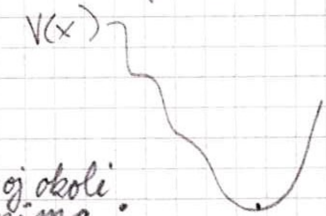
"Je treba svoj napor uporabit, de razumete." Ramšak

Harmonski oscilator

je kot nihalo v klasični fiziki – POMEMBNO

- Ker ga srečaš na vsakem koraku. Ko imaš minimum potenciala, razviješ. In dobiš, ka kšna so male nihanje v paraboličnem potencialu
- Poleg tega je to TOČNO REŠLJIVO.



razvoj okoli minima:

$V(x) = V_0 + \frac{1}{2}kx^2 + \mathcal{O}(x^4)$

$$H = -\frac{\hbar^2 \partial^2}{2m\partial x^2} + \frac{1}{2}kx^2 \quad ; \quad H\psi = E\psi \text{ ali pa } H|\psi\rangle = E|\psi\rangle$$

$$= \frac{p^2}{2m} + \frac{1}{2}kx^2$$

ni isto, ampak je pa isto ü

Enačbe v standardni obliki:

$$\ddot{x} + \omega^2 x = 0$$

$$\omega^2 = \frac{k}{m} \qquad x(t) = A\cos\omega t + B\sin\omega t$$

Zato želimo naš problem v standardno obliko prepisat:

$$H = \hbar\omega(a^+a + \frac{1}{2})$$ → standardna oblika za harmonični oscilator

Kako do tega?

najprej: v brezdim obliki: $H = \left(-\frac{1}{2}\frac{x_0\partial^2}{\partial x^2} + \frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$

kjer: $x_0 = \sqrt{\frac{\hbar}{\omega m}}$ → to ti ni treba na pamet znat

in če vzamemo:

$$a = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} + \frac{x_0\partial}{\partial x}\right)$$ → anhilacijski operator

$$a^+ = \frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - x_0\frac{\partial}{\partial x}\right)$$ → kreacijski operator

Ta je k a hermitsko adjungiran

Če ne vpeljem brezdim. spr.:

$$a = \frac{m\omega x + ip}{\sqrt{2m\omega\hbar}}$$

$$a^+ = \frac{m\omega x - ip}{\sqrt{2m\omega\hbar}}$$

$$x = \sqrt{\frac{\hbar}{2m\omega}}(a + a^+) = x^+$$

$$p = -i\sqrt{\frac{\hbar m\omega}{2}}(a - a^+) = p^+$$

$$\boxed{H = \frac{p^2}{2m} + \frac{1}{2}kx^2 = \hbar\omega\left(a^+a + \frac{1}{2}\right)}$$

$a^+a = \hat{n}$ → number operator
= operator števja

$\hat{n} = \hat{n}^+$

DN! Pokaži, da $[a, a^+] = 1$
veš: $[x, p] = i\hbar$

$\hat{n} = a^+a$ za najti lastne vr. $\hat{H}$ zadošča, če najdemo lastne vrednosti $\hat{n}$.

$\hat{n}\,\psi_\nu(x) = \nu\,\psi_\nu(x)$

$\hat{n}\,|\nu\rangle = \nu|\nu\rangle \quad / \langle\nu|$

$\langle a\psi_\nu | a\psi_\nu\rangle$

$\langle\nu|\hat{n}|\nu\rangle = \nu\langle\nu|\nu\rangle = \langle\nu|a^+a|\nu\rangle = \langle a\nu|a\nu\rangle \geq 0 \Rightarrow \nu \geq 0$

Lastna vrednost tega operatorja $\hat{n}$ je poz.

Poglejmo če je lahko $\nu = 0$:

če $\nu = 0$, potem: $\langle a\psi_\nu|a\psi_\nu\rangle = 0$ → torej je $a\psi_\nu = 0$

$a|\nu\rangle = 0 = a|0\rangle$

Tudi: $a^+a\psi_0 = \hat{n}\psi_0 = 0$ ← $a\psi_0 = 0$ = za osnovno stanje $|\nu\rangle$, je $a|\nu\rangle = 0$. operator a

$|\nu\rangle = |0\rangle$

"Ni delca, ki ga ne bi bilo." Bregar

$$\frac{x_0}{\sqrt{2}}\left(\frac{\partial}{\partial x} + \frac{x}{x_0^2}\right)\psi_0(x) = 0$$

$$\frac{\partial\psi_0}{\partial x} = -\frac{x}{x_0^2}\psi_0$$

$$\int\frac{d\psi_0}{\psi_0} = -\int\frac{x}{x_0^2}dx$$

$$\boxed{\psi_0(x) = C e^{-\frac{x^2}{2x_0^2}}}$$

v x reprezentaciji

= osnovno stanje, ki ima lastno vrednost k operatorju a 0.

$C = \frac{1}{\sqrt{\sqrt{\pi}\,x_0}}$ da je $\langle 0|0\rangle = 1$

$\langle x|0\rangle = \psi_0(x)$

Poglejmo: $[\hat{n}, a^+] = [a^+a, a^+] = a^+\overbrace{[a, a^+]}^{1} + \overbrace{[a^+, a^+]}^{0}a = a^+$

$[\hat{n}, a] = [a^+a, a] = a^+\underbrace{[a, a]}_{0} + \underbrace{[a^+, a]}_{-1}a = -a$

$\hat{n}\psi_\nu = \nu\psi_\nu$

$\hat{n}\, a^+\psi_\nu = (a^+\hat{n} + \underbrace{a^+}_{[\hat{n},a^+]})\psi_\nu = (\nu+1)\overbrace{a^+\psi_\nu}^{\text{sorazmerno je } \psi_{\nu+1}}$

$[\hat{n}, a^+] = \hat{n}a^+ - a^+\hat{n}$

↳ $a^+\psi_\nu$ je lastni vektor operatorja $\hat{n}$. Ima lastno vrednost $\nu+1$.

$\langle a^+\psi_\nu | a^+\psi_\nu\rangle = \langle\psi_\nu|\underbrace{aa^+}_{a^+a+1}|\psi_\nu\rangle = \langle\psi_\nu|(\overbrace{a^+a}^{\hat{n}}+1)|\psi_\nu\rangle =$

$\underbrace{[a, a^+]}_{1} = aa^+ - a^+a$

$= (\nu+1)\langle\psi_\nu|\psi_\nu\rangle$

če: $\langle\psi_\nu|\psi_\nu\rangle = 1$

Potem: $\langle a^+\psi_\nu | a^+\psi_\nu\rangle = \nu+1$

Oziroma normiramo stanje: $|\psi_{\nu+1}\rangle = \frac{1}{\sqrt{\nu+1}}\, a^+|\psi_\nu\rangle$ ker zdaj je $\langle\psi_{\nu+1}|\psi_{\nu+1}\rangle = 1$

npr: $\hat{n}|1\rangle = |1\rangle$

$|1\rangle = a^+|0\rangle$

$$\psi_1(x) = \underbrace{\frac{1}{\sqrt{2}}\left(\frac{x}{x_0} - x_0\frac{\partial}{\partial x}\right)}_{a^+}\psi_0(x)$$

$$H\psi_1(x) = \hbar\omega(\underset{\hat{n}}{1} + \tfrac{1}{2})\psi_1(x)$$

Torej: ν naj bo $\mathbb{N}$. in $\nu = n$

↳ $\psi_n(x) = \frac{1}{\sqrt{n}}\, a^+\psi_{n-1}(x)$ za $n>1$ za $n=0$ pa že vemo kakšen je ψ_0.

$\psi_2(x) = \frac{1}{\sqrt{2}}\, a^+\psi_1(x) = \frac{1}{\sqrt{2}}\frac{1}{\sqrt{1}}\, a^+a^+\psi_0$

In tako rekurzivno naprej:

$\psi_n(x) = \frac{1}{\sqrt{n!}}\, a^{+n}\psi_0(x)$ $\quad\langle n|n\rangle = 1$

Poglejmo: $\langle n_1|n_2\rangle = \frac{1}{\sqrt{n_1!}\sqrt{n_2!}}\langle 0|a^{n_1}a^{+n_2}|0\rangle$

Ker je $\hat{n}$ hermitski operator so lastne vr. ~~ort~~ realne.

Preverimo še ortogonalnost lastnih vr.

za: $n_2 > n_1$

$a^{n_1}a^{+n_2} = a^{n_1}\underbrace{(a^{+n_1})\,a^+\ldots a^+}_{a^{+n_2}}$ →

Stanje $|0\rangle$ pa se ne da več znižat, zato pride ničelno stanje, ko narediš $a|0\rangle$

vemo: $a|0\rangle = 0$ in $\langle 0|a^+ = 0$

$\langle 0|a^{n-1}(a^{+n-1})a^+ \dots a^+|0\rangle = 0$ → Premisli! So ortogonalni

Poglejmo:

$\hat{n}a\psi_\nu = (a\hat{n} - a)\psi_\nu = (\nu-1)a\psi_\nu$ → $a\psi_\nu \propto \psi_{\nu-1}$ → niže lastno vrednost.

podobno kot prej pridemo do:

$$\psi_n(x) = \frac{1}{\sqrt{n+1}}a\psi_{n+1}$$

Ali lahko ν ni $\mathbb{N}$? Je lahko recimo $\frac{3}{4}$?

Poglejmo: $\nu = n + \alpha$ $\quad 0<\alpha<1$

$\Rightarrow \hat{n}\psi_\nu = (n+\alpha)\psi_\nu$

drži, da: $\hat{n}a\psi_\nu = ((n-1)+\alpha)a\psi_\nu$

⋮

in spet delujemo z a in se lastna vrednost na vsakem koraku znižuje za 1. In pridem do neg. lastne vrednost po nekem številu korakov. To pa ni možno. Torej, α je lahko le $\in \mathbb{Z}$.

Torej:

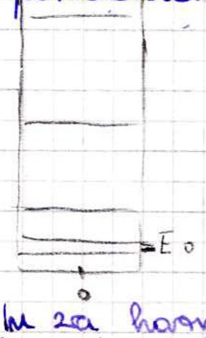
$$E_n = \hbar\omega\left(n+\tfrac{1}{2}\right)$$

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}\,x_0 2^n n!}} H_n\left(\frac{x}{x_0}\right) e^{-\frac{1}{2}\left(\frac{x}{x_0}\right)^2}$$

rešitve niso degenerirane. So Hermitovi polinomi.

imajo lepe lastnosti

∞ potencialna jama:



$E \propto n^2$

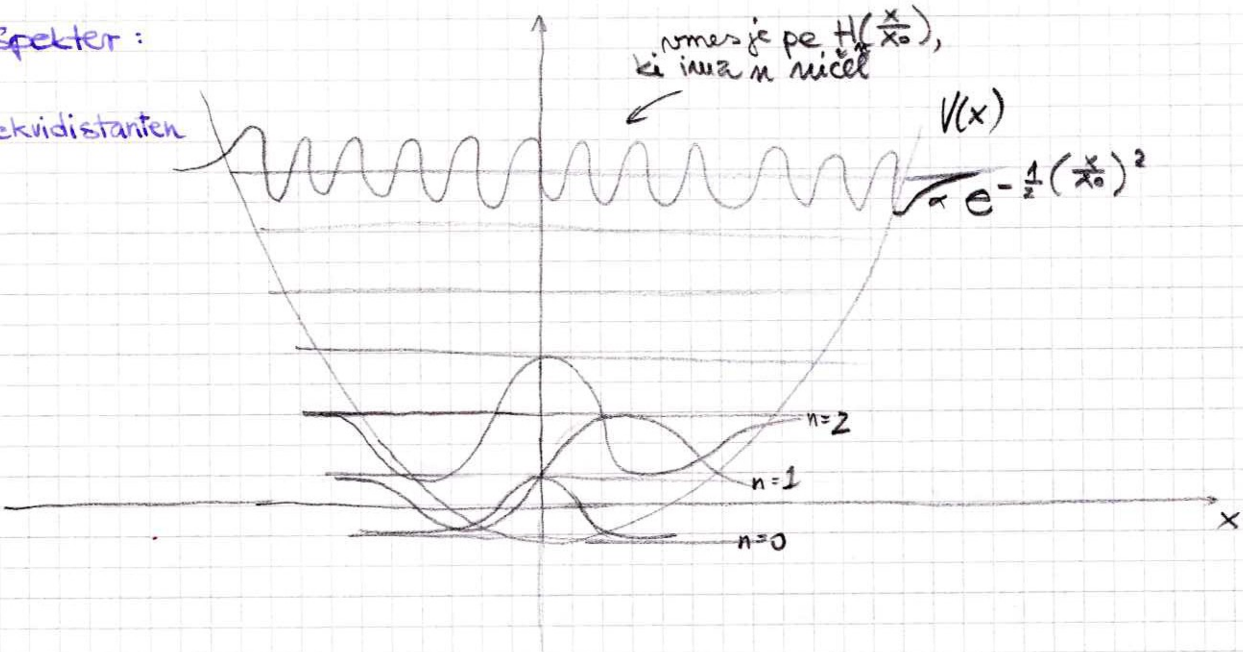
→ Funkcije so sode in lihe, se izmenjujejo ⇒ ker je $V(x)$ sod.

In za harmonski oscilator je tudi izmenjuje liho, sodo ... stanje ker je potencial sod.

↳ Energije so ekvidistantne pri harmoničnem oscilatorju.

Spekter:

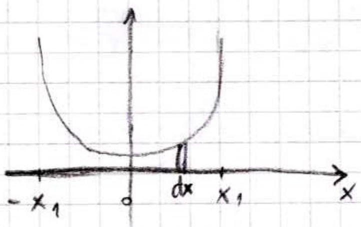
je ekvidistanten



za $n \gg 1$: Klasična limita

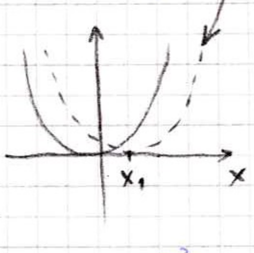
$x = x_1 \cos\omega t \qquad v = -\omega x_1 \sin\omega t$

$$dP = \underbrace{C\frac{dx}{v}}_{\text{verjetnost, da ga ujameno pri } dx} = D\frac{dx}{\sqrt{x_1^2 - x^2}},$$



Koherentno stanje = lastno stanje (osnovno) premaknjenega oscilatorja

Glejmo premaknjen oscilator



Poiščemo osnovnostanje premaknjenega oscilatorja.
Kaj se zgodi, če v prvotni oscilator damo delec z osnovnim stanjem, kot bi ga imel v premaknjenem?

$H = \frac{p^2}{2m} + \frac{1}{2}k(x - x_1)^2$

Rešitve premaknjenega oscilatorja želimo zapisati z nepremaknjenimi.

$x = \frac{x_0}{\sqrt{2}}(a + a^+)$

$a \to a + \alpha$ (število; treba ga je premaknit)

$\frac{\partial}{\partial x} = \frac{\partial}{\partial \tilde{x}}$ ker $\tilde{x} = x - x_1$

$\tilde{a}\psi_\alpha$

$(a - \alpha)\psi_\alpha = 0$

$a\psi_\alpha = \alpha\psi_\alpha$

star problem: $\langle x \rangle = 0$

premaknjen: $\langle novo | x | novo \rangle = x_1$

pričakovana vrednost x za to novo stanje, osnovno v premaknjenem

$\overset{a+\alpha}{\tilde{a}}|\alpha\rangle = \alpha|\alpha\rangle$

$|\alpha\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$

$\langle n | a | \alpha \rangle = \alpha \langle n | \alpha \rangle$

$\frac{1}{\sqrt{n!}}\langle 0 | a^{n+1} | \alpha \rangle = \frac{1}{\sqrt{n!}}\alpha^{n+1}\langle 0 | \alpha \rangle = \alpha \langle n | \alpha \rangle$

$$|\alpha\rangle = \sum_m |n\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\alpha^n |n\rangle\langle 0|\alpha\rangle$$ $\frac{a^{+n}|0\rangle}{\sqrt{n!}}$

$$= C\underbrace{\sum_{n=0}^{\infty} \frac{(\alpha a^+)^n}{n!}}_{\text{eksp. funkcija}}|0\rangle = Ce^{\alpha a^+}|0\rangle$$

normiramo:

$$\langle\alpha|\alpha\rangle = 1 = |C|^2\sum_n \frac{|\alpha|^{2n}}{n!} = |C|^2 e^{|\alpha|^2} \rightarrow \text{dobiš } C$$

$$\boxed{|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2+\alpha a^+}|0\rangle}$$

Pa poglejmo zdaj pričakovano vrednost x za ta $|\alpha\rangle$

$$\langle x\rangle = \frac{x_0}{\sqrt{2}}\langle\alpha|(a+a^+)|\alpha\rangle = \frac{x_0}{\sqrt{2}}\underbrace{(\alpha+\alpha^*)}_{2\mathrm{Re}\,\alpha} = x_1$$

$$\langle\alpha|a|\alpha\rangle = \alpha$$

$$\langle\alpha|a^+|\alpha\rangle = \alpha^*$$

ob : $t=0$ imamo torej $|\alpha\rangle$

zanima nas $t>0$: $|\alpha(t)\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-i\frac{E_n}{\hbar}t}|n\rangle$ $\quad E_n = \hbar\omega n + \frac{\hbar\omega}{2}$

$$= e^{-\frac{1}{2}|\alpha|^2}\sum\left(\frac{\alpha e^{-i\omega t}}{\sqrt{n!}}\right)^n e^{-\frac{i\omega}{2}t}|n\rangle$$

$$\alpha(t) = \alpha e^{-i\omega t}$$

$$|\alpha(t)\rangle = e^{-\frac{1}{2}|\alpha(t)|^2 + \alpha(t)a^+}|0\rangle e^{-\frac{i\omega}{2}t}$$

in pričakovana vrednost:

$$\langle x\rangle = \frac{x_0}{\sqrt{2}}(\alpha(t)+\alpha^*(t)) = \frac{x_0}{\sqrt{2}}2|\alpha|\cos(\omega t+\delta)$$

kjer: $\alpha = |\alpha|e^{i\delta}$

torej: pričakovana vrednost harmonsko niha.

to ustreza klasični rešitvi

Širina ostaja enaka. Srednja lega harmonično niha.

za več dimenzij ali več delcev:

$$H_i = \frac{p_i^2}{2m} + \frac{1}{2}k_i x_i^2$$

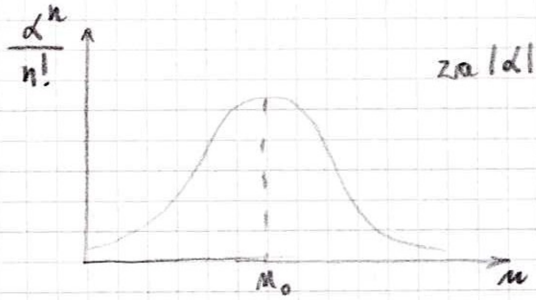
$$i = 1 \dots N$$

da se $\rightarrow \sum_{i=1}^{N}\hbar\omega_i(a_i^+a_i+\frac{1}{2})$

$$[a_i, a_j^+] = \delta_{ij}$$

Odmik x_1 je odvisen od α. $\rightarrow$

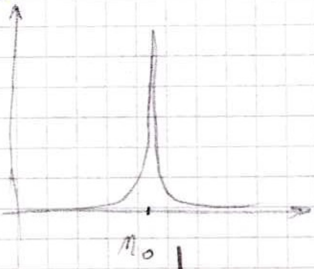
$\langle E \rangle = \frac{1}{2} k x_1^2$ → odmik



za $|\alpha| \gg 1$

Klasična limita

za $n \to \infty$
$\alpha \to \infty$



Delec je natančno lokaliziran in miru.

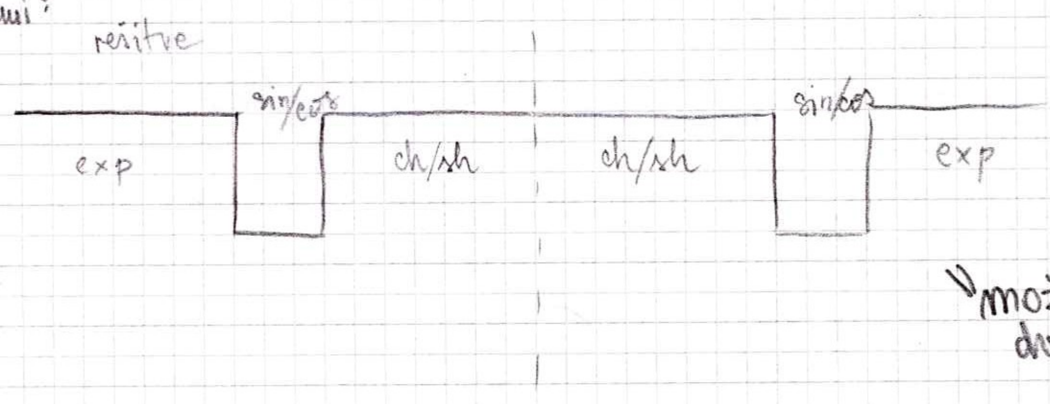
DVONIVOJSKI SISTEM

↳ mnogo primerov se da s tem aproksimirati

$|\psi_1\rangle$, $|\psi_2\rangle$

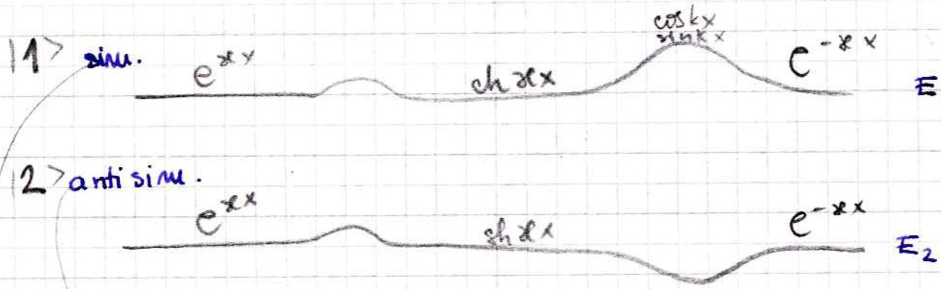
$|1\rangle$, $|2\rangle$

2 pot. jami:



možni sta dve stanji

simetrične rešitev ima nižjo energijo. Antisim. ima ničlo.



Da se naredit lin. komb. obeh stanj: $|\psi_\pm\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm |2\rangle)$

ψ_+ → delec v **stanju** ψ_+ → hribček bo v levi jami

ψ_2 → delec v **stanju** ψ_- → hribček bo v desni

naj bo: $|\psi_0\rangle = |\psi_+\rangle$ → naboj damo v levo jamo $= |\psi(t)\rangle\Big|_{t=0}$

↳ Med jamama je veliko prostora.

Kakšna je verjetnost, da gre delec v $|\psi_-\rangle$

$$|\psi(t)\rangle = \frac{e^{i\omega_1 t}|1\rangle + e^{-i\omega_2 t}|2\rangle}{\sqrt{2}}$$