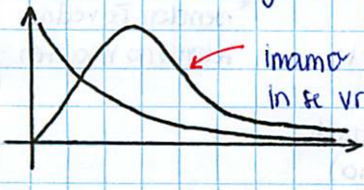
(c) $\omega_0 = \zeta \Rightarrow \omega = 0$ oz. $\sqrt{\zeta^2 - \omega_0^2} = 0$
Torej rešitve so oblike $e^{\lambda t}$ in $\lambda e^{\lambda t}$ (ker imamo dvojno ničlo)
Linearna kombinacija teh dveh predstavlja nihanje.



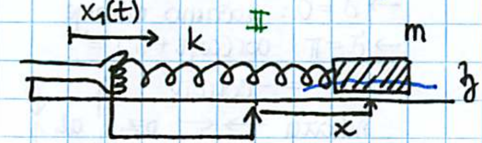
imamo neko začetno hitrost (potem se ustavi - IZNIHA in se vrne v ravnovesno lego)

Zanimivost: amortizerji imajo ζ malo manjši od ω_0. Če bi bil $\zeta < \omega_0$, bi bila guma pretrda, če $\omega_0 < \zeta$ bi pa avto tolj poskakoval.

$$\sum_{n=0}^{N} x^{(n)} c_n = 0$$
$$\sum_{n=0}^{N} \lambda^n c_n = 0$$
m-kratne ničle!
m: $t^0 e^{\lambda t}, t^1 e^{\lambda t}, \ldots, t^{m-1} e^{\lambda t}$
Ta števila (↑) vsa rešijo zgornjo enačbo!

VSILJENO NIHANJE

Zanima nas rešitev po dolgem času, tj. ko „x_{hom}" s časom zamre in nas zanima le „x_{part}".

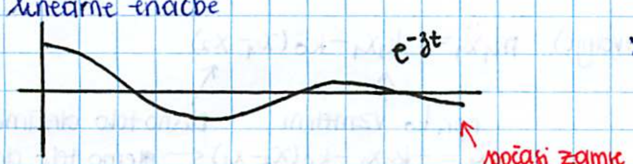


I: $m\ddot{x} + 6\pi R\eta\dot{x} + kx = F_z(t)$ ← zunanja sila odvisna od časa ; $2\zeta = \frac{6\pi R\eta}{m}$
↑ sila v nasprotni smeri $\dot{x}$; ↖ sila v nasprotni smeri od x

II: $m\ddot{x} + 6\pi R\eta\dot{x} + k(x - x_1) = 0.$ $\Leftrightarrow$ $\ddot{x} + 2\zeta\dot{x} + \omega_0^2 x = \omega_0^2 x_1(t)$ ← v našem poskusu: $x_1 = x_0 \cos\omega_1 t$ (lahko tudi sin)
↑ projekcija vrtenja

$x(t) = x(t)_{hom} + x(t)_{part}$
↑ rešitev homogene linearne enačbe ; ↖ premik

$x(t)_{hom}$



$x(t)_{part}$ odvisna od $x_1(t)$!

Nastavek:
$z(t)$, $x_{part}(t) = \mathrm{Re}\, z$ (= rešitev partikularne bo realni del z)
$z_1(t) = x_0 e^{i\omega_1 t} = x_0(\underbrace{\cos\omega_1 t}_{\text{vsiljeno nihanje } \in \mathbb{R}} + i\sin\omega_1 t)$

$x_1 = \mathrm{Re}\, z_1$
$\ddot{z} + 2\zeta\dot{z} + \omega_0^2 z = \omega_0^2 z_1$
$z = A e^{i\omega_1 t}$ → nihanje z isto frekvenco
↓
neznana amplituda odvisna od ω_1 (= znana RESONANČNA krivulja)

$-\omega_1^2 A e^{i\omega_1 t} + 2\zeta i\omega_1 A e^{i\omega_1 t} + \omega_0^2 A e^{i\omega_1 t} = x_0\omega_0^2 e^{i\omega_1 t}$; $e^{i\omega t} > 0$, zato pokrajšamo! Je tudi ena izmed rešitev!

$A(\omega_0^2 - \omega_1^2 + 2i\zeta\omega_1) = \omega_0^2 x_0$

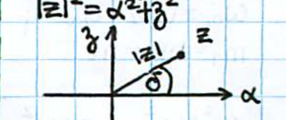
$$A = \frac{\omega_0^2 x_0 (\omega_0^2 - \omega_1^2 - 2i\zeta\omega_1)}{(\omega_0^2 - \omega_1^2 + 2i\zeta\omega_1)(\omega_0^2 - \omega_1^2 - 2i\zeta\omega_1)} = A_R + A_I =$$
$$= \frac{\omega_0^2 x_0 (\omega_0^2 - \omega_1^2 - 2i\zeta\omega_1)}{(\omega_0^2 - \omega_1^2)^2 + 4\zeta^2\omega_1^2} = |A| e^{i\delta} = |A|(\cos\delta + i\sin\delta)$$

$$|A| = \frac{\omega_0^2 x_0}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\zeta^2\omega_1^2}} = \sqrt{z \cdot \bar{z}}$$

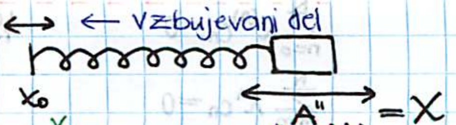
$$\tan\delta = \frac{\mathrm{Im}\, z}{\mathrm{Re}\, z} = \frac{\mathrm{Im}\, A}{\mathrm{Re}\, A} = -\frac{2\zeta\omega_1}{\omega_0^2 - \omega_1^2}$$

$$z(t) = A e^{i\omega_1 t} = |A| e^{i\delta + i\omega_1 t} = |A|(\cos(\omega_1 t + \delta) + i\sin(\omega_1 t + \delta))$$
$$x_p(t) = \mathrm{Re}\, z = \frac{\omega_0^2 x_0}{\sqrt{(\omega_0^2 - \omega_1^2)^2 + 4\zeta^2\omega_1^2}} \cos(\omega_1 t + \delta) = X\cos(\omega_1 t + \delta)$$

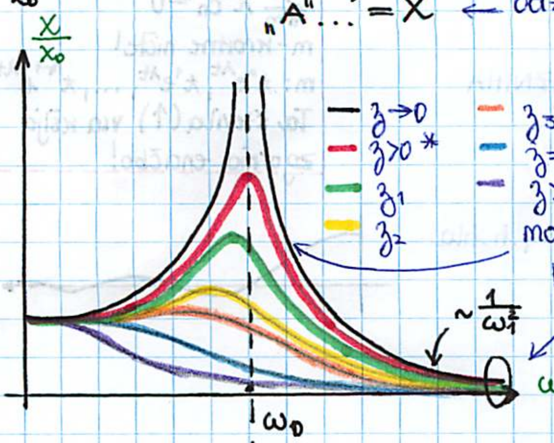
$z = \alpha + i\zeta$ $|z| = \sqrt{z \cdot \bar{z}}$
$|z|^2 = \alpha^2 + \zeta^2$



$z = |z|(\cos\delta + i\sin\delta) =$
$= |z| e^{i\delta}$ ← Taylor



} odvisno s katero frekvenco nihamo, ω_1, γ, dušenje



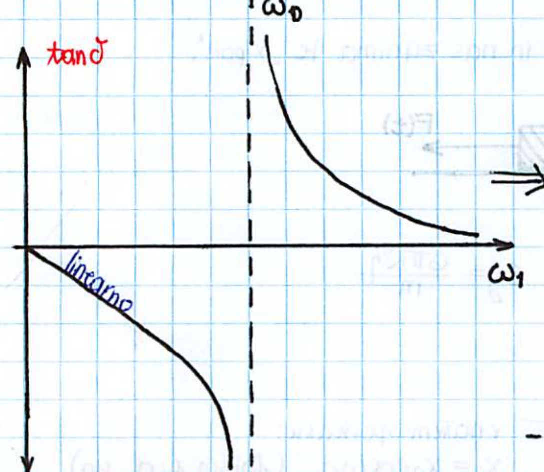
nihalo „kritično" dušeno (vrne se le v ravnovesno lego, ne niha)

amplituda je enaka amplitudi roke! (samo po sebi ne niha)

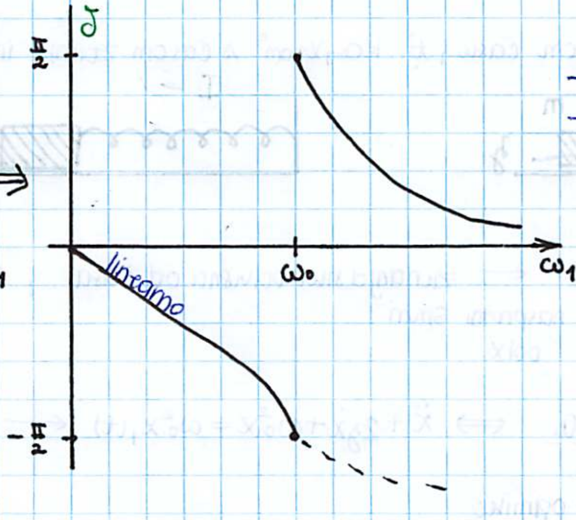
* vendar še vedno relativno majhen

maximum je ravno $\omega = \sqrt{\omega_0^2 - \gamma^2}$ (maximum se pomika navznoter)

frekvenca ZELO velika, vendar kroglica niha z majhno amplitudo, ker ne more slediti.



$\Longrightarrow$



$x_1 = x_0 \cos\omega_1 t$

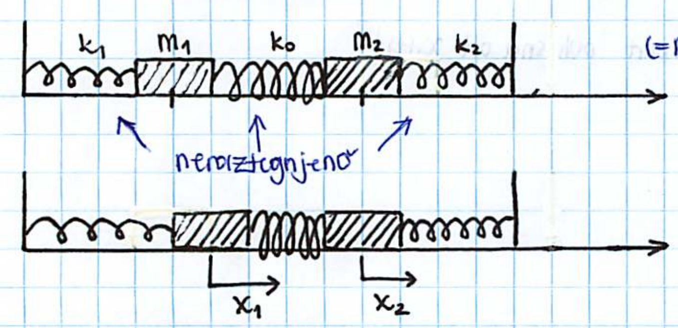
$x = X\cos(\omega_1 t + \delta)$

$\to \delta = 0$: sočasno nihanje

$\to \delta = \pi$: $\cos(\omega_1 t + \pi) = -\cos\omega_1 t$

nihata $\rightarrow \leftarrow$ oz. $\uparrow\downarrow$ ali $\downarrow\uparrow$

SKLOPLJENO NIHANJE



(=mirovanje) $m_1\ddot{x}_1 = -k_1 x_1 - k_0(x_1 - x_2)$

npr. k_0 vzmeti ni — desno telo držimo pri miru

$m_2\ddot{x}_2 = -k_2 x_2 - k_0(x_2 - x_1)$ ← levo telo držimo pri miru

$\ddot{x}_1 + \omega_1^2 x_1 + \omega_{01}^2 x_1 - \omega_{01}^2 x_2 = 0$

$\ddot{x}_2 + \omega_2 x_2 + \omega_{02}^2 x_2 - \omega_{02}^2 x_1 = 0$

$\omega_{01}^2 = \frac{k_0}{m_1}$, $\omega_{02}^2 = \frac{k_0}{m_2}$, $\omega_1^2 = \frac{k_1}{m_1}$, $\omega_2^2 = \frac{k_2}{m_2}$

Dobimo linearni (sklopljeni) homogeni diferencialni enačbi 2. reda.

Začetni pogoji:

$x_1 = 0$

$\dot{x}_1 = 0$

$x_2 = x_0$ ← nihalo 2 odmaknemo

$\dot{x}_2 = 0$

Nastavek:

(a) $x_1 = A_1 e^{i\omega t}$; $\mathrm{Re}\, A_1 e^{i\omega t} = |A_1|\cos(\omega t + \delta)$; $A_1 = |A_1| e^{i\delta}$

$2\gamma_1 \dot{x}_1$ ← predstavlja dušenje, potem nujno delamo v $\mathbb{C}$, sicer lahko delamo v $\mathbb{R}$ ((*) dobimo negativen λ, če imamo ta člen)

(b) $x_1 = A_1\cos(\omega t)$
$x_2 = A_2\cos(\omega t)$ $\longrightarrow$ iščemo $A_1 A_2$ in ω!

$A_1(-\omega^2 + \omega_1^2 + \omega_{01}^2) - A_2\omega_{01}^2 = 0$ — $\cos\omega t > 0$, zato pokrajšamo!

$A_2(-\omega^2 + \omega_2^2 + \omega_{02}^2) - A_1\omega_{02}^2 = 0$

$\omega_{01}^2 = \omega_{02}^2 = \omega_0^2$; $\omega_0^2 = \frac{k}{m}$

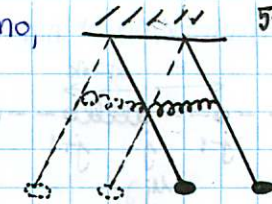
$m_1 = m_2 = m$

$$\begin{bmatrix} \omega_1^2 + \omega_0^2 & -\omega_0^2 \\ -\omega_0^2 & \omega_2^2 + \omega_0^2 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \lambda \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}; \ \lambda = \omega^2$$

Pozitivno definitna matrika (hermitsko simetrična).

λ se imenuje lastna kotna hitrost, $\binom{A_1}{A_2}$ pa lastni vektor.

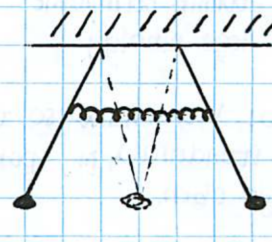
Če nihali sta enaki, pravtako vzmeti, tj. $k_1 = k_2$, potem je iz eksperimenta razvidno, da sta kotni hitrosti enaki in da je en lastni vektor, ne glede za koliko odmaknemo: $\begin{pmatrix} A_{1I} \\ A_{2I} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \underset{\substack{\uparrow \\ \text{normiramo}}}{=} \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$; $A_{1I}^2 + A_{2I}^2 = 1$



$\omega_I = \omega_1$

Če sta nihali še vedno enaki, pravtako tudi vzmeti, tj $k_1 = k_2$, razlika je le v smeri, v kateri sta nihali odmaknjeni. Odmaknjeni sta pa v različni smeri. Zatorej je ω_{II} večji, ker vzmet med nihaloma pospeši nihali eno proti drugemu (manjši nihajni čas).

Lastni vektor: $\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ $\qquad \omega_{II} = \sqrt{\omega_I^2 + (\Delta\omega)^2}$



$$\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 - \lambda_0^2 = 0 \quad (*)$$

$$\lambda_{I,II} = \frac{(\lambda_1+\lambda_2) \pm \sqrt{(\lambda_1+\lambda_2)^2 - 4(\lambda_1\lambda_2 - \lambda_0^2)}}{2} \geq 0 \quad \text{(poz. definitna)}$$

$\leftarrow$ dobimo splošno rešitev, ki jo vstavimo v enačbo, da dobimo $\frac{A_1}{A_2}$!

$k_1 = k_2$: $\lambda_1 = \lambda_2$

$$\lambda_{I,II} = \frac{2\lambda_1 \pm \sqrt{4\lambda_1^2 - 4(\lambda_1^2 - \lambda_0^2)}}{2} = \lambda_1 \pm \sqrt{\lambda_1^2 - \lambda_1^2 + \lambda_0^2} = \lambda_1 \pm \lambda_0 = \omega_1^2 + \omega_0^2 \pm \omega_0^2 > 0$$

$$= \lambda_1 \pm \lambda_0 = \begin{cases} \lambda_I : \omega_1^2 + 2\omega_0^2 \\ \lambda_{II} : \omega_1^2 \end{cases} > 0 = \omega_{I,II}^2$$

I: $\omega_I = \sqrt{\omega_1^2 + 2\omega_0^2}$; $\lambda_1 = \lambda_2$

$$\begin{bmatrix} \lambda_1 & -\lambda_0 \\ -\lambda_0 & \lambda_1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = (\lambda_1 + \lambda_0) \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \Rightarrow \begin{bmatrix} -\lambda_0 & -\lambda_0 \\ -\lambda_0 & -\lambda_0 \end{bmatrix} \cdot \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0 \Rightarrow A_{2I} = -A_{1I}, \text{ tj. lastni vektor } c \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

kjer je $(\lambda_1 + \lambda_0) = \begin{bmatrix} \lambda_1 + \lambda_0 & 0 \\ 0 & \lambda_1 + \lambda_0 \end{bmatrix}$.

II: $\omega_{II} = \omega_1$ $\leftarrow$ nihali nihata s frekvenco enega nihala

$$\omega_0 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = 0 \Rightarrow d \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ je lastni vektor.}$$

se da zapisati tudi kot $\cos(\omega t + \delta)$

Splošna rešitev:

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos \omega_1 t + B \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{\omega_1^2 + 2\omega_0^2}\, t) + C \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos \omega_1 t + D \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{\omega_1^2 + 2\omega_0^2}\, t)$$

$\hookrightarrow$ vektor, kjer so komponente odmiki posameznih nihal

ker je rešitev kosinus, je potem rešitev tudi sinus

A... poljubna konstanta,

... vektorji so nenormirani.

linearna kombinacija lastnih vektorjev

$t = 0$: odmik: $\begin{bmatrix} 0 \\ x_0 \end{bmatrix} = \begin{bmatrix} A + B \\ A - B \end{bmatrix} \Rightarrow$ $A = -B \Rightarrow B = -\frac{x_0}{2}$

$x_0 = A - B = A + A = 2A$

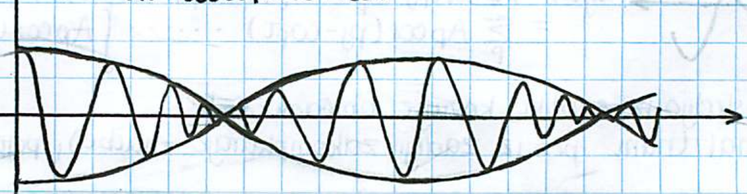
$A = \frac{x_0}{2}$

hitrost: $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = -\omega_1 \cdot 0 + \ldots \Rightarrow C = D = 0$

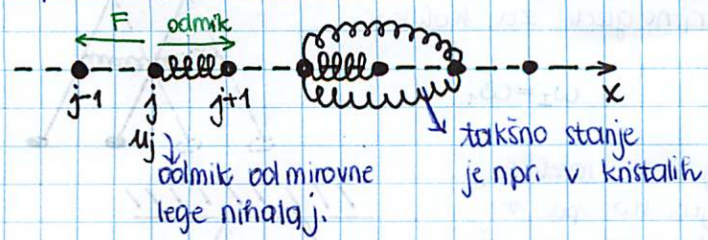
Dokončna rešitev:

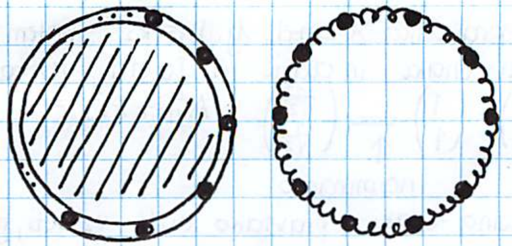
$$\boxed{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \frac{x_0}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos \omega_1 t - \frac{x_0}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \cos(\sqrt{\omega_1^2 + 2\omega_0^2}\, t)}$$

$\Delta\omega = \omega_1 - \sqrt{\omega_1^2 + 2\omega_0^2}$



Posplošitev na N nihal





Privzemimo: Vsa nihala so enaka, prav tako so enake vse vzmeti. Ker prvega odvoda ni (Newtonove enačbe - standard!), ni potrebno računati v kompleksnem svetu. Vzamemo kosinuse, sinuse ali kombinacijo obeh.

$$e^{j\dots} \qquad e^{j} \qquad e^{j+1\dots} \qquad e^{j+1+\dots}$$

$$m\ddot{u}_j = -k(u_j - u_{j-1}) - k(u_j - u_{j+1}) = -2ku_j + k(u_{j-1} + u_{j+1}) \;;\; \omega^2 = \frac{k}{m}$$

$$u_j(t) = A\cos(pj - \omega t \underbrace{}_{+\varphi})$$

p: parameter, ki ga ne poznamo

$$-\omega^2\cos(pj-\omega t) + 2\omega_0^2\cos(pj-\omega t) - \omega_0^2\left(\cos(p(j-1)-\omega t) + \cos(p(j+1)-\omega t)\right) = 0$$

$$-\omega^2\cos\cancel{(pj-\omega t)} + 2\omega_0^2\cos\cancel{(pj-\omega t)} = 2\omega_0^2\cos\cancel{(pj-\omega t)}\cos p \quad , \text{ če } \cos(pj-\omega t) \neq 0$$

$$\omega^2 = 2\omega_0^2(1-\cos p) = 4\omega_0^2\sin^2\frac{p}{2}$$

$$\omega^2 = 4\omega_0^2\sin\frac{p}{2} \implies \omega = 2\omega_0\left|\sin\frac{p}{2}\right| = 2\omega_0\left|\sin\frac{\pi}{\lambda}\right|$$

$$u_j(t) = A\cos(pj - \omega t + \varphi)$$

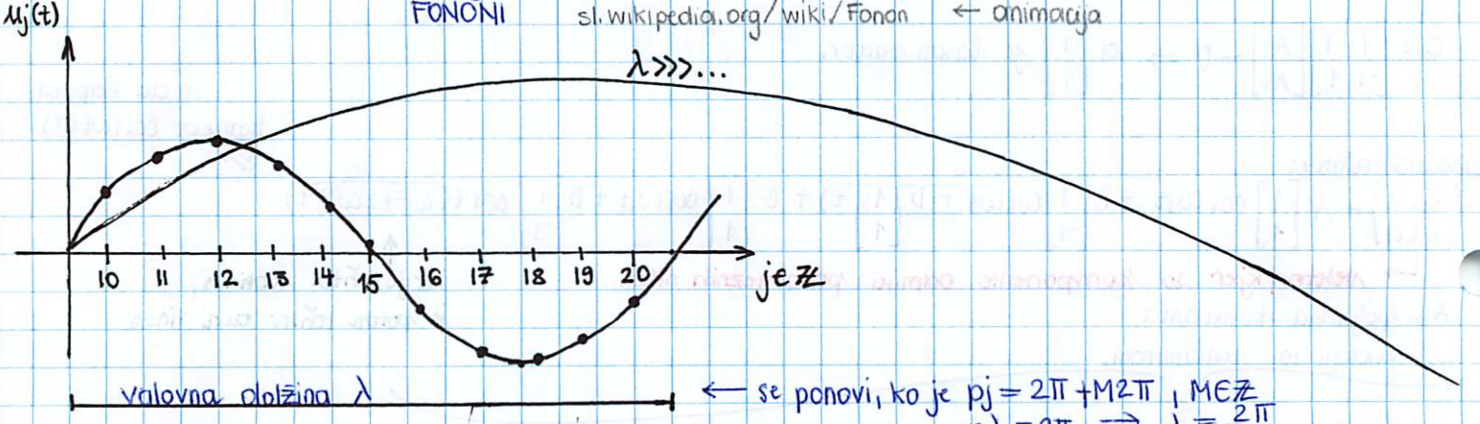
$p = ? \in \mathbb{R}$

$\left.\begin{matrix} A = ? \\ \varphi = ? \end{matrix}\right\}$ začetna pogoja

Če je $p=0$, pomeni, da mesto nihala ni pomembno, ker vsa nihala nihajo enako z frekvenco enega nihala. V našem primeru bi se obroč vrtel, ne da bi se vzmeti krčile oz. raztegnili med tem vrtenjem. Tj. $p=0 \implies \omega_p = 0$. (Goldston)

nihanje s frekvenco 0 je le translacija.

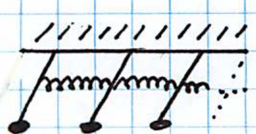
FONONI sl.wikipedia.org/wiki/Fonon ← animacija



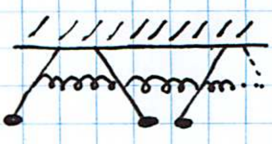
← se ponovi, ko je $pj = 2\pi + M2\pi$, $M \in \mathbb{Z}$

$$p\lambda = 2\pi \implies \lambda = \frac{2\pi}{p}$$

$$p = \frac{2\pi}{\lambda}$$

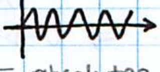


najnižja frekvenca, saj je enaka 0



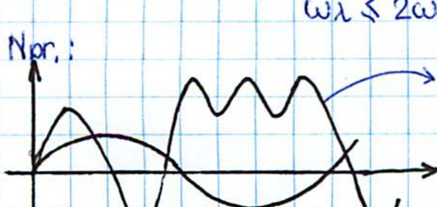
najvišja frekvenca : $\omega_\lambda = 2\omega_0\left|\sin\frac{\pi}{\lambda}\right|$ (vsa nihala nihaja s frekvenco ω_λ)

$\omega_\lambda \leq 2\omega_0$ ($\left|\sin\frac{p}{2}\right| \leq 1$)



← absolutna vrednost pa zato, ker je lahko val naprej in nazaj

Npr.:



$u_j(t)$ = poljubna linearna kombinacija rešitev homogene enačbe

$$u_j(t) = A_p\cos(pj-\omega_p t) + A_{p_1}\cos(p_1 t - \omega_{p_1} t) + A_{p_2}\cos(p_2 t - \omega_{p_2} t) + \dots + \sin\dots$$

$$= \sum_p A_p\cos(pj - \omega_p t) \;\cdots\cdots\; \int A_p\cos(pj-\omega t)\,dp \quad \text{za } \forall p \in \mathbb{R}$$

$A_p \rightarrow u_j(0)$ oz. A_p sledijo iz oblike krivulje ob času $t=0$

Če imamo končno nihal (nam prvi i zadnji zakomplicirajo enačbe), potem je $p \in \mathbb{Q}$ (če $N = N_0 < \infty$)