University *of Ljubljana*
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Seminar

The Weak Measurement in Quantum Mechanics

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Abstract

The topic of this seminar is the weak measurement and its corresponding weak value. The normal and double Stern-Gerlach experiment is explained, the latter being used as an example of how to conduct the weak measurement and what we get as a result. The difference between the weak and the strong measurement is pointed out and for the end, an experiment where a transverse wave function of a photon was being measured directly using the weak measurement is presented.

Contents

1 Introduction

In the classical world that we experience in everyday life, everyone can imagine what a measurement is. You take a measuring tool, obtain the desired quantity and that's it. The measured object, the measuring device and the person conducting the measurement (student) are the same as they were before. Well, perhaps the student's mood changes, depending on whether or not he got what he was hoping for. But things get a little bit complicated when we go down to the quantum world. Suddenly, our observation (measurement) of a particle disturbs it. As it is common to say, the wave function collapses into the state we measured the particle to be in. From that we learn nothing about what the wave function was before the measurement. It was comprised of many states with corresponding probabilities. Once we measured it in one particular state, the others have no meaning. In other words, their probability is zero, because the one's we've just measured became 1. In the moment of the measurement, the particle is in that particular state and no other, hence the probability 1. The quantum mechanics shows us, how to calculate the probable result of any measurement that one wishes to carry out. We can calculate the average value of an observable A by writing

$$\overline{A} = \int \psi^* A \psi dx, \tag{1.1}$$

where ψ is the wave function of the system under investigation [1].

The basic idea of the weak measurement is that the interaction (or disturbance) between the measuring apparatus and the observed system or particle is so weak, that the wave function does not collapse but continues on unchanged. In other words, a weak measurement is one in which the coupling between the measuring device and the observable to be measured is so weak that the uncertainty in a single measurement is large compared with the separation between the eigenvalues of the observable [2].

2 The difference between strong and weak measurement

The easiest way to illustrate this difference is with some help of the well known Stern-Gerlach experiment. Let us refresh our memory on that topic. When particles travel through an inhomogeneous magnetic field, they get deflected either in the same or the opposite direction, in which the field is inhomogeneous. The deflection depends on the spin of the particle. Given enough time (far from magnetic field), the initial wave packet will separate into two packets, one with the spin $\frac{1}{2}$ and the other with spin $-\frac{1}{2}$. As shown in the figure 1, we would get two spots on the screen, one above and the other below the place where the straight line of the initial path of the wave packet would meet the screen.

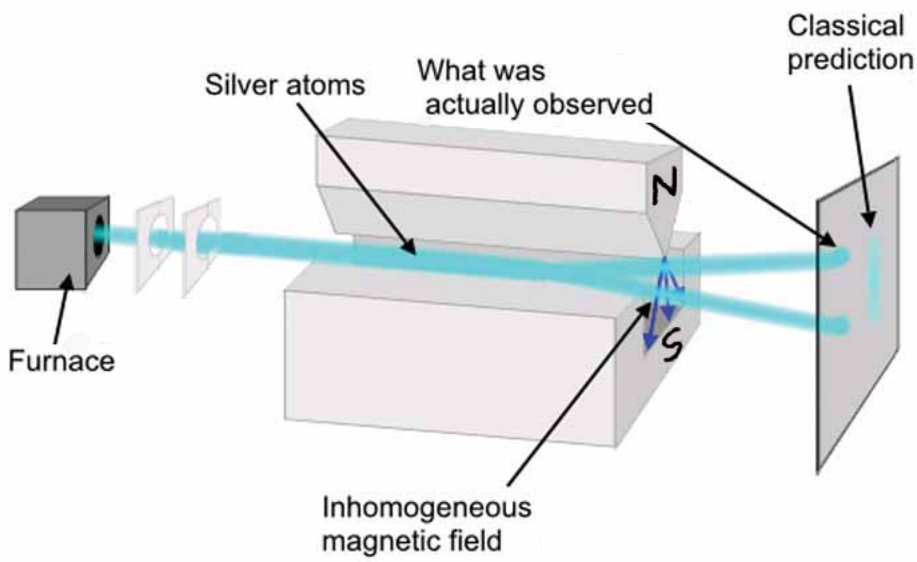

Figure 1: Standard Stern-Gerlach experiment with silver atoms [3].

What we actually do with this experiment is measure the spin of a particle. One of those spots are particles with spin up and the others with spin down. To determine exactly which is which we must know how the magnetic field is oriented. Let's say, that the ones that deflected up are spin 1/2 particles. To be sure, we got only those with spin 1/2, we must let them travel a sufficiently long enough distance, in order to insure they get separated enough. When we achieve that, we can be sure that we have the right ones. When we look only at the top spot and forget about the bottom one, the initial wave function, comprised of spin up and spin down components, collapses. What remains is only spin up component. That is called a strong measurement.

$$|\psi_i\rangle = \psi_\uparrow|\uparrow\rangle + \psi_\downarrow|\downarrow\rangle \tag{2.1}$$

$$|\psi_f\rangle = \psi_\uparrow|\uparrow\rangle \tag{2.2}$$

The distance of the screen in order to get good separation, depends on the strength of interaction. The stronger the interaction, the greater deflection, smaller the distance. Therefore, we can place the screen closer than we would in case of a weaker interaction. If we manage to make the interaction small enough, so it does not cause any or very little deflection, we get the weak measurement. So in the case of a Stern-Gerlach experiment the strength, or should we say weakness of the measurement, is defined by the intensity of the gradient of the inhomogeneous magnetic field, $\frac{\partial B}{\partial z}$ [4].

3 The weak value

In quantum mechanics the interaction Hamiltonian of the standard measurement procedure is

$$H = -g(t)qA, \tag{3.1}$$

where $g(t)$ is a normalized function with a support near the time of measurement, q is a canonical variable of the measuring device with it's conjugate momentum p and A is the variable we desire to measure. Ideally, the initial state of the measuring device is Gaussian in the q as well as in the p representation: $\phi(q) = \left(1/(\sqrt{2\pi}\sigma)\right)^{1/2} \exp\left(-q^2/(4\sigma^2)\right)$. What we will be focusing on is weak interaction, while having a well defined initial state and also a well defined final state, or should we say a desired final state. This method is called post-selection. After our interaction we have

$$|\psi(t)\rangle = \exp\left(\frac{-iHt}{\hbar}\right)|\psi_i\rangle|\phi\rangle, \tag{3.2}$$

and because our interaction is weak, we can assume that the term in the exponent is small or close to zero. By expanding it into a series, inserting H into our term and noticing, that q and A are actually operators, we get

$$|\psi(t)\rangle = |\psi_i\rangle|\phi\rangle - \frac{igt}{\hbar}\hat{A}|\psi_i\rangle\hat{q}|\phi\rangle - ... \tag{3.3}$$

Applying the final state from the left, that is post-selecting our system, the term yields:

$$\langle\psi_f|\exp\left(\frac{-iHt}{\hbar}\right)|\psi_i\rangle|\phi\rangle = \langle\psi_f|\psi_i\rangle|\phi\rangle - \frac{igt}{\hbar}\langle\psi_f|\hat{A}|\psi_i\rangle\hat{q}|\phi\rangle, \tag{3.4}$$

where we took only parts of the lowest order in gt. After we re-normalize we get

$$|\phi_{fi}\rangle = |\phi\rangle - \frac{igt}{\hbar}\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\hat{q}|\phi\rangle, \tag{3.5}$$

where $|\phi_{fi}\rangle$ labels the final state of the measuring device. At this point we could calculate expectation values for $\hat{q}$ and $\hat{p}$ and we would get that they are proportional to either imaginary or real part of some new quantity called weak value of the operator (or variable) $\hat{A}$, which is defined as [4]:

$$\hat{A}_w \equiv \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}. \tag{3.6}$$

It is known that the uncertainty of p (Δp) for each of the measuring devices is much bigger than the measured value. In that case, the weak value can not be distinguished. For the upper calculations to be valid, the following must apply:

$$\sigma \ll \max_n \frac{|\langle\psi_f|\psi_i\rangle|}{|\langle\psi_f|\hat{A}^n|\psi_i\rangle|^{1/n}}, \tag{3.7}$$

where $0 \leq n \leq N$ and N is our ensemble size. From $\Delta p = \frac{1}{2\sigma}$ we see, that

$$\Delta p \gg A_w. \tag{3.8}$$

When we take an ensemble of N devices, the uncertainty of the average of p is decreased by the factor of $1/\sqrt{N}$. So if N is large enough, we can achieve $(1/\sqrt{N})\Delta p \ll A_w$. Only then can the weak value be ascertained with arbitrary accuracy [4].

Probably the most interesting thing about the weak value is, that it is not bounded by the minimal and the maximal eigenvalues of variable A. A_w can actually be anything. It is also complex. As we can see from equation 3.6, the weak value strongly depends on how the initial and final states are oriented. If they are close to being orthogonal, then the denominator is close to zero and therefore the weak value increases. So we see that it is the final and initial states and their orientation that guides the weak value. The probability has only a minor role.

If we understand the measurement as a coupling between the particle and the measuring device or it's pointer, we can give some meaning to real and imaginary part of A_w. The pointer's position shift is proportional to $\mathrm{Re}(A_w)$ and it receives a momentum kick proportional to $\mathrm{Im}(A_w)$.

4 The weak measurement

When considering a weak measurement, one must take into account several factors in order for it to succeed. The main is of course the weakness of the interaction itself. The criteria for one interaction to be considered weak are written in equations 3.7 and 3.8. The other factors that contribute to a successful weak measurement will be explained later on an example. Usually the measurement is conducted in two parts, using the post-selection method, like described before. The point is to have a well defined initial state and also a desired final state. First we measure weakly and then conduct a strong measurement, with which we actually apply our post-selection. If we tuned our system just right, we should be able to read out the weak value and thus learn something about our system. The best way to better understand the process of weak measurement, we shall examine an example made by Aharonov, Albert and Vaidman in their article from 1988 [4].

4.1 Weak measurement of the z component of a spin-$\frac{1}{2}$ particle

The measurement of a spin is done by a Stern-Gerlach device, just like the one described in section 2. The difference here is, that for us to measure the weak value, we must have two Stern-Gerlach devices, perpendicular on each other. The first one will measure the spin weakly in the z direction. The requirement of weakness is fulfilled by making the gradient of the magnetic field sufficiently small. The second device conducts a strong measurement of spin in the x direction. This measurement splits the beam into two beams corresponding to the two values of σ_x. We keep only the one with $\sigma_x = 1$, which continues to move freely to the screen. We must ensure that the beam has split enough, so that we can distinguish between two different values of σ_x. That is done by placing the screen at the right distance. Not only for the separation in the x direction, but for the displacement in the z direction as well. It must be far enough, so that δz will be larger than the initial uncertainty Δz.

Note that when we talk about the z direction, we mention displacement and in the x direction separation. That is not a mistake. Due to the weakness of the interaction, the separation does not occur in the z direction. The gradient of the magnetic field was not strong enough to effectively separate the two σ_z values. This weak measurement causes the spatial part of the wave function to change into a mixture of two slightly shifted functions in the p_z representation, correlated to the two values of σ_z [4].

Our beam of particles is moving in the y direction with a well defined velocity, or should we say momentum. The particles are initially localized in the x-z plane and have their spins pointed in ξ direction. That is what we call a well defined initial state. And if we remember something from before, this well defined initial state is very important for our weak value and post-selection.

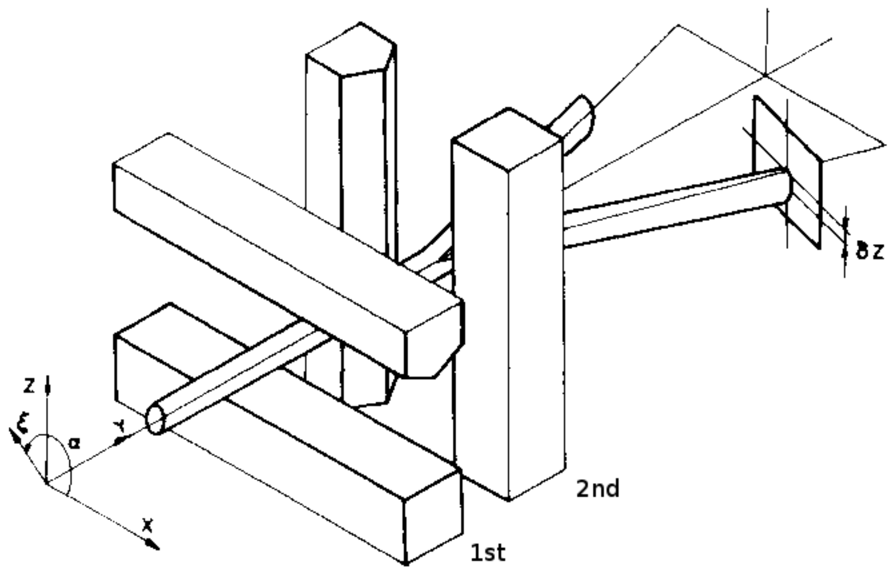

Figure 2: Here we can see the layout of the experiment. The first SG device measures the spin weakly in the z direction and the second one splits the beam according to the corresponding σ_x. On the screen we observe a displacement in the z direction, δz. From this we can calculate the weak value, as will be explained below [4].

The most interesting thing happens on the screen. What we are interested in is the displacement of the wide spot in the z direction. This happens due to our weak measurement with the first SG device. From that shift, we can calculate our weak value:

$$\sigma_{z,w} = \frac{\langle \uparrow_x | \sigma_z | \uparrow_\xi \rangle}{\langle \uparrow_x \mid \uparrow_\xi \rangle} = \tan\frac{\alpha}{2}, \tag{4.1}$$

where α is the angle between x axis and the ξ direction.

Let's take a look at a brief mathematical description of the experiment [4]. The particles of mass m, magnetic moment μ and average momentum p_0 in the y direction are in an initial state of $|\psi_i\rangle \propto e^{-x^2/4\sigma^2} e^{-y^2/4\sigma^2} e^{-z^2/4\sigma^2} e^{-ip_0 y}(\cos\frac{\alpha}{2}|\uparrow_x\rangle + \sin\frac{\alpha}{2}|\downarrow_x\rangle)$. This state changes under the influence of the first Hamiltonian of the weak interaction:

$$H_1 = -\mu \frac{\partial B_z}{\partial z} z \sigma_z g(y - y_1), \tag{4.2}$$

where $g(y-y_1)$ has a compact support at the location of the weak SG device. The function g is actually a function of time, since $y \cong (p_0/m)t$. From equation 4.2 we can see, that the canonical variable of the equation 3.1 is $q = \mu \frac{\partial B_z}{\partial z} z$. The gradient of the magnetic field is sufficiently weak. As we can see from equation 3.7 and the relation $\Delta p = \frac{1}{2\sigma}$, the requirement for sufficient weakness is

$$\mu |\frac{\partial B_z}{\partial z}| \max\left[|\tan\frac{\alpha}{2}|, 1\right] \ll \Delta p_z = \frac{1}{2\sigma}. \tag{4.3}$$

Our beam continues on from the first SG to the second SG device and there it undergoes an interaction described by the second Hamiltonian: $H_2 = -\mu \frac{\partial B_x}{\partial x} x \sigma_x g(y - y_2)$. This interaction separates the beam into two parts, according to the corresponding values of σ_x. The requirement for the splitting of the beam is

$$\mu |\partial B_x / \partial x| \gg \Delta p_x = \frac{1}{2\sigma}. \tag{4.4}$$

From here on our beam travels freely to the screen. As we mentioned before only the beam with $\sigma_x = 1$ is of interest to us. The wave function just before the collapse on the screen, is approximately [4]:

$$\exp\left[-\sigma^{-2}\left(\frac{p_0}{l}\right)^2 \left(z - \frac{l\mu}{p_0}\frac{\partial B_z}{\partial z}\tan\frac{\alpha}{2}\right)^2\right], \tag{4.5}$$

where

$$\frac{l\mu}{p_0}\frac{\partial B_z}{\partial z}\tan\frac{\alpha}{2} = \delta z. \tag{4.6}$$

Here we recognize our weak value from equation 4.1. It is now clear, that the weak value is responsible for the displacement in the z axis, indeed. Therefore, by measuring the displacement on the screen, we actually measure the weak value of the z component of the spin. It is also clear as predicted, that the weak value is not bounded by the minimal or maximal eigenvalues of σ_z.

4.2 Computer simulation of the experiment

By now it is clear, that the most interesting thing about this experiment is the displacement of the spot on the screen in the z direction. We shall examine this a little further with some help of a computer simulation program made in Mathematica [5]. The program enables us to see what happens on the screen with the wave packet. It has 5 parameters which can be manipulated: t, $px0$, $pz0$, $th0$ and $D0$. Parameter t has the role of time or it can also be interpreted as a distance where we place the screen. It is best to be set on maximum, for the further the screen is, the greater separation. Here I would like to point out, that this program plots the beam with $\sigma_x = -1$ also. The parameter $D0$ stands for the width of the initial Gaussian packet. It must be tuned carefully, for being to small or too large can result in too big expansion of the wave packet, which we do not want. It is best set on 4. Weak interaction of the first SG device is hidden in parameter $pz0 = \mu(\partial B_z/\partial z)\sigma_z$. In the same way is the second strong interaction hidden in $px0 = \mu(\partial B_x/\partial x)\sigma_x$. Finally, the parameter $th0$ stands for the angle between the z axis and the direction of ξ.

First, let's see what we get if we turn off both of the interactions, so that $px0 = pz0 = 0$.

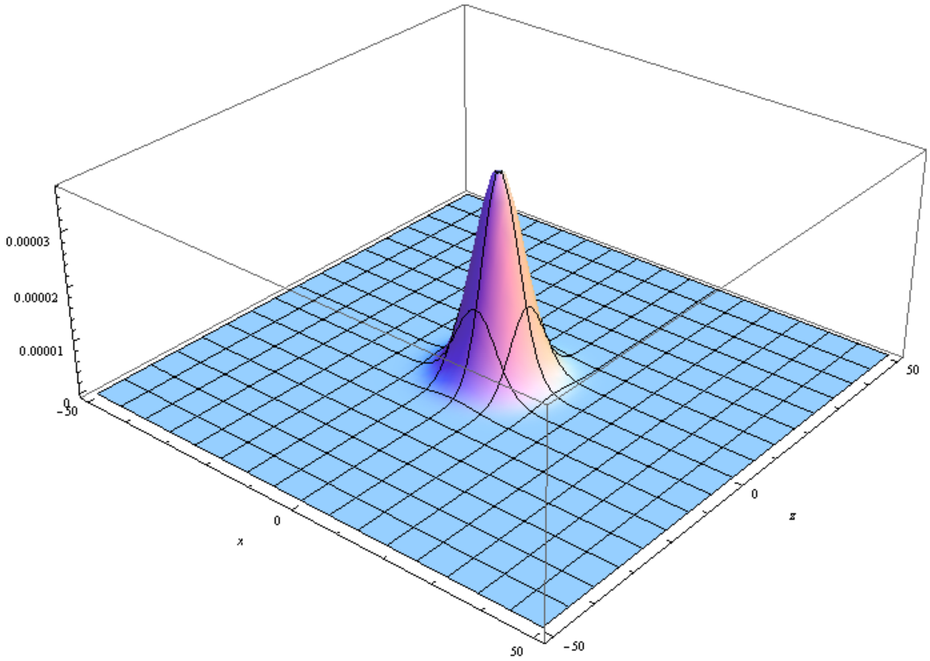

Figure 3: No interaction at all, we get a wave packet at the center of the screen as expected. $t = 10$, $px0 = 0$, $pz0 = 0$, $th0 = -1$, $D0 = 4$.

What we get is nothing special, the wave packet in the center of the screen and it seems that it's width is still quite narrow, so we have good resolution. Now let's try turning on just one SG device, the second one.

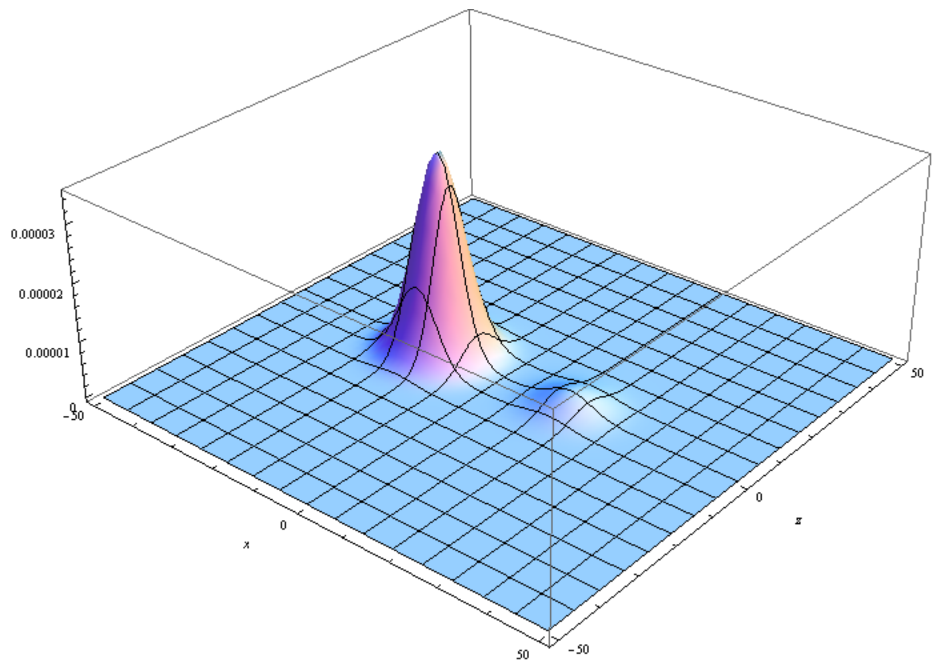

Figure 4: Second SG device turned on. We see, that the beam has separated into two beams. $t = 10$, $px0 = 0.75$, $pz0 = 0$, $th0 = -1$, $D0 = 4$.

It looks like it should, for a standard SG device. We get two peaks. We see that one has much bigger intensity than the other. That is due to the chosen initial direction of ξ, so when they pass through the second SG device, they rather deflect left than right (Figure 2). Just for fun I turned down the time or in other words, brought the screen closer to the SG device. In figure 5 we can see that the beam didn't have enough time to separate, so what we have here is one peak that is about to separate. We can notice the little one moving away at the bottom.

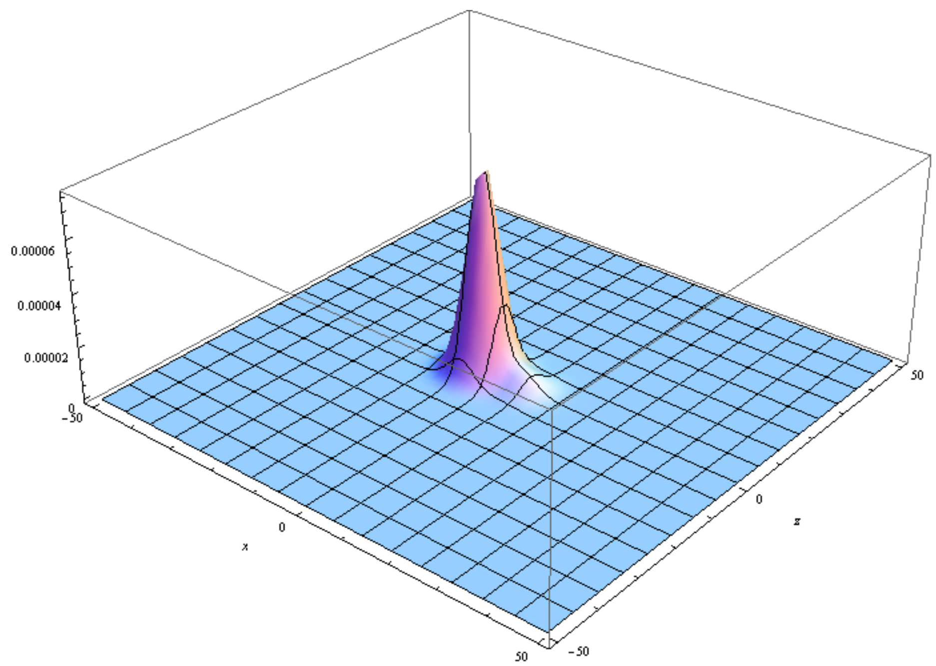

Figure 5: The screen brought closer to the SG device. We see the separation still taking place. $t = 3$, $px0 = 0.75$, $pz0 = 0$, $th0 = -1$, $D0 = 4$.

Now let's see what happens when we turn on our weak measurement. For this purpose we shall increase the strong interaction as well, just to get bigger separation than before.

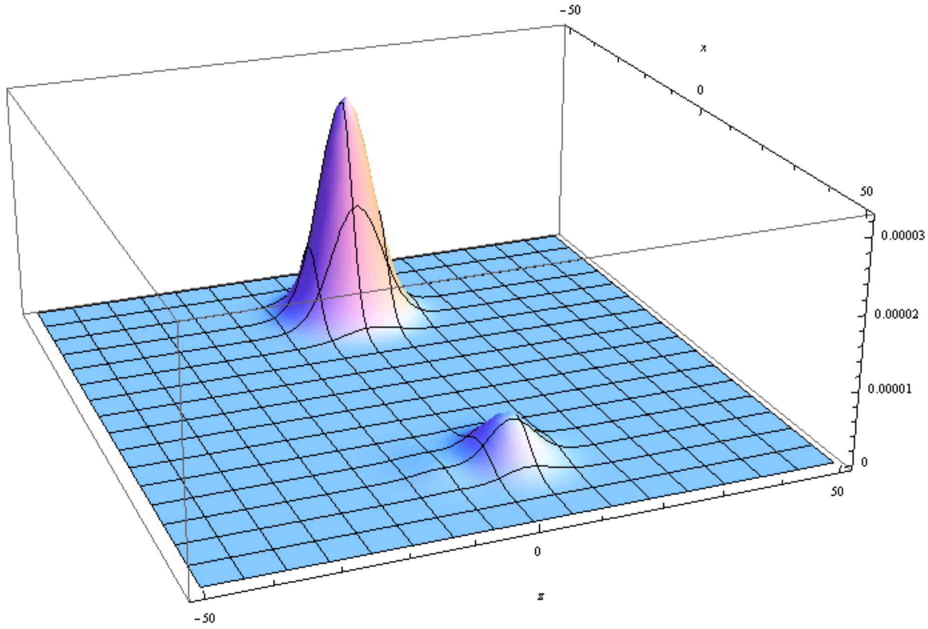

Figure 6: Turning on the weak measurement, the displacement in the z direction is observed. $t = 10$, $px0 = 1.5$, $pz0 = 0.1$, $th0 = -1$, $D0 = 4$.

If we look closely we can see, that the small one, the big one is not of our interest anyway, has moved up the z axis. It has moved for that δz we described earlier, that is proportional to the weak value of the z component of the spin. And what happens if the weak interaction isn't weak enough anymore? That is clearly seen in the figure 7 below. Besides the separation in the x direction, we also get a separation in the z direction. It is a common Stern-Gerlach experiment conducted in sequence.

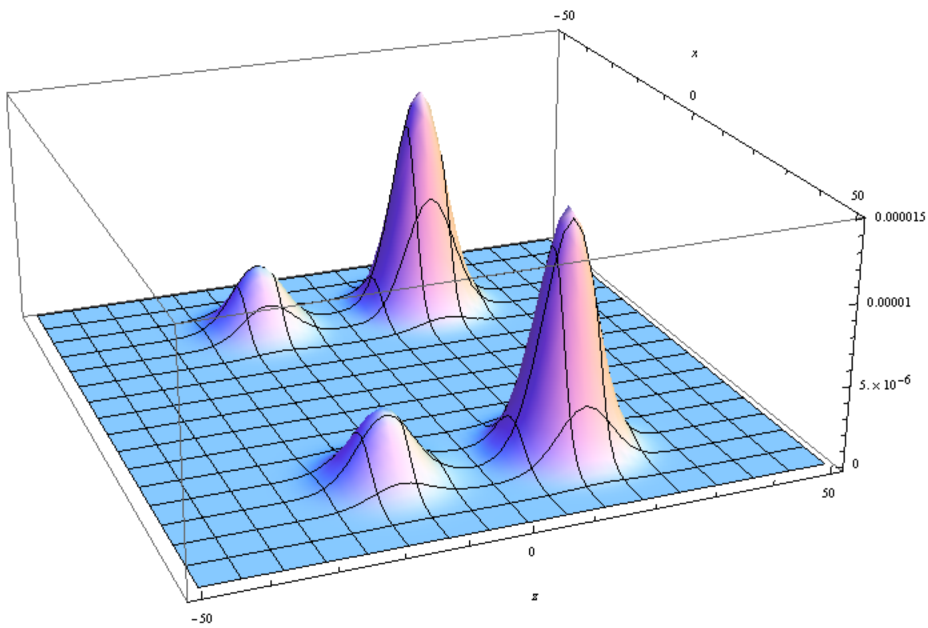

Figure 7: Turning up the gradient of the magnetic field in the z direction causes separation. $t = 10$, $px0 = 1.5$, $pz0 = 0.75$, $th0 = -1$, $D0 = 4$.

This program has shown us, exactly how a system on which we want to conduct the weak measurement behaves. We see that many factors play important roles on how to set up the experiment and how to tune it just right, so that we can get the best results.

5 Practical usage of the weak measurement

I will describe how the transverse quantum wave function of a photon was measured via weak measurement. The experiment was done by Jeff S. Lundeen, Brandon Sutherland,...[6]. The paper was published in 2011, so it is quite recent. I will not go into details on how the optical side of the experiment was carried out due to lack of information provided by the authors. For a deeper understanding the initial polarization should be know, but unfortunately it is not. Also it is my personal opinion that in our case, where the focus is on the weak value and its usage, it is really not that important.

Just like in the experiment we dealt with before, we have some weak measurement, followed by a strong measurement which post-selects our system into the desired state. If we take a look at the equation 3.6 and consider we measure weakly the position ($A = \pi_x \equiv |x\rangle\langle x|$), followed by a strong measurement of momentum p we get

$$\langle \pi_x \rangle_w = \frac{\langle p|x\rangle\langle x|\psi\rangle}{\langle p|\psi\rangle} = \frac{e^{ipx/\hbar}\psi(x)}{\phi(p)}. \tag{5.1}$$

In the case when post-selecting $p = 0$ this simplifies to

$$\langle \pi_x \rangle_w = k\psi(x), \tag{5.2}$$

where $k = 1/\phi(0)$ is a constant, which is eliminated by normalization. From here we can see that in this case, the weak value of position is directly proportional to the wave function of the particle at x. All we have to do now is scan the weak value through x and we should get the whole $\psi(x)$. The setup of the experiment is shown in the figure 8.

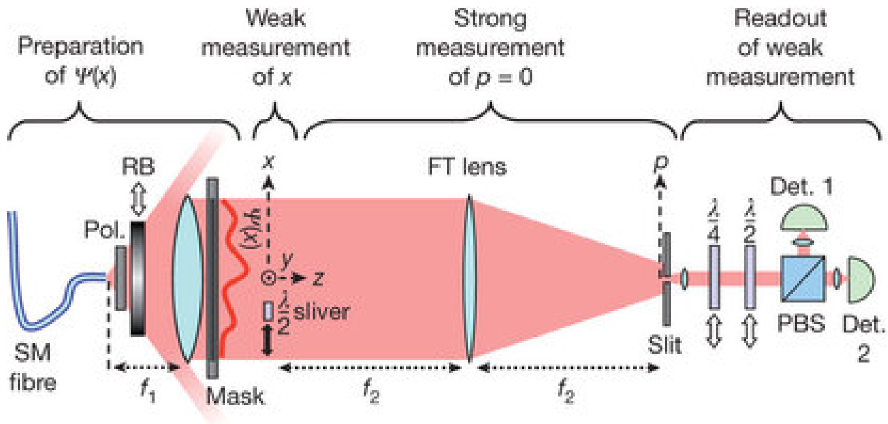

Figure 8: The setup of the experiment for the direct measurement of the transverse wave function of a photon [6].

First we have to prepare the wave function, our initial state. That is done by some photon source (SPDC or attenuated laser beam), SM fibre and some other optical elements. After that the weak measurement of position of a photon follows. This is done by coupling it to an internal degree of freedom of the photon, its polarization. This is done with the $\lambda/2$ sliver. We measure the strength of the measurement with the polarization angle α. If for example α is set to $\pi/2$, one can perfectly discriminate whether a photon had position x because it is possible to perfectly discriminate between orthogonal polarizations 0 and $\pi/2$. This is a strong measurement. Reducing the strength of a measurement corresponds to reducing α [6]. The next step is to post-select only those photons with momentum $p = 0$. This is done using the Fourier transform lens and a slit at around $p = 0$. After that it is just a matter of some more optical elements, $\lambda/4$ and $\lambda/2$ slivers and polarizing beam splitters that channel our photons to either detector 1 or 2. With the ratio of the signal from the detectors we can extract the imaginary and real part of our desired weak value.

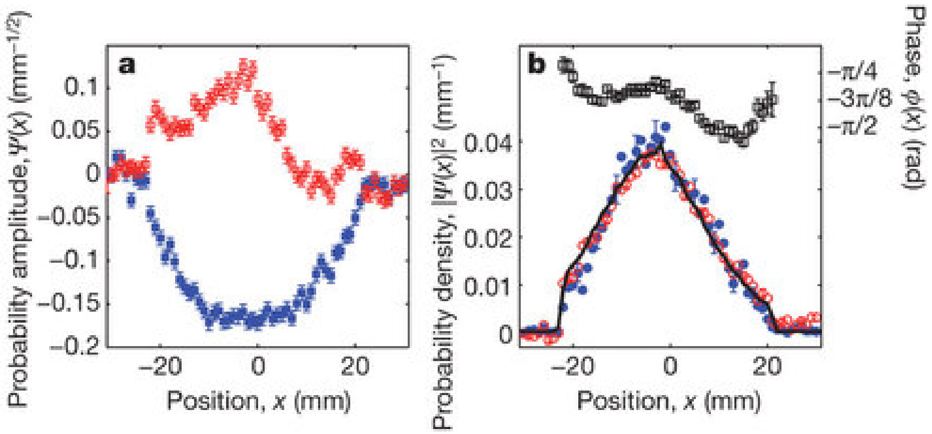

Figure 9: The results of their measurements are plotted. a) $Re\psi(x)$ (solid blue squares) and $Im\psi(x)$ (open red squares). b) phase $\phi(x) = \arctan(Re\psi(x)/Im\psi(x))$ (grey squares, right axis), $|\psi(x)|^2$ (solid blue circles, left axis), prob(x) (solid line), $Re\psi(x)$ without post-selection (open red circles) [6].

From the above graph we can see, that there is a good agreement between calculated $|\psi(x)|^2$ from the weak value and the measured prob(x), that they got by scanning a detector along x. This is a very important result that confirms that this direct measurement works. When there is no post-selection applied, the weak value becomes standard expectation value. The plotted Re$\psi(x)$ after it was renormalized also shows good agreement with prob(x).

This experiment shows what great potential the weak measurement has, indeed. By being able to directly measure the quantum wave function, it shines a whole new light on how we see it and whether it is just a mathematical tool or maybe it also has some deeper physical meaning.

6 Conclusion

To conclude with I would like to point out, that in my opinion the weak measurement is becoming more and more important in the quantum world. Its biggest advantage is, that the disturbance induced on the system is minimal, so it does not collapse the wave function. The weak value has some surprising characteristics such as, that it does not lie in between the minimal and maximal eigenvalues. In this seminar it was shown, how the weak measurement can be performed and how the corresponding weak value can be read out and interpreted. The interesting thing is also the fact, that with the help of the weak measurement, we can measure the quantum wave function directly. I am certain that this is not all that the weak measurement has in store for us, we'll see.

7 References

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