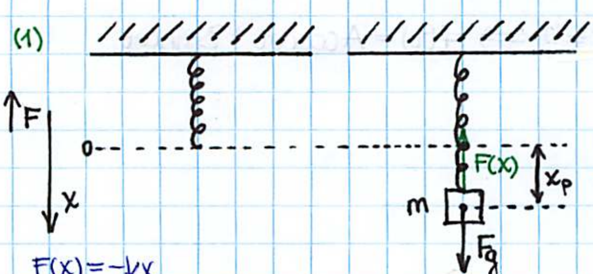


na veliki skali $\Longrightarrow$



PRIMERI NIHANJA

(1)



izhodišče (neraztegnjena vzmet)

odmik = odmik težišča

$F(x) = -kx$

$m\ddot{x} = -kx + mg$

$\ddot{x} + \omega^2 x = g$ Nehomogena linearna diferencialna enačba 2. reda s konstantnimi koeficienti (= koeficienti niso odvisni, ker niso funkcija časa; npr. ω ni konstanta, če matematičnemu nihalu spreminjamo dolžino)

$x(t) = ?$

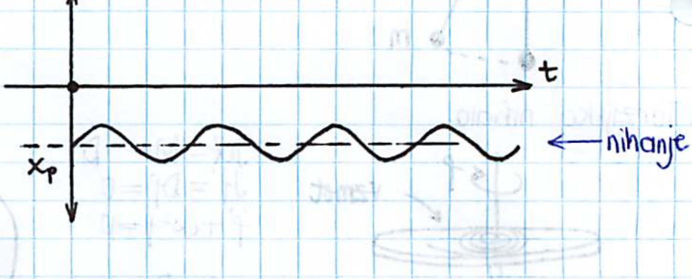
$x_p = \frac{g}{\omega^2}$ partikularna rešitev (konstantna in „ena sama") = predstavlja odmik (težišča)

$x(t) = A\cdot\cos\omega t + B\sin\omega t + x_p = x_H + x_p$ (Alg 1)

$\ddot{x}_H + \omega^2 x_H = 0$

$\ddot{x}_p + \omega^2 x_p = g$

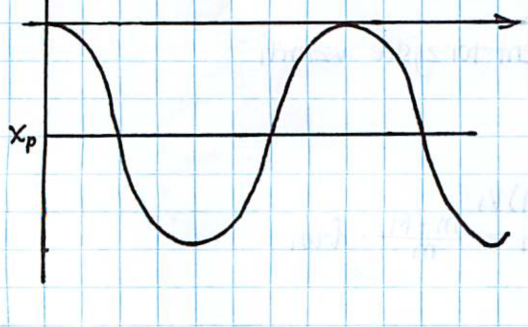
$x = x_H + x_g$ ✓



Začetni pogoji:

$x(0) = 0$

$\dot{x}(0) = 0$ (= vzmet ni raztegnjena)



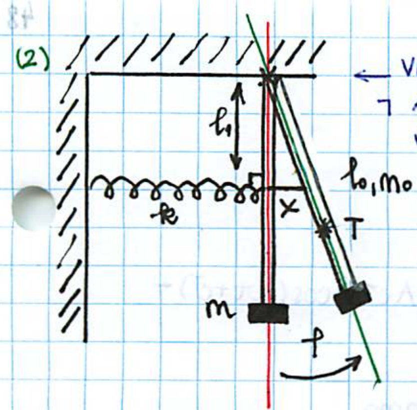
$x(t) = A\cos\omega t + B\sin\omega t + x_p$

$x(0) = A + x_p = 0 \Longrightarrow A = -x_p$

$\dot{x}(0) = \omega B = 0$

$x(t) = -x_p\cos\omega t + x_p = x_p(1-\cos\omega t)$

(2)



← vrtljiva palica
→ pri majhnih odmikih
vzmet je neraztegnjena

$J\ddot{\varphi} = -(m_0+m)l^* g\sin\varphi - kxl_1$; $x = l_1\varphi$

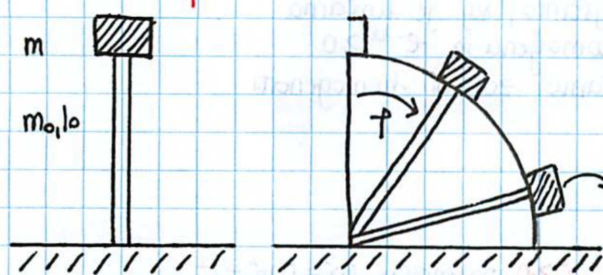
$$l^* = \frac{\frac{m_0 l_0}{2} + m l_0}{m + m_0}$$

$J = \frac{1}{3}m_0 l_0^2 + m l_0^2$ ← palica, ki se vrti okrog svojega roba

$\ddot{\varphi} + \omega^2\varphi = 0$

$$\omega^2 = \frac{(m_0+m)l^* g + k l_1^2}{\frac{1}{3}m_0 l_0^2 + m l_0^2}$$

(3)



$\varphi = A\,\mathrm{ch}\,\omega t + B\,\mathrm{sh}\,\omega t$

skočiš (sestopiš)

$v = ?$ ✓ (vaje - opica)

$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$; $i^2 = -1$

$\sin(ix) = ix - \frac{(ix)^3}{3!} + \frac{(ix)^5}{5!} - \frac{(ix)^7}{7!} + \dots = ix + \frac{ix^3}{3!} + \frac{ix^5}{5!} + \frac{ix^7}{7!} + \dots = i\,\mathrm{sh}\,x$

$\cos(ix) = 1 - \frac{(ix)^2}{2!} + \frac{(ix)^4}{4!} - \frac{(ix)^6}{6!} + \dots = 1 + \frac{x^2}{2!} + \dots = \mathrm{ch}\,x$

$\mathrm{ch}\,x = \frac{e^x + e^{-x}}{2}$

$\mathrm{ch}\,ix = \cos x$ (cos x soda funkcija)

$$\left.\begin{aligned} \cos x &= \mathrm{ch}\,ix = \frac{e^{ix} + e^{-ix}}{2} \\ \sin x &= \frac{\mathrm{sh}\,ix}{i} = \frac{e^{ix} - e^{-ix}}{2i} \end{aligned}\right\} \begin{aligned} &/1 \\ &/i \end{aligned} \quad \oplus$$

$$\boxed{\begin{aligned} e^{ix} &= \cos x + i\sin x \\ e^{-ix} &= \cos x - i\sin x \\ e^{x} &= \mathrm{sh}\,x + \mathrm{ch}\,x \end{aligned}}$$ ← Eulerjevi formuli

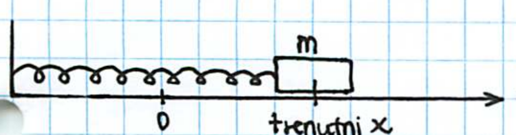
$e^{ix} = \cos x + i\sin x$

$|e^{ix}|^2 = e^{ix}\overline{e^{ix}} = e^{ix}\cdot e^{-ix} = 1 = (\cos x + i\sin x)(\cos x - i\sin x) = \cos^2 x + \sin^2 x$

$\overline{e^{ix}} = e^{-ix}$

e^z, $z \in \mathbb{C}$ (= enake formule kot $\mathbb{R}$)

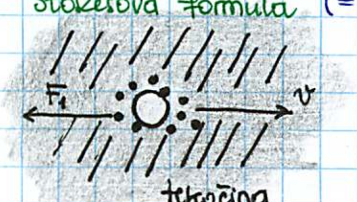
DUŠENO NIHANJE



podlaga s trenjem

$m\ddot{x} = -kx + F(x(t))$ ← sila je odvisna od tega, kje telo je in od tega kako se giblje

Stokesova formula (= 0 viskoznosti)

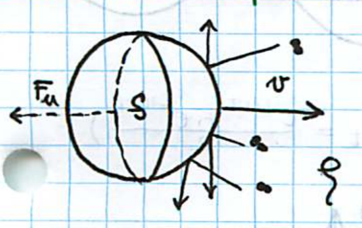


$F_1 = 6\pi\eta R v \quad \propto \dot{x}$

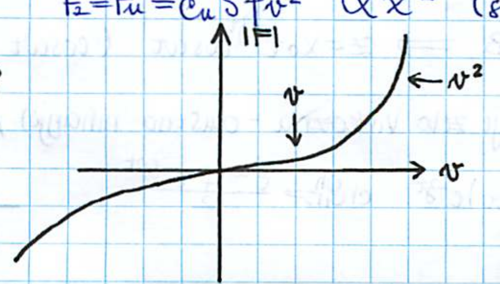
η ↓ koeficient viskoznosti

Kvadratni zakon upora (= krogla „odmika" molekule zraka)

F_u - sila upora zraka



$F_2 = F_u = c_u S \rho v^2 \quad \propto \dot{x}^2$ (sign $\dot{x}$ ← hude težave = zanimivo)



$m\ddot{x} = -kx - 6\pi\eta R\dot{x}$ linearna funkcija

$\ddot{x} + 2\zeta\dot{x} + \omega_0^2 x = 0$

$\omega_0^2 = \frac{k}{m}$

$2\zeta = \frac{6\pi\eta R}{m}$

Nastavek (= predpostavka: dušeno nihanje):

$x(t) = Ae^{-\lambda t}\cos(\omega t + \delta)$ $(-\lambda = 0$, če ni dušeno nihanje)

$\dot{x}(t) = -\lambda Ae^{-\lambda t}\cos(\omega t+\delta) - \omega Ae^{-\lambda t}\sin(\omega t+\delta)$

$\ddot{x}(t) = +\lambda^2 Ae^{-\lambda t}\cos(\omega t+\delta) - \lambda\omega Ae^{-\lambda t}\sin(\omega t+\delta) + \lambda\omega Ae^{-\lambda t}\sin(\omega t+\delta) - \omega^2 Ae^{-\lambda t}\cos(\omega t+\delta) -$

$= Ae^{-\lambda t}[\cos(\omega t+\delta)\cdot(\lambda^2-\omega^2) + 2\lambda\omega\sin(\omega t+\delta)]$

$\sin(\omega t+\delta) = 2\lambda\omega - 2\zeta\omega = 0$

$\zeta = \lambda$

$\cos(\omega t+\delta) = (\lambda^2-\omega^2) - 2\zeta\lambda + \omega_0^2 = 0$

$(\zeta^2-\omega^2) - 2\zeta^2 + \omega_0^2 = 0$

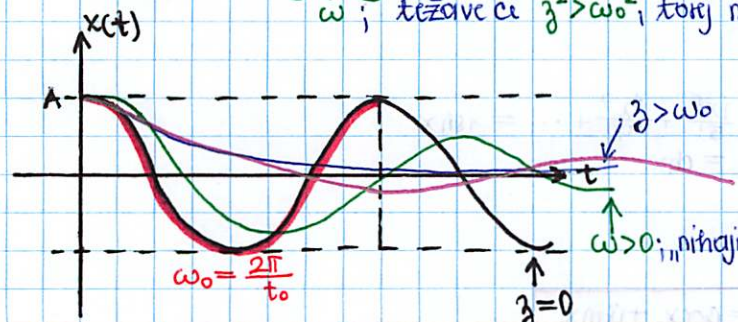
$\omega^2 = \omega_0^2 - \zeta^2$

+ $e^{-\lambda t}$ pokrajšamo, saj je linearna enačba homogena in $e^{-\lambda t} > 0$

+ A pokrajšamo zaradi homogenosti

$x(t) = Ae^{-\zeta t}\cos(\underbrace{\sqrt{\omega_0^2-\zeta^2}}_{\omega}\, t + \delta)$

ω; težave či $\zeta^2 > \omega_0^2$, torej mora biti $\omega^2 \geq 0$ po enačbi $\omega^2 = \omega_0^2 - \zeta^2$



$\omega = 0 \leftarrow$ to ni nihanje, ampak eksponentno padanje (torej primer kroglice v glicerinu – primer viskoznosti – kroglica se hitro vrne v ravnovesno lego.

V $\mathbb{C}$:

$\ddot{x} + 2\zeta\dot{x} + \omega_0^2 x = 0$

$z(t)$ kompleksna funkcija

$z(t) = x_R(t) + i x_I(t) \in \mathbb{C}$, $x_R, x_I \in \mathbb{R}$

x_R, x_I reši enačbo $\Longrightarrow$ z reši enačbo ✓ $\Longrightarrow$ ker so vsi deli linearni

$\ddot{z} + 2\zeta\dot{z} + \omega_0^2 z = 0 \Longrightarrow \mathrm{Re}\, z = x_R$

$\mathrm{Re}\, z = x_I$

$\xi \Leftrightarrow$ ceta

Nastavek:

$z = Ae^{i\xi t}$

$\dot{z} = i\xi Ae^{i\xi t}$

$\ddot{z} = Ai^2\xi^2 e^{i\xi t} = -A\xi^2 e^{i\xi t}$

$-\xi^2 + 2i\zeta\xi + \omega_0^2 = 0$ karakteristični polinom $\Longrightarrow \xi_{1,2}$ rešitvi

← realni del je enak kot zgoraj

$\Longrightarrow$ karakteristični polinom $\lambda^2 e^{\lambda t} + 2\zeta\lambda e^{\lambda t} + \omega_0^2 e^{\lambda t} = 0$, ker $e^{\lambda t} > 0$ pokrajšamo!

$\lambda^2 + 2\zeta\lambda + \omega_0^2 = 0$

$\lambda_{1,2} = \frac{-2\zeta \pm \sqrt{4\zeta^2 - 4\omega_0^2}}{2} = -\zeta \pm \sqrt{\zeta^2 - \omega_0^2}$

$z(t) = e^{\lambda t} = e^{-\zeta t} e^{\pm\sqrt{\zeta^2-\omega_0^2}\, t}$

Posebni primeri

(a) $\zeta < \omega_0$

$z = Ae^{-\zeta t}e^{i\omega t} + Be^{-\zeta t}e^{-i\omega t}$; $\omega = \sqrt{\omega_0^2 - \zeta^2} > 0$ $(\in \mathbb{R})$

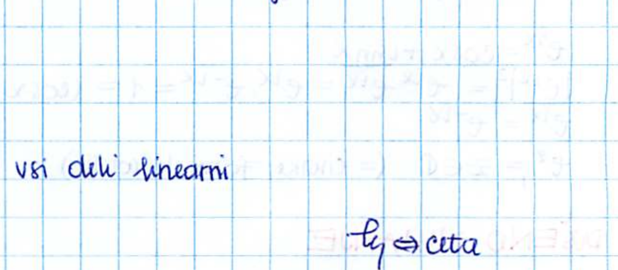
$z(0) = x_0 \in \mathbb{R}$

$\dot{z}(0) = v_0 \in \mathbb{R}$

$z(0) = x_0 = A + B$

$\dot{z}(0) = v_0 = A(-\zeta + i\omega) + B(-\zeta - i\omega) = 0$ ← poseben primer

$A(-\zeta + i\omega) = B(\zeta + i\omega)$ .... $A, B \Longrightarrow z = x_0 e^{-\zeta t}\cos\omega t$ $\left(\cos\omega t = \frac{e^{i\omega t} + e^{-i\omega t}}{2}\right)$



(b) $\zeta > \omega_0$ (razmet je zelo tanka in tekočina je zelo viskozna – dušeno nihanje)

$\omega = \sqrt{\zeta^2 - \omega_0^2} \in \mathbb{R}$

$z = Ae^{-\zeta t}e^{\omega t} + Be^{-\zeta t}e^{-\omega t} = (C\,\mathrm{ch}\,\omega t + D\,\mathrm{sh}\,\omega t)e^{-\zeta t}$ $\mathrm{ch}\,\omega t = \frac{e^{\omega t} + e^{-\omega t}}{2}$