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Feynman's formulation of Quantum mechanics

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Abstract

This paper is about Feynman's path integral formulation of quantum mechanics. It mostly deals with the basic concept of the kernel, both for one–particle and for multiparticle systems. The paper also includes a brief overview of the classical action and the wave function formulation of quantum mechanics, which will be needed throughout our discussion. The treatment of path integrals is separated from Feynman's formulation, so both topics can be presented in a more precise manner. Also, some brief arguments for the reduction of Feynman's formulation of quantum mechanics to classical mechanics in the macroscopic limit are made. Finally, the Schrödinger equation for a single particle in one dimension is derived from Feynman's formulation.

Contents

Introduction

By the end of the 19th century, it had become apparent that a considerable amount of experimental evidence directly contradicted the prevalent physical theories of that time. What ensued was a new kind of physics, which was so much different, we label the post–1900 physics as *modern physics*, as opposed to *classical physics*. Even *Newtonian mechanics*, the flagship of classical physics, was supplanted by *Quantum Mechanics* and *Einstein's theory of relativity*.

In 1900 Max Planck hypothesized that atoms emit energy in "discrete energy elements". This enabled him to work around the problem in the calculation of black body radiation, which was falsely predicted to be infinite by the classical theory of that time. In 1905 Albert Einstein found an explanation for the photoelectric effect by postulating "discrete energy packets" of light called photons. Louis De Broglie was the first to propose that matter behaves as waves in 1923 [1]. This all brought about the idea of particle–wave duality: in some experiments, particles behaved like discrete packets, while in others they behaved like waves.

The first major theoretical framework that appropriately incorporated these ideas came in 1925, when physicists Werner Heisenberg, Max Born and Pascual Jordan developed the *matrix formulation of quantum mechanics*, while Erwin Schrödinger developed the *wave function formulation of quantum mechanics* in 1926 [1]. In matrix mechanics, the physical properties of a particle were interpreted in terms of a time dependent matrix. On the other hand, wave mechanics interpreted particle properties through a function — the wave function — which is the solution of a partial differential equation called the *Schrödinger equation*. At first, that was a puzzling situation, because the two approches to quantum mechanics were seemingly distinct, yet both made accurate

predictions. Soon, Schrödinger managed to show the equivalence of the two approaches, with the final reconciliation coming from Paul Dirac in 1930 [2]. By 1932, the first mathematically coherent theory had been fully developed by mathematician John von Neumann, building on the earlier work done by David Hilbert on function spaces [3]. Von Neumann realized that the wave function could be thought of as a point in an infinitely dimensional Hilbert space of functions. And due to Hermann Weyl's work on Lie groups and representation theory, the matrix form was identified as a representation of the wave function in a certain basis of the Hilbert space. Thus the two formulations of quantum mechanics, the matrix formulation and wave formulation, were proved to be equivalent in a rigorous manner.

In 1948, a thrid approach to quantum mechanics was introduced by the American physicist Richard Feynman, building on the work previously done by Paul Dirac [4]. The new development is now called *the path integral formulation of quantum mechanics*, or alternatively *Feynman's formulation of quantum mechanics*. This approach is an extension of the notion of *classical action*.

In classical mechanics, we can define the *action* as an integral of the *Lagrangian*, a function which "summarizes the dynamics of a system" [5]. The classical trajectory of a particle (or any physical system in general) can be obtained by *Hamilton's principle*, which states that the trajectory is such that the action (as a function of trajectory) is stationary [6]. This formulation of classical mechanics is called *Lagrangian mechanics*, and it's main tool is the *calculus of variations* (a part of mathematical analysis).

Feynman's approach to quantum mechanics is trajectory based. Unfortunately, the trajectory of a physical system is not unique. Instead, every trajectory is taken into account, even the ones that would seem impossible in the classical sense. Then a sum of all these paths is calculated via path integrals [4]. A sum will represent the probabilitiy of finding the particle in a certain location. Although it is far easier to use the Schrödinger equation for calculations in most cases, the importance of Feynman's formulation should not be underestimated, since his approach is of great importance in theoretical physics. Because it uses the notion of action, it has benefits similar to those found in Lagrangian mechanics, when one can have a quick overview of the underlying physics by examining the structure of the Lagrangian and its symmetry properties. Feynman's formulation also gives a better intuitive understanding of quantum mechanics by linking it with classical mechanics in a way far more apparent than the wave function and matrix formulations do.

An established formal mathematical theory of Feynman's formulation of quantum mechanics does exist, but is quite complex, mostly due to the notion of the path integral. The Lebesgue approach to integration cannot be performed for path integrals in a straight–forward manner, due to the trouble of finding a suitable measure on the infinite–dimensional space of functions. A reader mostly interested in path integrals should refer to [7, 8].

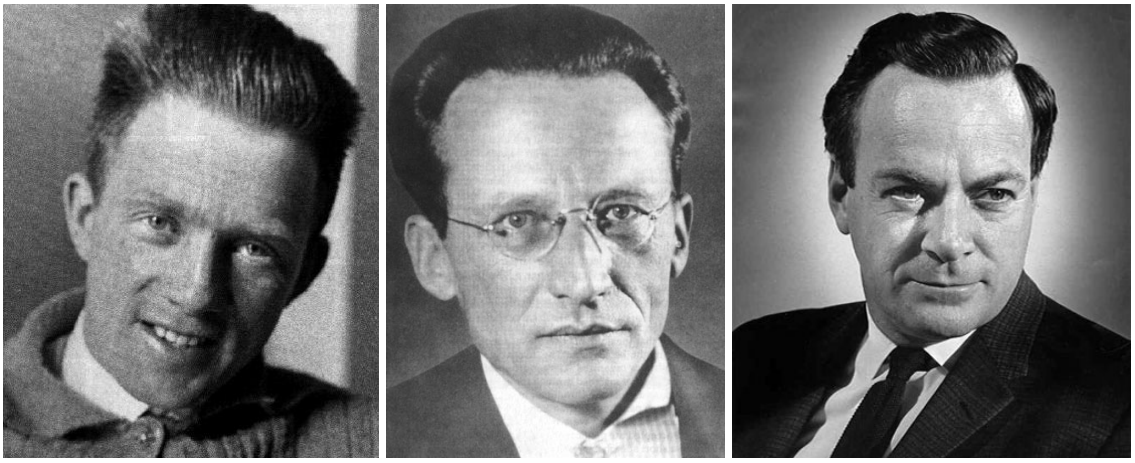

Figure 1: The men behind the three formulations of quantum mechanics (from left to right): Werner Heisenberg (1901-1976) [9], Erwin Schrödinger (1887-1961) [10] and Richard Feynman (1918-1988) [11].

1 Preliminaries

1.1 On Variational Calculus and the Classical Action

Variational calculus deals, among other things, with the extremal values of a functional. Consider a function $\mathscr{L} \in \mathscr{C}^2([t_a, t_b]\times\mathbb{R}\times\mathbb{R})$, by which we mean a twice continuously differentiable function $[t_a, t_b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$, where $t_a, t_b \in \mathbb{R}$. For a real function $q \in \mathscr{C}^1([t_a, t_b])$ we define

$$S(q) \;\coloneqq\; \int_{t_a}^{t_b} \mathscr{L}(t, q(t), \dot{q}(t))\, dt. \tag{1}$$

The functional S is a mapping $\mathscr{C}^1([t_a, t_b]) \longrightarrow \mathbb{R}$. At this point we fix the end points of the functions q under consideration: we are interested only in such functions q that $q(t_a) = a$ and $q(t_b) = b$ for a given pair of real numbers a and b. The problem is posed as follows: for which funtion q does the functional S take a minimal value? There is a theorem, which states the necessary condition for q (see [12], p. 47): if such a minimizer funtion exists (we shall denote this function $\bar{q}$), then it must satisfy the differential equation

$$0 \;=\; \frac{\partial \mathscr{L}}{\partial q}\Big(t, \bar{q}(t), \dot{\bar{q}}(t)\Big) - \frac{d}{dt}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}}\Big(t, \bar{q}(t), \dot{\bar{q}}(t)\Big)\right]. \tag{2}$$

We call equation (2) the Euler–Lagrange equation, and it is a second order ordinary differential equation for $\bar{q}$. In a more general case, we could define a real function $f_q : \mathbb{R} \longrightarrow \mathbb{R}$, with $f_q(\alpha) \coloneqq S(\bar{q} + \alpha q)$. We say that the functional S has an extremal value in $\bar{q}$, when for all real functions f_q the expression $f_q'(0) = 0$ holds. A more general theorem (than the one stated previously) identifies the Euler-Lagrange equation (2) as the necessary condition for the extremal of S (see [13], p. 37). Sometimes, the search for the *stationary path* $\bar{q}$ is summed up by the mathematically somewhat dubious equation $\delta S = 0$, and the expression of *the first variation of the functional* being zero.

The formulation of classical mechanics in terms of variational methods is called Lagrangian mechanics. In this framework, S is called *the classical action* and $\mathscr{L}$ is called the Lagrangian. The Lagrangian of a system is a function of the generalized coordinates q_i, which we choose by applying all the holonomic constraints (those of the form $f(x_i, \ldots, t) = 0$) on the regular coordinates x_i (for a N–particle system in three dimensions, x_i are the $3N$ space coordinates of the particles). When the generalized coordinates q_i are independent (with no nonholonomic constraints), and the forces acting on the system are conservative, so that their potential V is explicitly dependent only on q_i (and not $\dot{q}_i$ or t), the Lagrangian can be written as $T - V$, where T is the expression for the total kinetic energy of a system. The main postulate here is *Hamilton's principle*, which states that the actual trajectory $\bar{q}$ of the system, descibed by a Lagrangian, is the one in which the action $S(q)$ is extremal ([13], p. 36). This means the trajectory in physical reality is the solution of the Euler–Lagrange equation.

A more general situation represents a system described by many variables q_i, all with specified end–points. Let $i \in \{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Then the Lagrangian is expressed in terms of *generalized coordinates* q_i and their first derivatives. In this case, the action is expressed as

$$S(q_1, \ldots, q_n) \;\coloneqq\; \int_{t_a}^{t_b} \mathscr{L}[t, q_1(t), \dot{q}_1(t), \ldots, q_n(t), \dot{q}_n(t)]\, dt. \tag{3}$$

Hamilton's principle for such a system again states, that the trajectories $\bar{q}_i$ are extremal values of the action S. Every trajectory thus obeys its own Euler–Lagrange equation ([13], p. 31):

$$\forall i \in \{1, \ldots, n\}: \qquad \frac{\partial \mathscr{L}}{\partial q_i}\Big(t, \bar{q}_i(t), \dot{\bar{q}}_i(t)\Big) \;=\; \frac{d}{dt}\left[\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\Big(t, \bar{q}_i(t), \dot{\bar{q}}_i(t)\Big)\right]. \tag{4}$$

1.2 On Schrödinger's Formulation of Quantum Mechanics

In Schrödinger's formulation, the states of a quantum system are described by a wave function Ψ. The wave function $\Psi(\mathbf{r}, t)$ for a single particle is a function $\mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{C}$, where $\mathbf{r}$ will denote a point in space and t a point in time. If $c \in \mathbb{C}$, then $c\Psi$ represents the same state as Ψ ([13], p. 247). The squared absolute value $|\Psi(\mathbf{r}, t)|^2$ expresses the probability density for the particle at time t to be near $\mathbf{r}$. Because the probability of finding a particle in all space is 1, spatial normalization of the wave functions is demanded:

$$\int_{\mathbb{R}^3} |\Psi(\mathbf{r}, t)|^2\, dV \;=\; 1 \qquad \forall t \in \mathbb{R}. \tag{5}$$

We should remember from section 1.1, that in classical mechanics, a differential equation determines the actual trajectory for a particle. In Schrödinger's formulation of quantum mechanics, the notion of a trajectory is meaningless, since the wave function can provide only probabilities. In fact, due to Heisenberg's uncertainty principle $\delta x \delta p \geq \hbar/2$, we cannot know at any given time both a particle's position x and its momentum p to an arbitrary degree of accuracy. Because the notion of a trajectory implies we know both the exact position and velocity, it is inconsistent with the quantum mechanics of Schrödinger. However, the wave function must satisfy its own differential equation. We call this equation *the Schrödinger equation* and it takes the following form (see [14], p. 27):

$$i\hbar\frac{\partial\Psi}{\partial t} \quad = \quad \hat{\mathbf{H}}\Psi. \tag{6}$$

The symbol i denotes the imaginary unit, $\hbar = h/2\pi$, where h is Planck's constant, and $\hat{\mathbf{H}}$ is the Hamiltonian operator, acting on the space of functions Ψ. If we write Hamilton's operator for a single particle with mass m in the potential $V(\mathbf{r})$ explicitly, we get the following well known form of equation (6):

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{r},t) \quad = \quad \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{r})\Big]\,\Psi(\mathbf{r},t). \tag{7}$$

The symbol $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ denotes the Laplace differential operator.

In Schrödinger's quantum mechanics, the functions Ψ form a *Hilbert space*; a Hilbert space is a vector space equipped with an inner product (and it is complete as a metric space, where the metric is naturally defined by the norm, which in turn is naturally defined by the inner product). The inner product between two functions Ψ_1 and Ψ_2 is defined by an integral:

$$\langle\Psi_2 \mid \Psi_1\rangle\,(t) \quad \coloneqq \quad \int_{\mathbb{R}^3}\Psi_2^*(\mathbf{r},t)\Psi_1(\mathbf{r},t)\,dV. \tag{8}$$

The symbol $*$ denotes complex conjugation.

Physical quantities, which are functions in classical mechanics, become hermitian operators in quantum mechanics. We define the average value of an operator $\hat{\mathbf{A}}$ as $\langle\hat{\mathbf{A}}\rangle \coloneqq \langle\Psi \mid \hat{\mathbf{A}}\Psi\rangle$ ([14], p. 27).

Up until now, we have been describing the quantum mechanics of a single particle. Suppose we have n particles ($n \in \mathbb{N}$), each with its own coordinate $\mathbf{r}_i$, $i = 1,\ldots,n$. The states of a multiparticle system are described by a wave function $\Psi : \mathbb{R}^{3n}\times\mathbb{R} \longrightarrow \mathbb{C}$. The squared absolute value $|\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_n,t)|^2$ expresses the probability density for finding at a given time t each of the particles i near $\mathbf{r}_i$. The wave function is again spatially normalized:

$$\int_{\mathbb{R}^{3n}}|\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_n,t)|^2\,dV_1\ldots dV_n \quad = \quad 1 \qquad \forall t\in\mathbb{R}. \tag{9}$$

The Schroödinger equation in the form (6) still holds, if one takes the Hamiltonian operator for multiple particles. In its explicit form, the Schrödinger equation for n–particles is the following ([14], p. 28):

$$i\hbar\frac{\partial\Psi}{\partial t}(\mathbf{r}_1\ldots,\mathbf{r}_n,t) \quad = \quad \left[\sum_{i=1}^{n}\Big(-\frac{\hbar^2}{2m_i}\nabla_i^2\Big) + V(\mathbf{r}_1,\ldots,\mathbf{r}_n)\right]\Psi(\mathbf{r}_1,\ldots,\mathbf{r}_n,t). \tag{10}$$

Here, m_i signifies the mass of the i–th particle, while $\nabla_i^2 = \partial^2/\partial x_i^2 + \partial^2/\partial y_i^2 + \partial^2/\partial z_i^2$.

1.3 A Heuristic Approach to Path Integrals

Path integrals are a type of integral, where we integrate in a space of continuous functions (paths). They are usually defined alongside Feynman's formulation of quantum mechanics; we will treat them here separately as a mathematical subject. The integration "over all paths" is in stark contrast with the well known Riemann integral, where we integrate in the space of real numbers $\mathbb{R}$ or a multidimensional space $\mathbb{R}^n$, for $n \in \mathbb{N}$. An alternative to the Riemann integral is the Lebesgue integral, which allows us to integrate real functions with their domains in a measurable space. In case of path integrals, we are hard pressed to define a measure in the space of paths $\mathbb{R} \longrightarrow \mathbb{R}$, mainly because the paths do not form a finite dimensional vector space over $\mathbb{R}$. Consequently, we shall attempt to describe path integrals with an analogy to Riemann integrals. Our approach will be heuristic in nature (based on [15], p. 31–35), since the intent is to give the reader an intuitive feel for path integrals. The reader should not be discouraged by the lack of mathematical rigor in the following paragraphs.

We shall first consider the Riemann integral of functions $\mathbb{R} \longrightarrow \mathbb{R}$. The main idea of this integral is to calculate the area under the curve in a plot of $y = f(x)$ of the integrated function. Suppose we are interested only in the area, where $x \in [x_a, x_b]$. We shall follow a series of steps, where we calculate a regular sum in each of the steps. In the n–th step, we divide the interval $[x_a, x_b]$ into a partition of n equally long intervals. For a given n we define

$$\epsilon := \frac{x_b - x_a}{n}, \tag{11}$$

$$x_i := x_a + i\epsilon \qquad i \in \{0, 1, 2, \ldots, n\}. \tag{12}$$

We can interpret ϵ as the length of the intervals in the n–th step, while x_i are partition points. The given definitions also imply $x_0 = x_a$ and $x_n = x_b$. The Riemann integral of f is in this case (somewhat informally) defined as the following:

$$\int_{x_a}^{x_b} f(x)\,dx := \lim_{n\to\infty} \epsilon \sum_{i=0}^{n-1} f(x_i). \tag{13}$$

Of course the limit on the right–hand side of the above equation does not necessarily exist. The Riemann integral is defined only for such functions f where the limit exists. It turns out that all functions $[a, b] \longrightarrow \mathbb{R}$, which have finitely many points of discontinuity on $[a, b]$, can be integrated (see [16], p. 126).

Now we turn our attention to path integrals. Our first problem will be, how to characterize a continuous function $q(t) : \mathbb{R} \longrightarrow \mathbb{R}$ in a finite manner. The objects we will be integrating will be functionals $F(q) : \mathbb{R}^{\mathbb{R}} \longrightarrow \mathbb{R}$. As in the Riemann integral, we shall specify the endpoints of the region over which we integrate. We specify the path endpoints by fixing $q(t_a) = a$ and $q(t_b) = b$. We again divide the limiting process into steps. In the n–th step, we divide the time interval into n smaller intervals in a similar manner as we did with the Riemann integral (see Figure 2):

$$\epsilon := \frac{t_b - t_a}{n}, \tag{14}$$

$$t_i := t_a + i\epsilon \qquad i \in \{0, 1, 2, \ldots, n\}, \tag{15}$$

$$x_i := q(t_i) \qquad i \in \{0, 1, 2, \ldots, n\}. \tag{16}$$

The time values t_i are the partition points of the interval $[t_a, t_b]$. We have $t_0 = t_a$ and $t_n = t_b$. We characterize a path q by specifying the values x_i, or in other words, by specifying its value at specific equidistant points in time. Of course, we already know that $x_0 = a$ and $x_n = b$. This means we must specify $n-1$ values: x_i for $i \in \{1, \ldots, n-1\}$. These values can be any real number. However, the functions q are specified by the values in all time points between t_a and t_b. We now define values $q(t)$ for times other than t_i in such a way, that the motion of a particle moving along such a path between x_i and x_{i+1} is uniform (its velocity between t_i and t_{i+1} is constant, the path "consists of straight lines"). We can write that explicitly as

$$\forall i \in \{0, \ldots, n-1\}: \qquad q(t) \;=\; x_i + t \cdot \frac{x_{i+1} - x_i}{\epsilon} \qquad \text{for } t \in [t_i, t_{i+1}]. \tag{17}$$

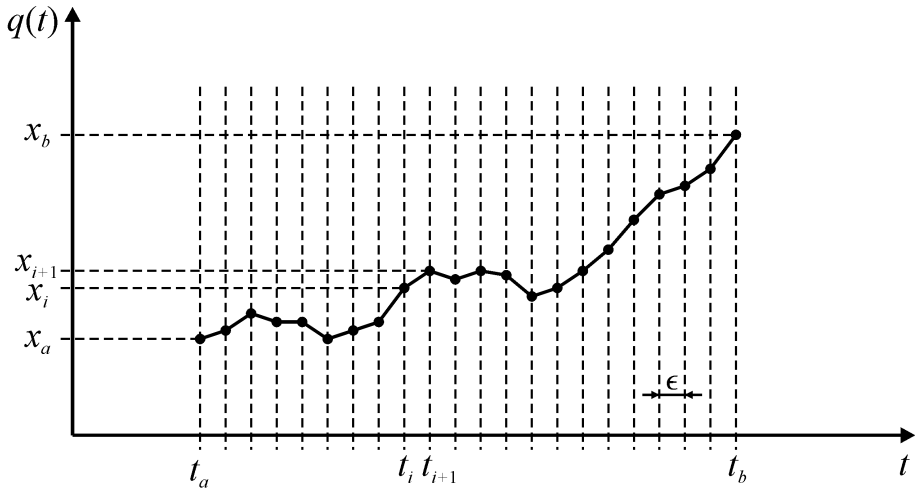

Figure 2: We characterize a path by splitting the time interval into a partition of equally long intervals, and specifying the values x_i at times t_i, while demanding uniform motion between them.

A sum over all possible paths (actually all paths with line segments) in step n will thus translate into a sum for all possible values x_i for each $i \in \{1, \ldots, n-1\}$. Because x_i can change by an infinitesimal amount, a sum for each x_i will be calculated by integration over x_i. Analogous to the normalizing constant ϵ (it changes from step to step) in the Riemann integral, we will here use the symbol A for the normalizing constant in path integrals. This constant ensures, that the limit, where the number of steps approaches ininity, exists. This means that A depends on ϵ, which in turn depends on n.

We now define the path integral (the reader should note the notation used on the left–hand side) as

$$\int_{a'}^{b'} F_{[b',a']}(q)\; \mathscr{D}q \;\coloneqq\; \lim_{n\to\infty} \frac{1}{A} \int_{\mathbb{R}^{n-1}} F_{[b',a']}(q)\; \frac{dx_1}{A} \ldots \frac{dx_{n-1}}{A}. \tag{18}$$

In the above definition, $a' = (t_a, a)$ and $b' = (t_b, b)$, and by $F_{[b',a']}$ we mean the functional F restricted to such functions q, for which $q(t_a) = a$, $q(t_b) = b$ and q consists of line segments with specified values in partition points x_i.

In general, we do not know how A relates to ϵ, but we will calculate it for a special case in section 3.2. We will otherwise not concern ourselves too much with A, because in practice we usually do not calculate the path integral by the form given in (18), but rather use tricks on a case by case basis ([15], p. 41–74).

Taking uniform motion for the path between two points x_i could present difficulties, if the integrated functional F assumes a twice differentiable function (path) q in its domain, or in other words, if F depends on $\ddot{q}$. Namely, our path with segments of uniform motion is not in general twice differentiable, because the "acceleration" at times t_i is infinite, therefore the second derivative at those times does not exist. Since all functionals used will depend on the classical action S, which in turn demands only the existence of the first derivative in the Lagrangian, the problem of taking uniform paths between points x_i will not be an issue.

Also, it is mathematically especially convenient, that $F(q)$ is an oscillatory functional (its real values oscillate around zero), so that many different paths cancel each other out. This makes it easier to achieve the finitness of the path integral.

2 Path Integral Formulation of Quantum mechanics

2.1 A Justification Through Thought Experiments

The two slit experiment is a famous thought experiment in quantum physics (see Figure 3). Suppose we have a source of electrons at plane A. The electrons come out of this source with roughly the same energies, but in different directions. We have a screen at B, with two narrow holes (slits). Further away, we have a plane C, where we detect the coming electrons. We move the detector in plane C, so that we measure the intensity of the current of electrons at different positions in the plane. In classical physics, we would expect most of the electrons to be near two distinct locations at C, depending on which slit in the screen B they passed through. If we actually perform such an experiment, it turns out the result in plane C is an interference pattern. That is an unexpected result from the perspective of classical physics: interference patterns are formed by interfering waves, while we expect the electron to behave like a point particle. Therefore, we have to find a way to give the electron wave–like properties.

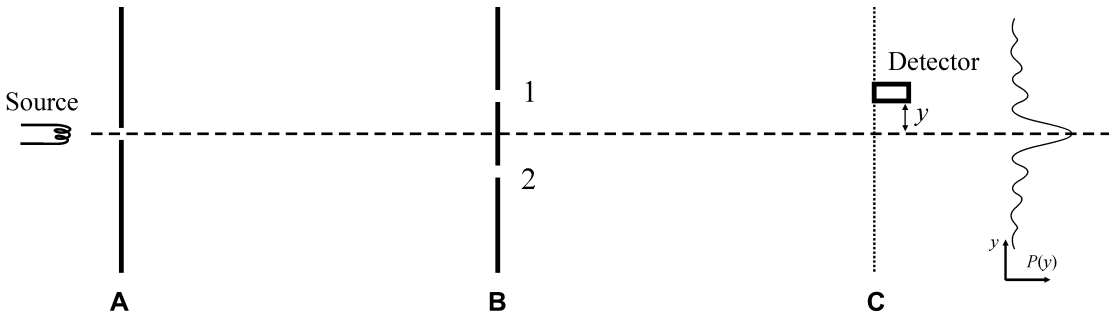

Figure 3: The configuration of the two slit experiment, with the source of electrons at A, the screen with the two slits at B, and the detector at C. The experimental result for the probability density $P(y)$ of detecting electrons at y is on the far–right — it is an interference pattern.

Perhaps our first attempt at explaining the interference pattern at C would be by proposing interference between different electrons, while they pass through the two slits. If the intensity of

the electron source at A is sufficiently reduced, no more than a single electron would be passing through the screen at B toward the detector C at any given time. The detector would in this case record single pulses, corresponding to single particles. One could perform this variation of the experiment by counting how many particles come at different positions in the plane C in a given amount of time. By plotting the results, one would again obtain an interference pattern, even though single electrons were passing through the slits. This means each electron somehow "interferes with itself": it does not simply go through one of the two slits as a point particle in classical physics would. The mathematics describing such behaviour is known from the study of waves: each wave–source is attributed an amplitude function, say ϕ_1 for the first source and ϕ_2 for the second. In our experiment, the two slits in the screen B are our sources (for the "waves of the electron"). These amplitude functions take the known form of the exponential function $C \cdot e^{i2\pi s/\lambda}$, where C is the amplitude, i the imaginary unit, λ the wavelenght of the wave, and s the distance between the wave–source and the location, where the amplitude is being calculated. Note that the value of the amplitude function is a complex number. The *principle of superposition* states that the amplitude of the newly constructed wave is a sum of the amplitudes from the two separate sources: $\phi = \phi_1 + \phi_2$.

In the field of optics, a light detector does not measure the amplitude of the electric field, but rather the intensity, which is proportional to the square of the absolute value $|E|^2$ of the amplitude E. Analogously to optics, our detector at plance C measures the intensity of the coming electrons in the form of electrical current, this means we are not actually measuring ϕ, but $|\phi|^2$. The number of electrons detected at a specific location is thus directly proportional to $|\phi|^2$. We say that the number of electrons detected near a location divided by the number of all electrons detected in plane C is the *probability* of detecting the electron at that location. Therefore, the probability P (or rather probability density) of finding the electron at a specific location is proportional to $|\phi|^2$. By renormalizing the function ϕ we can get $P = |\phi|^2$. That is why we call ϕ the *probability amplitude*.

The main idea in the "two slit experiment", was to calculate the probability amplitude of the electron by adding the two separate probability amplitudes of the two possible paths the electron could take: the path through the first and second slit. Now we shall try to generalize the result. If we drill a third hole into the screen at B, the electron can take three possible paths: through the first, the second, or the third slit. That means we have 3 probability amplitudes, each for a corresponding slit, and the total probability amplitude at the detector as the sum $\phi = \phi_1 + \phi_2 + \phi_3$. If we have n holes in the screen, the total probability amplitude will be the sum of n separate amplitudes (corresponding to each of the holes).

An even further generalization would be to have multiple screens (see Figure 4), say B,D and E. A particle detected at C could have passed through a given screen at any of its holes. This means we have to take into account the multiple choices at each screen. Since the choices between different screens are independent, this means the number of possible paths through a series of screens is worked out by multiplying the number of choices for each screen. Again, the probability density at the detector C would be the sum of the separate amplitudes of all different paths the particle could take through the screens. By taking more and more screens, one would have to calculate a sum over more and more paths, where one would specify the path by attributing a hole for each of the screens. The number of possible paths is also increased by repeatedly drilling more and more holes into any of the screens.

If there was nothing but empty space between the source of the electrons and the detector, one could imagine that as a limiting case with infinitely many screens (arbitrarily close in every location), but each screen has infinitely many holes, so that the particle can pass through anywhere

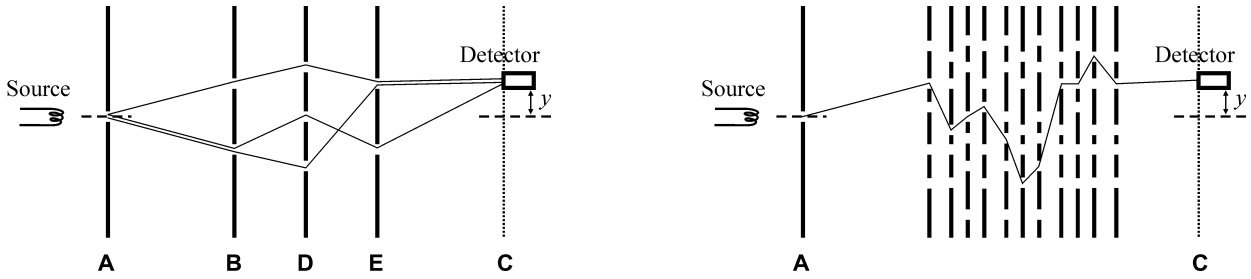

Figure 4: We can modify the two slit experiment by adding new screens and drilling new holes, which results in many possible paths (three such paths are drawn on the left–hand side). If the number of screens and holes increases, a possible path can become very complex (right–hand side).

in the screen. A sum over all possible paths through the screens would in this case be a sum over all possible paths in euclidean space; the sum would actually be a path integral. For each path q there would be a probability amplitude $\phi(q)$. The form of ϕ in the general case is not immediately apparent, even more so if the particle moves in fields of conservative forces. It turns out that the following case for ϕ is consistent with the regular quantum mechanics of Schrödinger:

$$\phi(q) \quad = \quad e^{\frac{i}{\hbar}S(q)}. \tag{19}$$

The contribution of each path thus has the same amplitude (the absolute value of each probability density is 1), but different paths contribute different phase shifts, which depend on the classical action S. The total probability amplitude in a point in space is calculated as $\sum_q \phi(q)$, where we take the sum over all possible paths q from the starting point to the point in question. The form in equation (19) is not inconsistent with our previous thought experiments on the probability amplitude in the case of screens: the Lagrangian is classically a constant of motion for a free particle, and in that case, we took the sum only over "the relevant paths" of uniform motion. The exponential of the action of such paths thus reduces to a simple phase factor.

2.2 The Kernel

2.2.1 Definition

We denote the previously discussed probability amplitude of a particle as $K_{[b',a']}$, where $a' = (t_a, a)$ and $b' = (t_b, b)$. Very loosely speaking, we could regard the square of the absolute value of the kernel as the probability density of finding the particle at the location b at time t_b, if the particle was located at a at time t_a. The function $K_{[b',a']}$ is also called *the kernel*.

According to Feynman's formulation, the kernel of a particle is determined in the following way ([15], p. 34):

$$K_{[b',a']} \quad = \quad \int_{a'}^{b'} e^{\frac{i}{\hbar}S_{[b',a']}(q)}\, \mathscr{D}q, \qquad \text{where} \quad S_{[b',a']}(q) = \int_{t_a}^{t_b} \mathscr{L}(t, q(t), \dot{q}(t))\, dt. \tag{20}$$

The domain of $S_{[b',a']}$ are continuous functions q with fixed endpoints $q(t_a) = a$ and $q(t_b) = b$. Due to the heuristic manner of our definition of the path integral, we will not enter into the issue of the

existence of the above integrals. Suffice it to say that it is the oscillatory nature of the integrated functionals that makes the evaluation of the path integrals possible.

It should be noted, the kernel in equation (20) relates to nonrelativistic quantum mechanics, also with no account for the spin of particles. In a relativistic quantum theory, one could still use the notion of a kernel as a sum over all paths. In this paper, we shall be content with Feynman's formulation of nonrelativistic quantum mechanics with no spin.

The notion of a probability of finding a particle in a cetrain location is strange in the context of classical physics, because classically the exact position can be determined. However, because the exact position of the particle is known at time t_a, the uncertainty of the position at that time is 0. That means, according to Heisenberg's uncertainty principle $\delta q \delta p_q \geq \hbar/2$, the uncertainty in velocity (which is proportional to momentum) is infinite. Thus the particle's unknown velocity can be of any value (in nonrelativistic physics the speed of light c is not the limit), and the particle can be found anywhere in space at a later time t_b.

2.2.2 Chain Rule for Events

Suppose we choose a time t_c, so that $t_a < t_c < t_b$, and $c' = (t_c, c)$. We can then write:

$$S_{[b',a']}(q) = \int_{t_a}^{t_b} \mathscr{L}(t, q(t), q'(t))\, dt, \tag{21}$$

$$S_{[b',a']}(q) = \int_{t_a}^{t_c} \mathscr{L}(t, q_a(t), q'_a(t))\, dt + \int_{t_c}^{t_b} \mathscr{L}(t, q_b(t), q'_b(t))\, dt, \tag{22}$$

$$S_{[b',a']}(q) = S_{[b',c']}(q_b) + S_{[c',a']}(q_a). \tag{23}$$

In the first term on the right–hand side of equation (23), we take the restriction $q_a = q|_{[t_a,t_c]}$ with endpoints $q(t_a) = a$ and $q(t_c) = c$, while the second term is the restriction $q_b = q|_{[t_c,t_b]}$ with endpoints $q(t_c) = c$ and $q(t_b) = b$. The differential can now be written as $\mathscr{D}q = dc\, \mathscr{D}q_a\, \mathscr{D}q_b$: we integrate over all paths from a' to c', integrate over all possible paths from c' to b', then integrate over all possible midpoints c at time t_c (see Figure 5). We can then write:

$$K_{[b',a']} = \int_{\mathbb{R}} \int_{c'}^{b'} \int_{a'}^{c'} dc\, \mathscr{D}q_b\, \mathscr{D}q_a\, e^{(i/\hbar)S_{[c',a']}(q_a)+(i/\hbar)S_{[b',c']}(q_b)}, \tag{24}$$

$$K_{[b',a']} = \int_{\mathbb{R}} dc \left(\int_{a'}^{c'} e^{\frac{i}{\hbar}S_{[c',a']}(q_a)}\, \mathscr{D}q_a \right) \left(\int_{c'}^{b'} e^{\frac{i}{\hbar}S_{[b',c']}(q_b)}\, \mathscr{D}q_b \right). \tag{25}$$

We recognize the two factors in the integral as kernels. We can write the so called *chain rule for two events* as

$$K_{[b',a']} = \int_{\mathbb{R}} K_{[b',c']} K_{[c',a']}\, dc. \tag{26}$$

The name kernel for K is now justified, for the above equation can be viewed as an integral transform with the kernel function (also called nucleus) $K_{[b',c']}$. Generalizing the chain rule result, by picking n specific times $t_a < t_{c_1} < \cdots < t_{c_n} < t_b$, with $c'_i = (t_{c_i}, c_i)$, we get

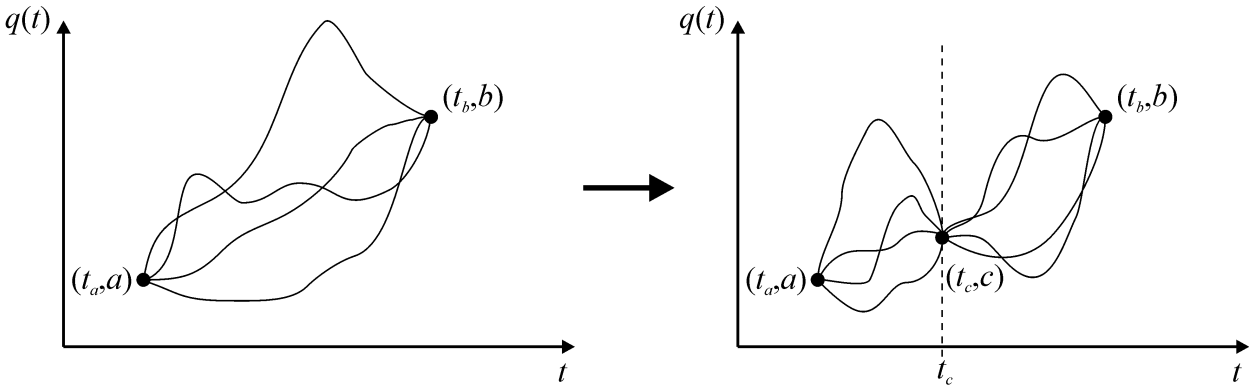

Figure 5: In this demonstration of the chain rule, a sum over all paths between two points is substituted by two sums over paths from an endpoint to the midpoint (t_c, c); then we also have to take the sum over all possible locations c for a specified t_c.

$$K_{[b',a']} = \int_{\mathbb{R}^n} \left(K_{[b',c'_n]} K_{[c'_n,c'_{n-1}]} \cdots K_{[c'_2,c'_1]} K_{[c'_1,a']} \right) dc_1 \ldots dc_n. \tag{27}$$

One could choose for times t_{c_i} the times t_i from the definition of path integrals in step n: $t_i = t_a + i(t_b - t_a)/n$. In step n, we would be integrating $(n-1)$–times over variables $c_1, \ldots, c_{n-1}$. By sending n to infinity, the $(n-1)$-time integral would actually become a path integral, and the infinite product of kernels would become the probability amplitude. We define $c'_0 = a'$ and $c'_n = b'$. Then one could write the probability amplitude alternatively as (see [15], p. 38)

$$\phi(q) = \lim_{n\to\infty} \prod_{i=0}^{n-1} K_{[c_{i+1},c_i]}. \tag{28}$$

The function q in the amplitude has specified values $q(t_i) = c_i$.

2.3 The Kernel of Multiparticle Systems

Until now, we have viewed the kernel $K_{[b',a']}$ as a function relating to a single particle. Moreover, the path of the particle was a continuous function into one–dimensional space $\mathbb{R}$. We are now interested in a general case, when we have multiple particles in multi–dimensional space, or perhaps even a general quantum system, described by generalized coordinates.

Suppose we have a system, which is classically described by n (independent) generalized coodrinates $q_1, \ldots, q_n$, and with the classical action $S(q_1, \ldots, q_n)$ (which is obtained by integrating the Lagrangian in the form of equation (3)). Again, we shall fix the endpoints of the functions in question: for $i \in \{1, \ldots, n\}$ we define $a'_i := (t_a, a_i)$, $b'_i := (t_b, b_i)$ and demand $q_i(t_a) = a_i$, $q_i(t_b) = b_i$. Note, that we specify endpoints for the same timepoints t_a and t_b for all functions q_i. We shall write the action of a system with specified endpoints as $S_{[b'_1,a'_1;\ldots;b'_n,a'_n]}(q_1, \ldots, q_n)$. The kernel (probability amplitude) can be expressed as a multiple path integral (a generalization of eq. in [15], p. 66):

$$K_{[b'_1,a'_1;\ldots;b'_n,a'_n]} = \int_{a'_1}^{b'_1} \cdots \int_{a'_n}^{b'_n} e^{\frac{i}{\hbar} S_{[b'_1,a'_1;\ldots;b'_n,a'_n]}(q_1,\ldots,q_n)} \, \mathscr{D}q_1 \ldots \mathscr{D}q_n. \tag{29}$$

Again, we can view this kernel as representing the wavefunction of the system at time t_b in the state $q_i(t_b) = b_i$, when the system was measured at time t_a of being in the state $q_i(t_a) = a_i$.

A notable special case of a system with multiple degrees of freedom is one with a separable action: the action S as a functional of multiple paths can be expressed as the sum of functionals S^i of one variable q_i:

$$S_{[b'_1,a'_1;\ldots;b'_n,a'_n]}(q_1,\ldots,q_n) = \sum_{i=1}^{n} S^i_{[b'_i,a'_i]}(q_i). \tag{30}$$

If such a separation is possible, then the kernel can be written in the following way (generalization of eq. in [15], p. 67):

$$K_{[b'_1,a'_1;\ldots;b'_n,a'_n]} = \int_{a'_1}^{b'_1} \cdots \int_{a'_n}^{b'_n} e^{\frac{i}{\hbar}\sum_{i=1}^{n} S^i_{[b'_i,a'_i]}(q_i)} \mathscr{D}q_1 \ldots \mathscr{D}q_n, \tag{31}$$

$$K_{[b'_1,a'_1;\ldots;b'_n,a'_n]} = \prod_{i=1}^{n} \int_{a'_i}^{b'_i} e^{\frac{i}{\hbar} S^i_{[b'_i,a'_i]}(q_i)} \mathscr{D}q_i, \tag{32}$$

$$K_{[b'_1,a'_1;\ldots;b'_n,a'_n]} = \prod_{i=1}^{n} K^i_{[b'_i,a'_i]}. \tag{33}$$

If the action can be separated into a sum of actions of a single variable, then the kernel K of a multi–variable system is the product of the kernels K^i, which correspond to actions of a single variable.

In Schrödinger's quantum mechanics, the wavefunction exhibits certain symmetries under particle exchanges. Namely, the wavefunction is symmetric under the exchange of two bosons (particles with integer spin) and antisymmetric under the exchange of two fermions (particles with half–integer spin). The product of kernels does not exhibit these types of symmetry under particle exchanges, but rather conserves the type of symmetry of a pre–existing wavefunction in an integral transformation, when we propagate the wavefunction to a later time with the kernel (see equation (36) and section 3.2 for an explanation).

3 Connections to other physical theories

3.1 The classical limit

Quantum mechanics should be structured in such a way, that it reduces to classical mechanics in the macroscopic limit. We shall make some brief arguments, why this is true in the case of Feynman's formulation of quantum mechanics.

In the path integral formulation of quantum mechanics, a particle takes no one path. Rather, the probability amplitude is calculated by a sum over all paths q, with each path contributing just a different phase factor (equation (20)). In classical physics, the particle takes a specific path $\bar{q}$, called the trajectory. An important question is, how is $\bar{q}$ any different from other paths in the macroscopic limit of Feynman's formulation.

In the macroscopic limit, masses, spatial dimensions and times are large, which also means the classical action S is large, at least compared to $\hbar$. One could also obtain the classical limit by

demanding $\hbar \to 0$. Either way, $S \gg \hbar$, which means $S/\hbar \gg 2\pi$. Considering only paths with the same endpoints, if one slightly changes the path from q to $q + \delta q$, then this can be observed in the change of the action from S to $S + \delta S$. The phase factor between the probability amplitudes of these two paths is $e^{i\frac{\delta S}{\hbar}}$. If q is not an extremal of S, then even relatively slight changes δq in the path q bring about a large enough δS, so that $\delta S/\hbar$ is comparative to 2π. Due to the periodic nature (the period is 2π) of the exponential function of imaginary numbers, the phase shift between the two paths is equally likely of being any value on the unit circle in complex numbers $\mathbb{C}$. The real and imaginary parts of the exponential function are the functions cosine and sine, respectively. Considering paths similar to a given non–stationary q, the sine and cosine oscillate quickly around the value 0, which means that neighboring paths cancel each other out: their net contribution to the kernel is close to zero (see Figure 6).

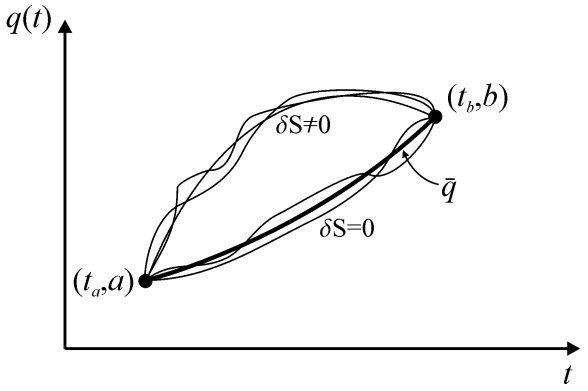

Figure 6: In the macroscopic limit, neighboring paths cancel each other out, unless they are in the neighborhood of the trajectory $\bar{q}$, where $\delta S = 0$.

One path is exempt from this analysis: the classical trajectory. For this path, $\delta S = 0$ in the first order, because this path is stationary. This means its neighboring paths don't have a change in the phase factor: all paths near the classical trajectory contribute a very similar probability amplitude. This fact has consequences on the kernel. Suppose we are interested in $K_{[b',a']}$, or ultimately in the probability density at time t_b at location b. If $b \approx \bar{q}(t_b)$, then the classical trajectory and its neighboring paths bring a substantial probability amplitude into the kerel, while all other paths cancel each other out. The net contribution off all paths is equal to the contribution of the classical trajectory, which means the particle is verly likely to be in the place specified by classical mechanics. On the other hand, if b is far from $\bar{q}(t_b)$, then no possible paths with the specified endpoints are stationary, which means they are cancelled out by their neighborhoods: the net contribution to the kernel is 0. That also means the probability density is near zero: the particle is unlikely to deviate from its trajectory in the classical limit.

It has already been mentioned that paths similar to the classical trajectory also contribute to the kernel. They contribute, as long as their action is similar to the action $S(\bar{q})$ in the units of $\hbar$; in other words, paths contribute, while roughly $\delta S < \hbar$. This means the kernel considerably differs from zero in a neighborhood of $\bar{q}(t_b)$: the path of a particle in the classical limit is still a little fuzzy. It is very likely that the particle will be observed near its classical trajectory to within around $\hbar$ of $S(\bar{q})$: this is consistent with Heisenberg's uncertainty principle of quantum mechanics, for we cannot know exactly a particle's position and velovity at the same time, and the measure for this uncertainty is $\hbar$.

When not in the classical limit, when we are on an atomic scale, the action S is comparable to $\hbar$, which means all paths have to be considered.

3.2 Equivalence to Schrödinger's formulation

It can be shown that Feynman's and Schrödinger's formulation of quantum mechanics are equivalent. It will not be discussed here, how to get from Schrödinger to Feynman, but it can be done (see [7], p. 154). We shall rather show, how to get to the wave functions and to the Schrödinger equation (we follow the work in [15], p. 76–78).

The kernel $K_{[b',a']}$ contains both the starting point and the endpoint of the particle. Suppose we are interested only in the endpoint, while we are not interested in the place of origin of the particle. This new concept will be the wave function (we define $x' = (t, x)$):

$$\Psi(x,t) \quad := \quad K_{[x',a']}. \tag{34}$$

Note, that the wave function written above is a wave function for a single particle in onedimensional space. The wave function for a kernel with multiple degrees of freedom x_i would be defined as:

$$\Psi(x_1,\ldots,x_n,t) \quad := \quad K_{[x'_1,a'_1;\ldots;x'_n,a'_n]}. \tag{35}$$

Again, $x'_i = (t, x_i)$ in the above definition. Ψ in the above definitions (34) and (35) are the mathematical objects familiar to us from Shrödinger's formulation. Now, we shall try to derive the Schrödinger equation (7) for a single particle in one dimension (the most simple case).

First, we can rewrite equation (26), by defining $x'_2 = b'$, $x'_0 = a'$, $x'_1 = c'$, $x'_2 = (t_2, x_2)$, $x'_0 = (t_0, x_0)$, $x'_1 = (t_1, x_1)$ and recognizing two kernels as wave functions: $K_{[x'_2,x'_0]} = \Psi(x_2, t_2)$ and $K_{[x'_1,x'_0]} = \Psi(x_1, t_1)$. We get:

$$\Psi(x_2,t_2) \quad = \quad \int_{\mathbb{R}} K_{[x'_2,x'_1]}\Psi(x_1,t_1)\,dx_1. \tag{36}$$

Again, we can interpret the kernel $K_{[x'_2,x'_1]}$ as the kernel of an integral transform; we could also view K as the *propagator*, because it tells us how to propagate the wavefuntion from an earlier time t_1 to a later time t_2. Alternatively, we can also view the kernel $K_{[x',x'_1]}$ as a wave funtion $\Psi(x, t)$ with the initial value $\Psi(x, t_1) = \delta(x - x_1)$, where $\delta(x)$ is the Dirac delta function. It should be noted that not all wavefunctions (as known from Schrödinger's quantum mechanics) are kernels, but only those wavefunctions with the peculiar initial condition of a delta function (these wavefunctions can not be normalized). However, equation (36) holds for all wavefunctions, even those we did not define in the Feynman formalism by equation (34).

The obtained equation (36) is an integral equation, while we would like to get a differential equation. That is why we take the two timepoints very close together: $t_1 = t$ and $t_2 = t + \epsilon$. We will use an approximation for the kernel with a short time–gap between the specified endpoints, as if the action on the short time interval is constant (evaluated in the spatial midpoint $(x_1 + x_2)/2$, with the velocity being equal to the average velocity $(x_2 - x_1)/\epsilon$, with only one path $x(t)$ in the sum—the straight path):

$$K_{[x_2',x_1']} = \int_{(t,x_1)}^{(t+\epsilon,x_2)} e^{\frac{i}{\hbar}S(q)}\,\mathscr{D}q \quad \approx \quad \frac{1}{A}e^{\frac{i}{\hbar}\int_t^{t+\epsilon}\mathscr{L}(t,x(t),\dot{x}(t))\,dt} \approx \frac{1}{A}e^{\frac{i}{\hbar}\epsilon\mathscr{L}(t+\frac{\epsilon}{2},\frac{x_2+x_1}{2},\frac{x_2-x_1}{\epsilon})}. \tag{37}$$

If we insert (37) into (36), while also inserting the Lagrangian for a single particle in one dimension $\mathscr{L}(t,x,\dot{x}) = m\dot{x}^2/2 - V(x,t)$ (m is the mass of the particle, while V is the potential), we get:

$$\Psi(x_2,t+\epsilon) = \int_{\mathbb{R}} \frac{1}{A}e^{\frac{i}{\hbar}\epsilon\mathscr{L}(t+\frac{\epsilon}{2},\frac{x_2+x_1}{2},\frac{x_2-x_1}{\epsilon})}\,\Psi(x_1,t)\,dx_1. \tag{38}$$

$$\Psi(x_2,t+\epsilon) = \int_{\mathbb{R}} \frac{1}{A}\left(e^{\frac{i}{\hbar}\epsilon\frac{m(x_2-x_1)^2}{2\epsilon^2}}\right)\left(e^{-\frac{i}{\hbar}\epsilon V(\frac{x_1+x_2}{2},t+\frac{\epsilon}{2})}\right)\Psi(x_1,t)\,dx_1. \tag{39}$$

The exponential of the first factor contains $(x_2-x_1)^2/\epsilon$, which is very large, if x_2 and x_1 are not very close. This means that for x_1 not very near x_2, the integrated function oscillates (all other functions in the integral are smooth) and the total contribution of the part of the integral far from x_2 is zero. We define $x_2 = x$ and make the substitution $x_1 = x_2 + \eta$, suspecting contributions only for η near 0.

$$\Psi(x,t+\epsilon) = \int_{\mathbb{R}} \frac{1}{A}\left(e^{\frac{i}{\hbar}\frac{m\eta^2}{2\epsilon}}\right)\left(e^{-\frac{i}{\hbar}\epsilon V(x+\frac{\eta}{2},t+\frac{\epsilon}{2})}\right)\Psi(x+\eta,t)\,d\eta. \tag{40}$$

We can see that the phase change of a few radians in the phase factor will require $\eta \sim \sqrt{2\hbar\epsilon/m}$. We will be interested only in approximations in the first order of ϵ, or consequently of order η^2 in η. The approximation will be done by taking only the first few terms in the Taylor series around the point (x,t) for functions Ψ and V:

$$\Psi(x,t) + \epsilon\frac{\partial\Psi}{\partial t} + \mathscr{O}(\epsilon^2) = \int_{\mathbb{R}} \frac{e^{\frac{i}{\hbar}\frac{m\eta^2}{2\epsilon}}}{A}e^{-\frac{i}{\hbar}\left(\epsilon V(x,t)+\frac{\epsilon^2}{2}\frac{\partial V}{\partial t}+\frac{\epsilon\eta}{2}\frac{\partial V}{\partial x}+\mathscr{O}(\epsilon^2)\right)}\left(\Psi(x,t)+\eta\frac{\partial\Psi}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2\Psi}{\partial x^2}+\mathscr{O}(\eta^3)\right)d\eta,$$

$$\Psi(x,t) + \epsilon\frac{\partial\Psi}{\partial t} = \int_{\mathbb{R}} \frac{1}{A}e^{\frac{i}{\hbar}\frac{m\eta^2}{2\epsilon}}\left(1-\frac{i\epsilon}{\hbar}V(x,t)\right)\left(\Psi(x,t)+\eta\frac{\partial\Psi}{\partial x}+\frac{\eta^2}{2}\frac{\partial^2\Psi}{\partial x^2}\right)d\eta. \tag{41}$$

If we send $\epsilon \to 0$, the equation (41) must still hold, and we can determine the constant A by comparing the two terms with Ψ:

$$\Psi(x,t) = \frac{\Psi(x,t)}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\,d\eta = \frac{\Psi(x,t)}{A}\left(\frac{2\pi i\hbar\epsilon}{m}\right)^2 \tag{42}$$

$$\implies A = \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{1/2}. \tag{43}$$

The integral in (42) looks similar to an integral of a Gaussian function, but with an imaginary variance. The integral is over an infinite region and does not converge in the mathematical sense. A meaningful value for this integral can still be obtained by taking the value around which the integral

$\int_{-R}^{R} e^{\frac{im\eta^2}{2\hbar\epsilon}}\, d\eta$ oscillates, as $R \to \infty$. This can be evaluated by a trick, if we integrate $\int_{-\infty}^{\infty} e^{-\lambda\eta} e^{\frac{im\eta^2}{2\hbar\epsilon}}\, d\eta$, which converges for all $\lambda > 0$, and sending $\lambda \to 0$ in the result. By the same principles, the values of two other integrals are obtained (the first is zero, because the integrated function is of odd parity in η; the result from the second one is from [15], p.78):

$$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\,\eta\, d\eta = 0, \tag{44}$$

$$\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\,\eta^2\, d\eta = \frac{i\hbar\epsilon}{m}. \tag{45}$$

Now, knowing the value of the above integrals, we return to equation (41):

$$\Psi(x,t) + \epsilon\frac{\partial\Psi}{\partial t} = \left(1 - \frac{i\epsilon}{\hbar}V(x,t)\right)\left(\Psi(x,t)\,\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\, d\eta + \frac{\partial\Psi}{\partial x}\,\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\,\eta\, d\eta + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}\,\frac{1}{A}\int_{-\infty}^{\infty} e^{\frac{im\eta^2}{2\hbar\epsilon}}\,\eta^2\, d\eta\right),$$

$$\Psi(x,t) + \epsilon\frac{\partial\Psi}{\partial t} = \left(1 - \frac{i\epsilon}{\hbar}V(x,t)\right)\left(\Psi(x,t)\cdot 1 + \frac{\partial\Psi}{\partial x}\cdot 0 + \frac{1}{2}\frac{\partial^2\Psi}{\partial x^2}\cdot\frac{i\hbar\epsilon}{m}\right), \tag{46}$$

$$i\hbar\frac{\partial\Psi}{\partial t} = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi. \tag{47}$$

We have obtained the desired end–result: equation (47) is the Schrödinger equation for a particle in 1 dimension in the presence of potentials, and we have shown it follows from Feynman's formulation of quantum mechanics.

Conclusion

Since its inception, Feynman's formulation has been of great importance in theoretical physics. One advantage over Shcrödinger's formulation is in the use of the Lagrangian instead of the Hamiltonian. In the Schrödinger equation, we relate the time derivative of the wavefunction with the Hamiltonian. However, in relativity, time is not independent of the reference frame. Thus the Hamiltonian is not very appropriate for a relativistic theory. On the other hand, the Lagrangian can be written in a manifestly relativistic form, making it the preferred choice in describing a relativistic quantum theory.

An extension of Feynman's approach has also been carried out in quantum field theory. In this case, the sum over all possible paths is replaced by a sum over all possible scalar fields $\phi : \mathbb{R}^3 \times \mathbb{R} \longrightarrow \mathbb{R}$. The action S now has a domain of all fields, with integration of the Lagrangian density over a region in space–time. A formal mathematical framework dealing with integration over all fields still remains to be developed [4].

Finally, path integrals have also established a remarkable link between quantum mechanics and stochastic processes. The first hint of similarity comes from the Schrödinger equation, since it is essentially a diffusion equation with an imaginary diffusion constant. Path integrals themselves take the similarity further, since they are also used in a number of areas in statistical physics, such as the study of Brownian motion (for the description of random walks of particles), in polymer science (for random chains), and even outside statistical physics in the study of financial markets. In the

1970s, there was a grand synthesis of quantum field theory and statistical field theory. Namely, the partition function in statistical mechanics can be obtained by performing the "Wick rotation" $t \to it$ on Feynman's path integrals [4].

In all likelihood, the path–integral will remain an important tool in many areas of theoretical physics in the future.

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