Partial overview of Dyson-Schwinger approach to QCD and some applications to structure of hadrons [a]

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[a] Talk at Mini-workshop BLED 2014: QUARK MASSES AND HADRON SPECTRA,

Bled, Slovenia, July 6-13, 2014

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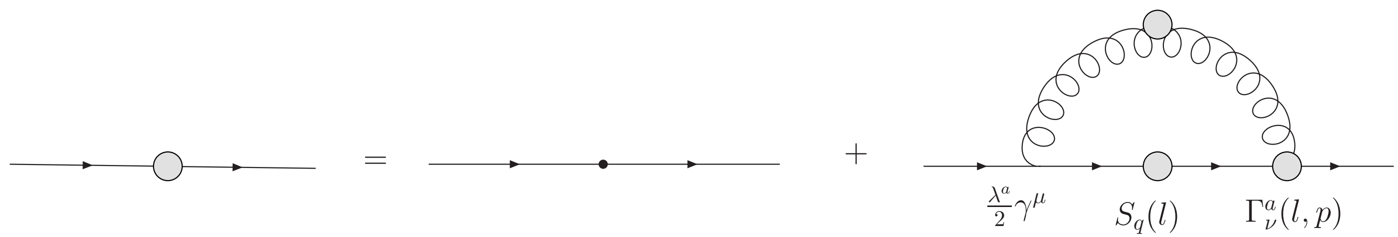
Dyson-Schwinger approach to quark-hadron physics

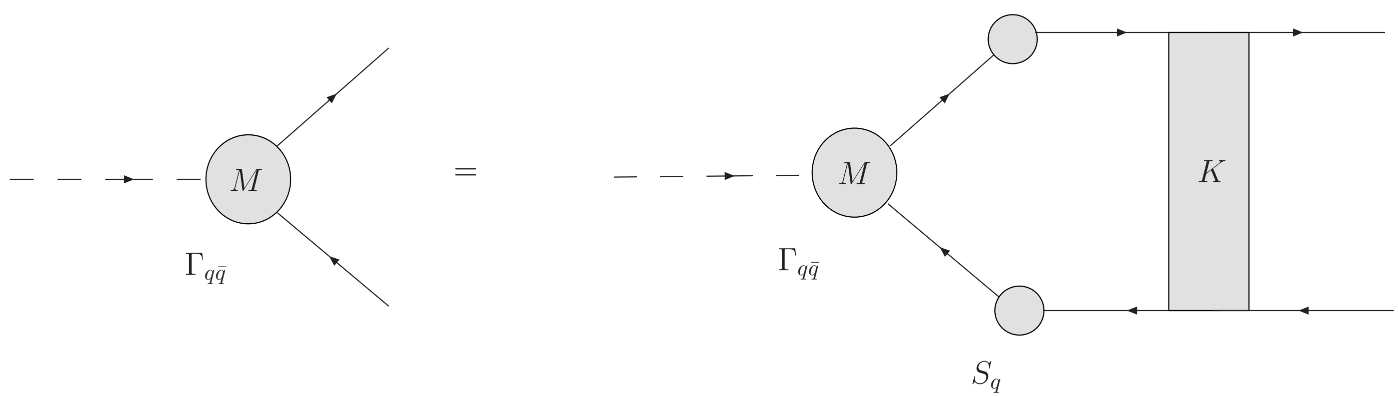
- = the bound state approach which is nopertubative, covariant and chirally well-behaved.
  e.g., GMOR relation: $\lim_{\widetilde{m}_q \to 0} M^2_{q\bar{q}}/2\widetilde{m}_q = -\langle \bar{q}q \rangle / f^2_\pi$
- a) direct contact with QCD through *ab initio* calculations
- b) phenomenological modeling of hadrons as quark bound states (used also here, for example)
- DS eq's: coupled system of integral equations for Green functions of QCD ... but ... equation for n-point function calls (n+1)-point function ... $\rightarrow$ cannot solve in full the growing tower of DS equations
- $\rightarrow$ various degrees of truncations, approximations and modeling is unavoidable (more so in phenomenological modeling of hadrons, as here)

Phenomenologically most important DS equations:

- Gap eq. for propagator S_q of dynamically dressed quark q



- Homogeneous Bethe-Salpeter (BS) equation for a Meson $q\bar{q}$ bound state vertex $\Gamma_{q\bar{q}}$



Gap and BS equations in rainbow-ladder truncation

$$S_q(p)^{-1} = i\gamma \cdot p + \widetilde{m}_q + \frac{4}{3}\int \frac{d^4\ell}{(2\pi)^4}\, g^2 G^{\rm eff}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell)\gamma_\nu$$

$$\rightarrow S_q(p) = \frac{1}{i\not p A_q(p^2) + B_q(p^2)} = \frac{-i\not p A_q(p^2) + B_q(p^2)}{p^2 A_q(p^2)^2 + B_q(p^2)^2} = \frac{1}{A_q(p^2)}\,\frac{-i\not p + m_q(p^2)}{p^2 + m_q(p^2)^2}$$

$$\Gamma_{q\bar{q}'}(p,P) = -\frac{4}{3}\int\frac{d^4\ell}{(2\pi)^4} g^2 G^{\rm eff}_{\mu\nu}(p-\ell)\gamma_\mu S_q(\ell+\frac{P}{2})\Gamma_{q\bar{q}'}(\ell,P)S_q(\ell-\frac{P}{2})\gamma_\nu$$

- Euclidean space: $\{\gamma_\mu,\gamma_\nu\} = 2\delta_{\mu\nu}$, $\gamma_\mu^\dagger = \gamma_\mu$, $a\cdot b = \sum_{i=1}^4 a_i b_i$
- P is the total momentum, $M^2 = -P^2$ meson mass^2
- $G^{\rm eff}_{\mu\nu}(k)$ an “effective gluon propagator” - modeled !

From the gap and BS equations ...

- solutions of the gap equation → the dressed quark mass function

$$m_q(p^2) = \frac{B_q(p^2)}{A_q(p^2)}$$

- propagator solutions $A_q(p^2)$ and $B_q(p^2)$ pertain to confined quarks if

$$m_q^2(p^2) \neq -p^2 \quad \text{for} \;\; \text{real} \quad p^2$$

- The BS solutions $\Gamma_{q\bar{q}'}$ enable the calculation of the properties of $q\bar{q}$ bound states, such as the decay constants of pseudoscalar mesons:

$$f_{PS}\, P_\mu \quad = \quad \langle 0|\bar{q}\, \frac{\lambda^{PS}}{2}\, \gamma_\mu\, \gamma_5\, q|\Phi_{PS}(P)\rangle$$

$$\longrightarrow \quad f_\pi P_\mu \quad = \quad N_c\, \mathrm{tr_s} \int \frac{d^4\ell}{(2\pi)^4}\, \gamma_5\gamma_\mu\, S(\ell + P/2)\, \Gamma_\pi(\ell; P)\, S(\ell - P/2)$$

Some renormalization-group improved interactions

$$\text{Landau gauge gluon propagator}: \quad g^2 G^{\text{eff}}_{\mu\nu}(k) = G(-k^2)(-g_{\mu\nu} + \frac{k_\mu k_\nu}{k^2}),$$

$$G(Q^2) \equiv 4\pi \frac{\alpha_s^{\text{eff}}(Q^2)}{Q^2} = G_{\text{UV}}(Q^2) + G_{\text{IR}}(Q^2), \qquad Q^2 \equiv -k^2 .$$

$$G_{\text{UV}}(Q^2) = 4\pi \frac{\alpha_s^{\text{perturbative}}(Q^2)}{Q^2} \approx \frac{4\pi^2 d}{Q^2 \ln(x_0 + \frac{Q^2}{\Lambda^2_{\text{QCD}}})} \{1 + b \frac{\ln[\ln(x_0 + \frac{Q^2}{\Lambda^2_{\text{QCD}}})]}{\ln(x_0 + \frac{Q^2}{\Lambda^2_{\text{QCD}}})}\},$$

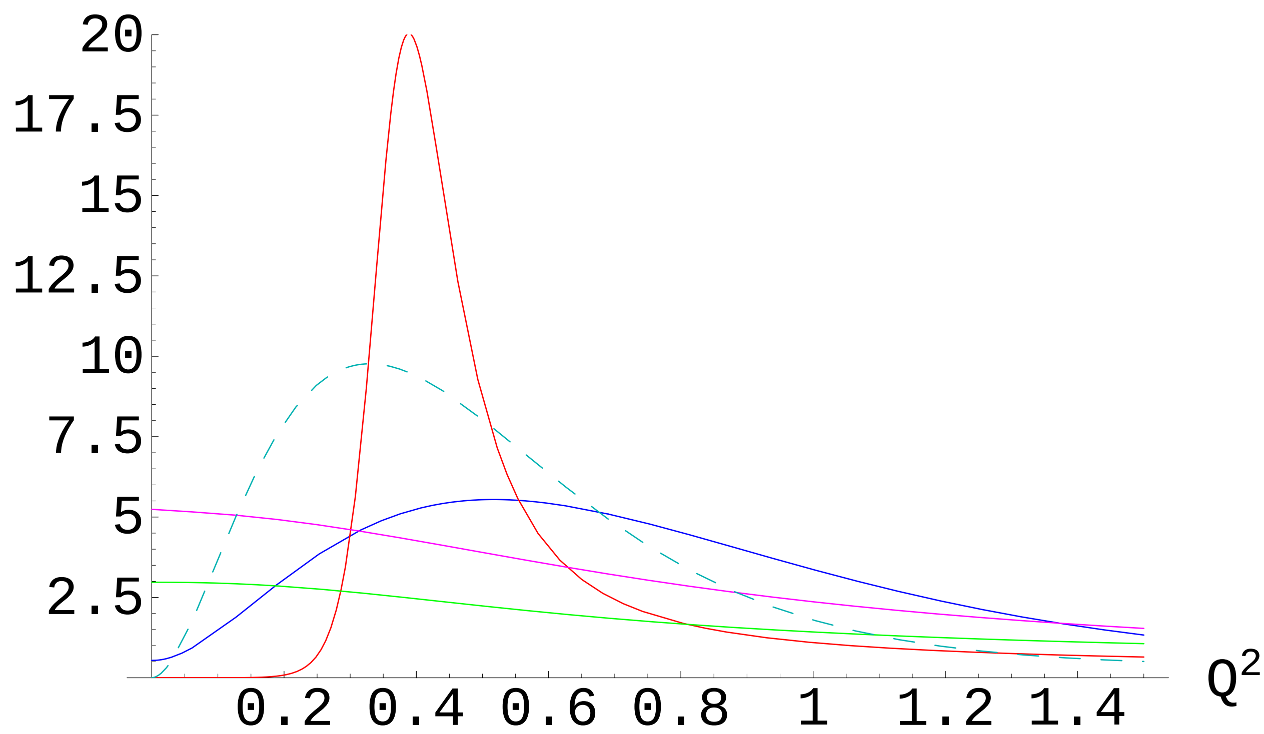
- but modeled non-perturbative part, e.g., Jain & Munczek:

$$G_{\text{IR}}(Q^2) = G_{\text{non-pert}}(Q^2) = 4\pi^2 \, a \, Q^2 \, \exp(-\mu Q^2) \quad (\text{similar}: \ \text{Maris, Roberts...})$$

- or, dressed propagator with dynamical gluon mass induced by dim. 2 gluon condensate $\langle A^2 \rangle$ (Kekez & Klabučar, PRD 71 (2005) 014004):

$$G(Q^2) = 4\pi \frac{\alpha_s^{\text{pert}}(Q^2)}{Q^2} \left(\frac{Q^2}{Q^2 - M^2_{\text{gluon}} + \frac{c_{\text{ghost}}}{Q^2}} \right)^2 \frac{Q^2}{Q^2 + M^2_{\text{gluon}} + \frac{c_{\text{gluon}}}{Q^2}} .$$

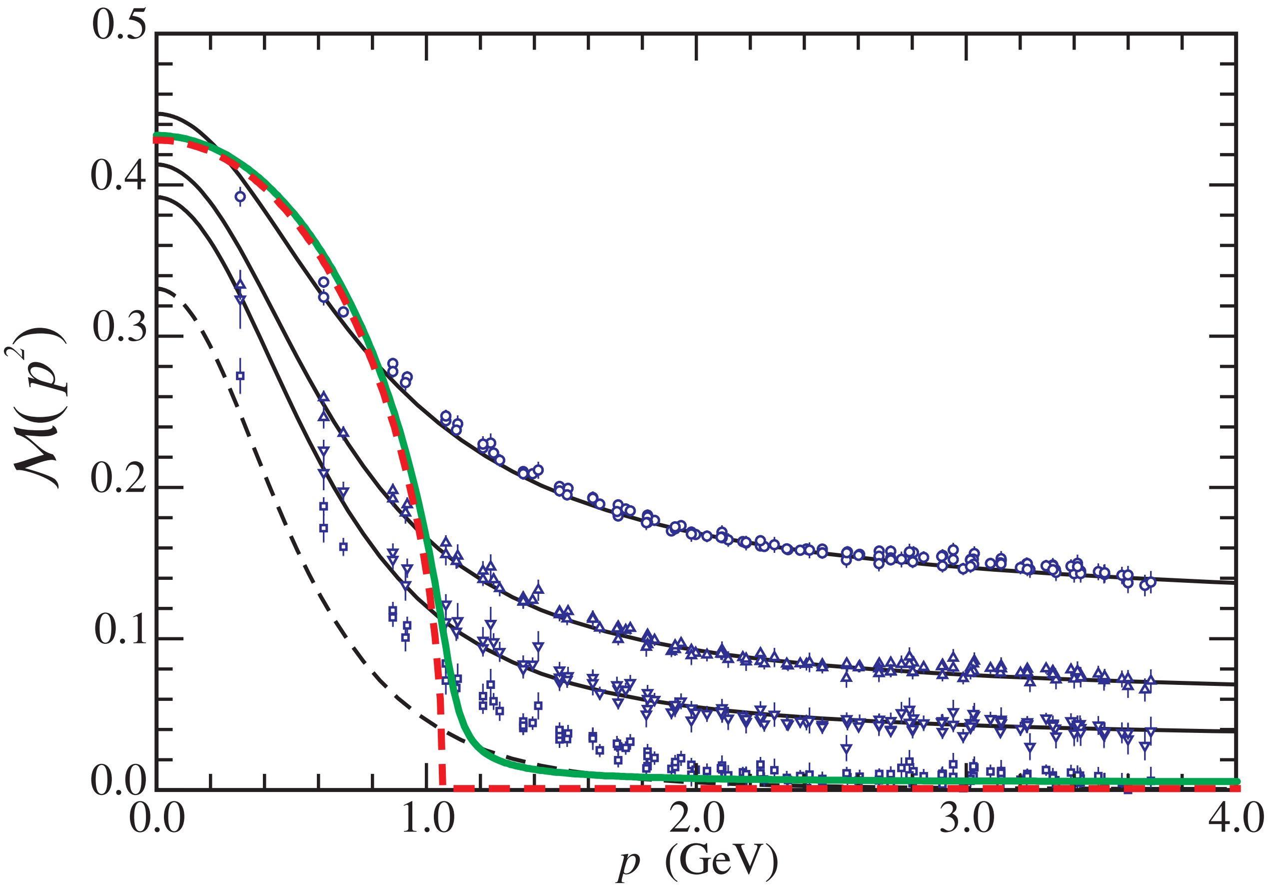
These effective strong couplings $\alpha_s^{\text{eff}}(Q^2) \equiv Q^2\, G(Q^2)/4\pi$



- Blue = Munczek & Jain model. Red = K & K propagator with $\langle A^2 \rangle$-induced dynamical gluon mass. Green = Alkofer. Magenta = Bloch. Turquoise dashed: Maris, Roberts & Tandy model. **Important**: integrated IR strength must be sufficient for **DChSB**!

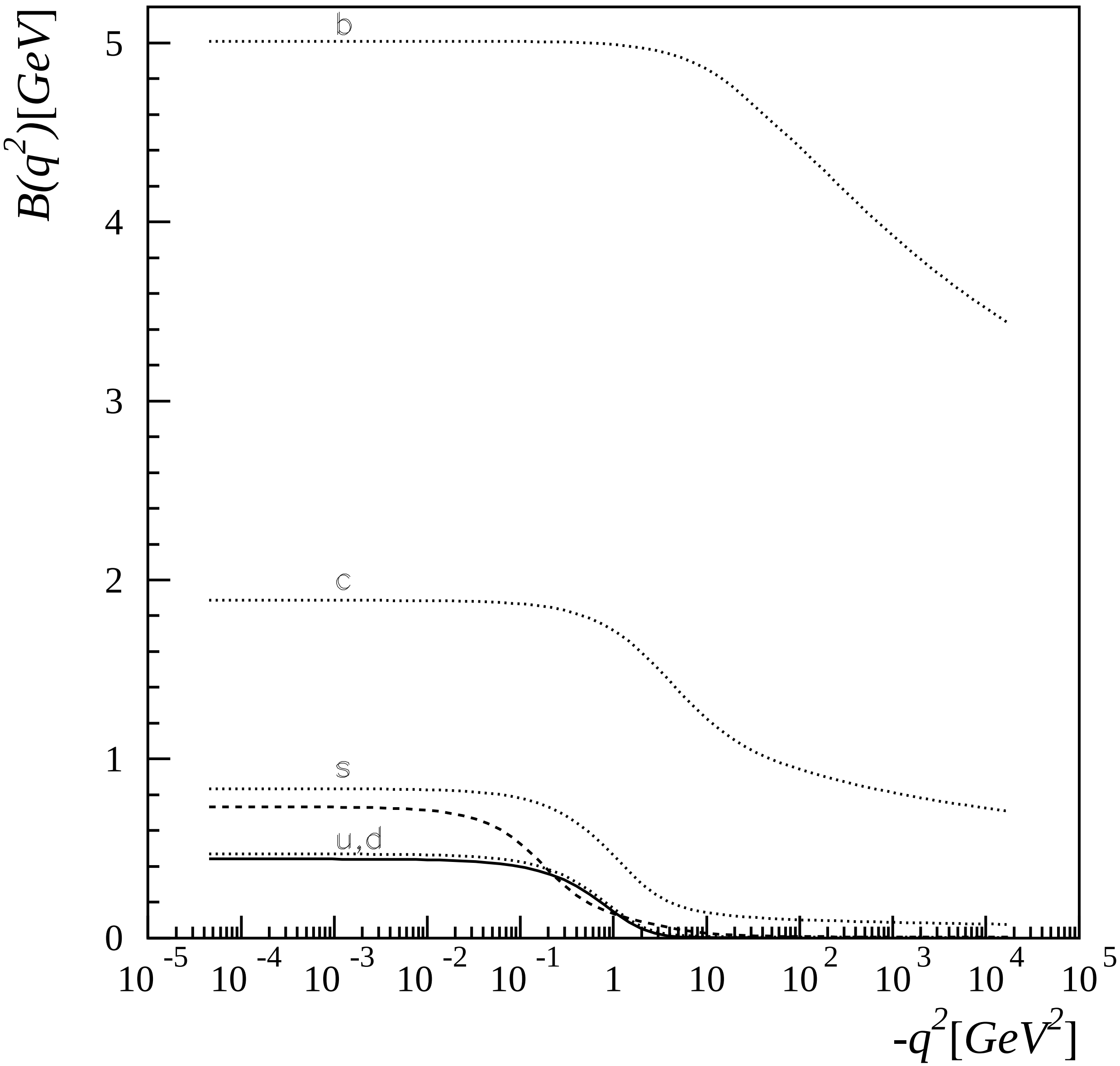
Agreement with lattice and with perturbative QCD

of **realistic DS approaches to QCD (also incorporating pQCD at high p^2)**



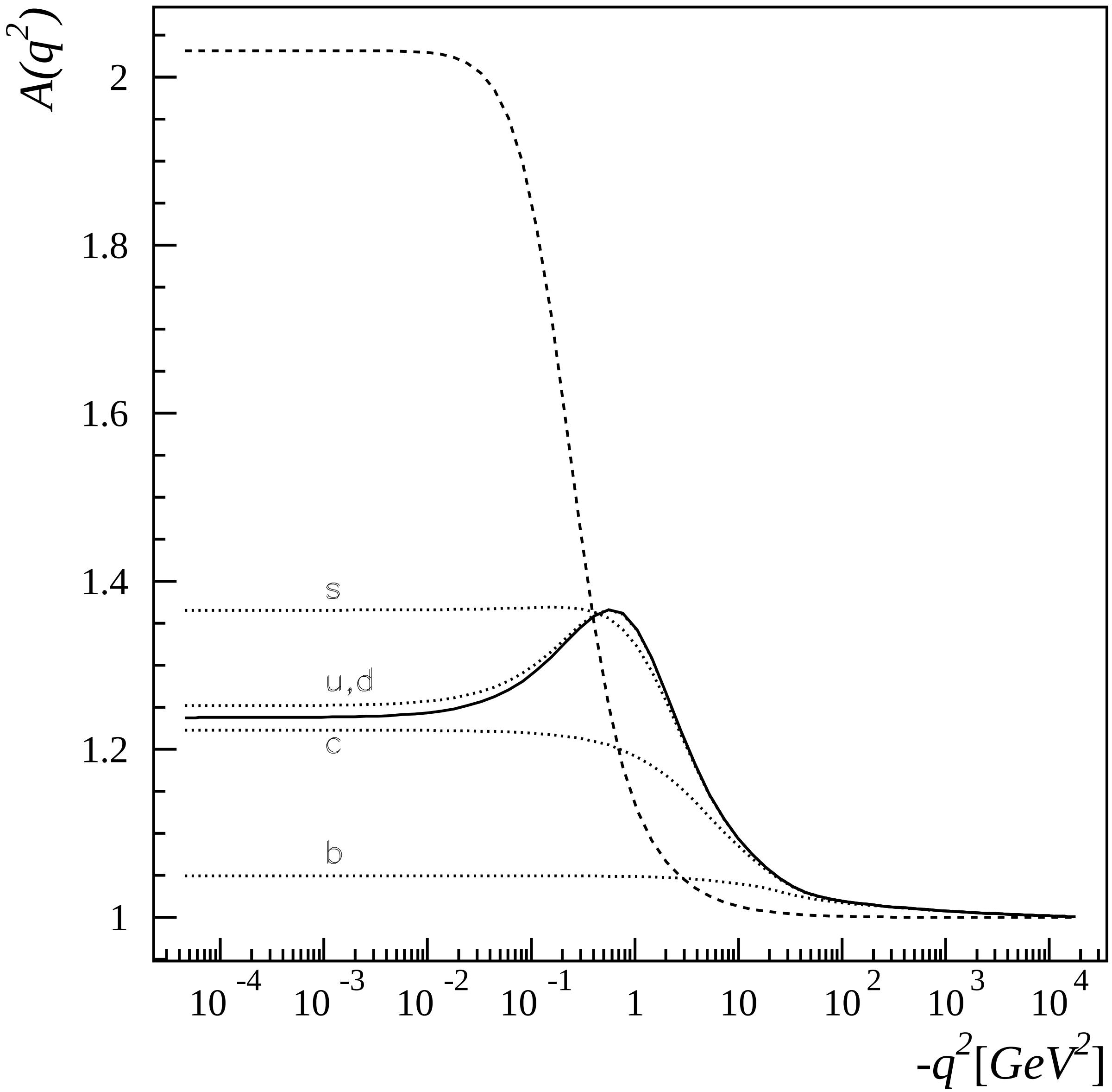
*Lattice data for $\mathcal{M}(p^2)$ compared with numeric sol'ns of gap eq. for realistic DS model in Bhagwat+al, PR***C 68***, 015203 (2003). Dashed curve: sol'n in chi.lim, $m = 0$. Solid crvs: sol'ns for $\mathcal{M}(p^2)$ for current-quark masses $m = 30$ MeV, 55 MeV, and 110 MeV. Red dashed curve is the chiral-limit solution for $\mathcal{M}(p^2)$ from the MN model with $\mathcal{G} = 0.281$ GeV^2, and the solid green curve is the corresponding numerical sol'n with $m = 5$ MeV.*

Results for $B(q^2)$ for the flavors u, d, s, c, b and chiral limit



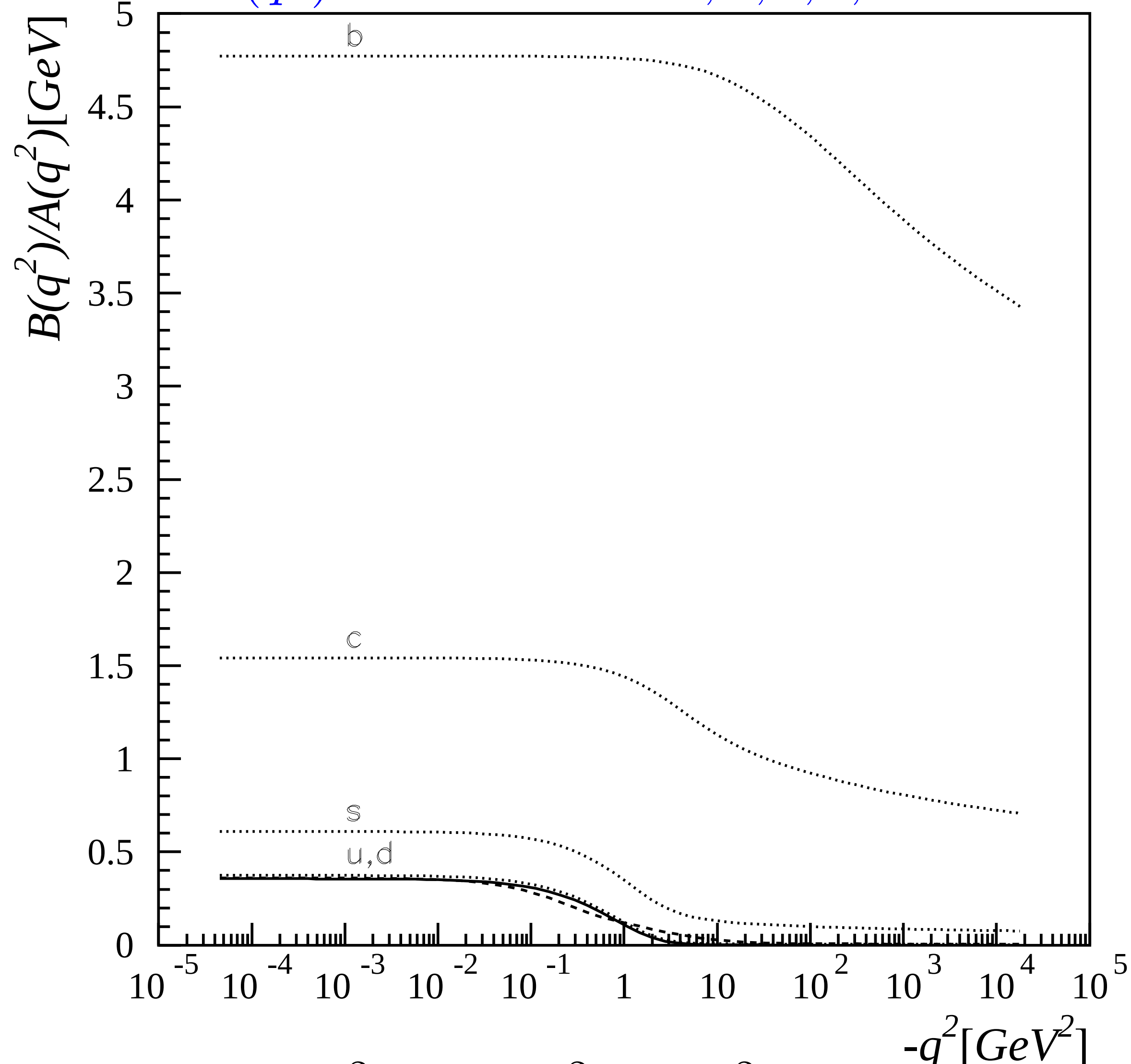
Our (Kekez+al, Int.J.Mod.Phys. A14 (1999) 161) solutions for propagator function $B(q^2)$ in chi. limit $\widetilde{m} = 0$ (solid curve) and for various flavors with various masses $\widetilde{m}(\Lambda) \neq 0$ (dotted curves). Roberts-Frank's Ansatz for u, d quarks is dashed.

Results for $A(q^2)$ for the flavors u, d, s, c, b and chiral limit



Comparison of our chiral-limit solution (solid curve) for the propagator function $A(q^2)$ with our massive solutions for various $\widetilde{m}(\Lambda) \neq 0$ given by the dotted lines marked by flavors, and with Roberts-Frank Ansatz for u, d-quarks (dashed line).

Results for $\mathcal{M}(q^2)$ for the flavors u, d, s, c, b and chiral limit



Our constituent masses $\mathcal{M}(q^2) = B(q^2)/A(q^2)$: solid curve is in chiral limit, while dotted ones (marked by pertinent flavors) denote our constituent quark mass functions for $\widetilde{m}(\Lambda) \neq 0$ (Kekez+al, Int.J.Mod.Phys. A14 (1999) 161). Dashed curve is $\mathcal{M}(q^2)$ following from $A(q^2)$ & $B(q^2)$ Ansätze of Roberts & Frank+al.

Dyson-Schwinger estimate of g_0^2/g_{24}^2 in GBE RCQM

In the chiral limit, f_π gives the normalization of the pseudoscalar $q\bar{q}$ bound-state vertex Γ_π, whereas its $\mathcal{O}(p^0)$ piece is proportional to the scalar propagator function $B(q^2)$, generalizing the GT relation:

$$\Gamma_\pi(q; p^2 = M_\pi^2 = 0) = \frac{2\,B(q^2)_{m=0}}{f_\pi}\,\gamma_5\,.$$

In some applications, this is a reasonable approximation also realistically away from the chiral limit, $B(q^2)_{m=0} \to B(q^2)_{u,d}, B(q^2)_s$. Following the GT analogy, i.e., assuming that the constant pseudoscalar couplings in GBE RCQM are approximations to low-energy magnitudes of the pseudoscalar $q\bar{q}$ vertices, yields $g_0^2/g_{24}^2 > 1$ due to $B(q^2)_s/f_{s\bar{s}} > B(q^2)_{u,d}/f_\pi$. Further, assuming non-anomalous nonet yields $g_0^2/g_{24}^2 \sim 1.5$, in agreement with Plessas, Day & Choi fitting GBE RCQM to phenomenology.

Illustrate this with $B(q^2)_{u,d}/f_\pi$ and $B(q^2)_s/f_{s\bar{s}}$ from the separable model

we used in Horvatić & al, Phys.Rev.D**76** (2007) 096009 [arXiv:0708.1260] as separable model → good fits, + easier to calculate, especially at $T > 0$:

- Calculations simplify with the separable Ansatz for $G^{\rm eff}_{\mu\nu}$:

$$G^{\rm eff}_{\mu\nu}(p-q) \longrightarrow \delta_{\mu\nu}\, G(p^2, q^2, p\cdot q)$$

$$G(p^2, q^2, p\cdot q) = D_0\ f_0(p^2) f_0(q^2) + D_1\ f_1(p^2)(p\cdot q) f_1(q^2)$$

- two strength parameters D_0, D_1, and corresponding form factors $f_i(p^2)$. In the separable model, gap equation yields

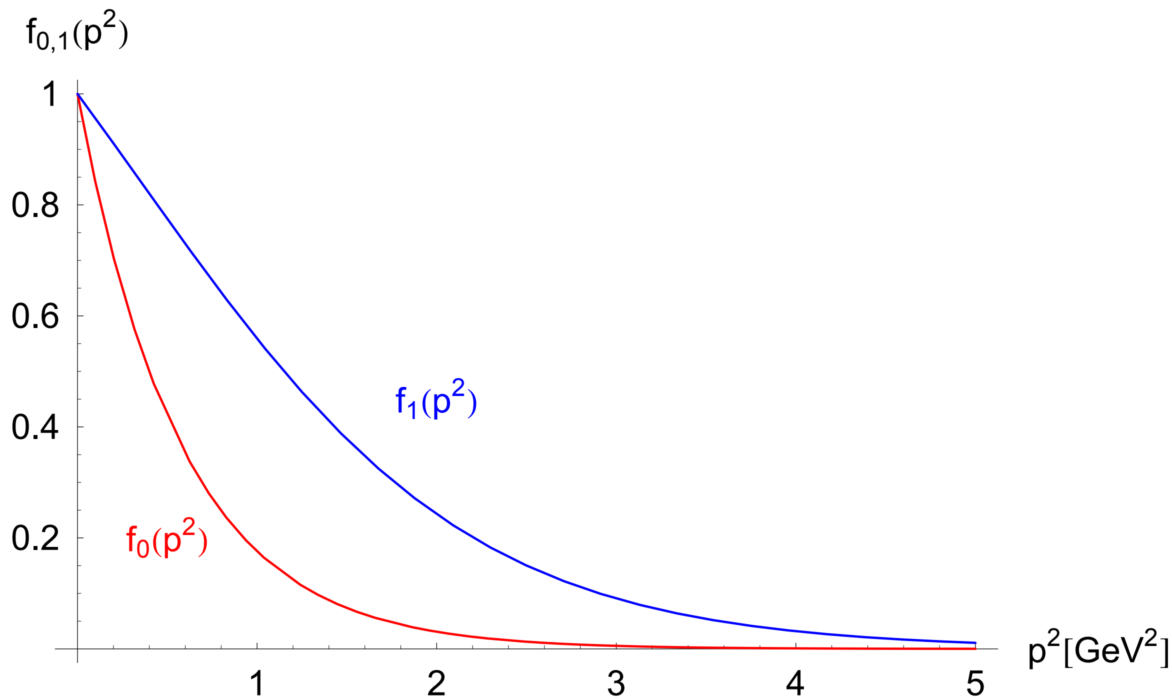
$$\begin{aligned} B_f(p^2) &= \widetilde{m}_f + \frac{16}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{B_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)} \\ \left[A_f(p^2) - 1\right] p^2 &= \frac{8}{3}\int \frac{d^4q}{(2\pi)^4} G(p^2, q^2, p\cdot q) \frac{(p\cdot q) A_f(q^2)}{q^2 A_f^2(q^2) + B_f^2(q^2)} \,. \end{aligned}$$

- This gives $B_f(p^2) = \widetilde{m}_f + b_f\ f_0(p^2)$ and $A_f(p^2) = 1 + a_f\ f_1(p^2)$, reducing to nonlinear equations for constants b_f and a_f.

A simple choice for 'interaction form factors' of the separable model

(no perturbative part, but omitting it is not important at low energies):

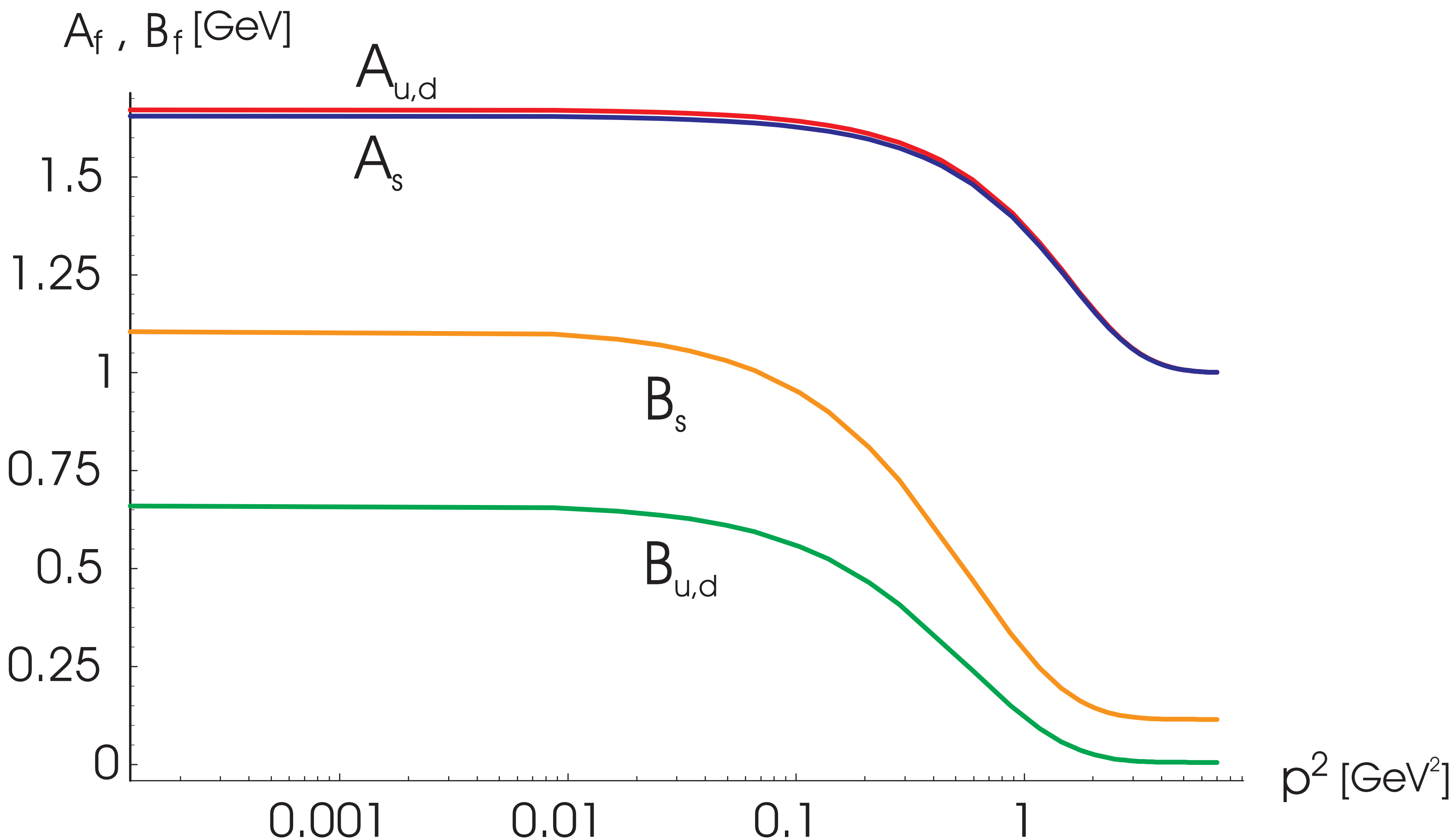
- $f_0(p^2) = \exp(-p^2/\Lambda_0^2)$
- $f_1(p^2) = [1 + \exp(-p_0^2/\Lambda_1^2)]/[1 + \exp((p^2 - p_0^2))/\Lambda_1^2]$
  gives good description of pseudoscalar properties if the interaction is strong enough for realistic DChSB, when $m_{u,d}(p^2 \sim small) \sim$ the typical constituent quark mass scale $\sim M_\rho/2 \sim M_N/3$.



Nonperturbative dynamical propagator dressing

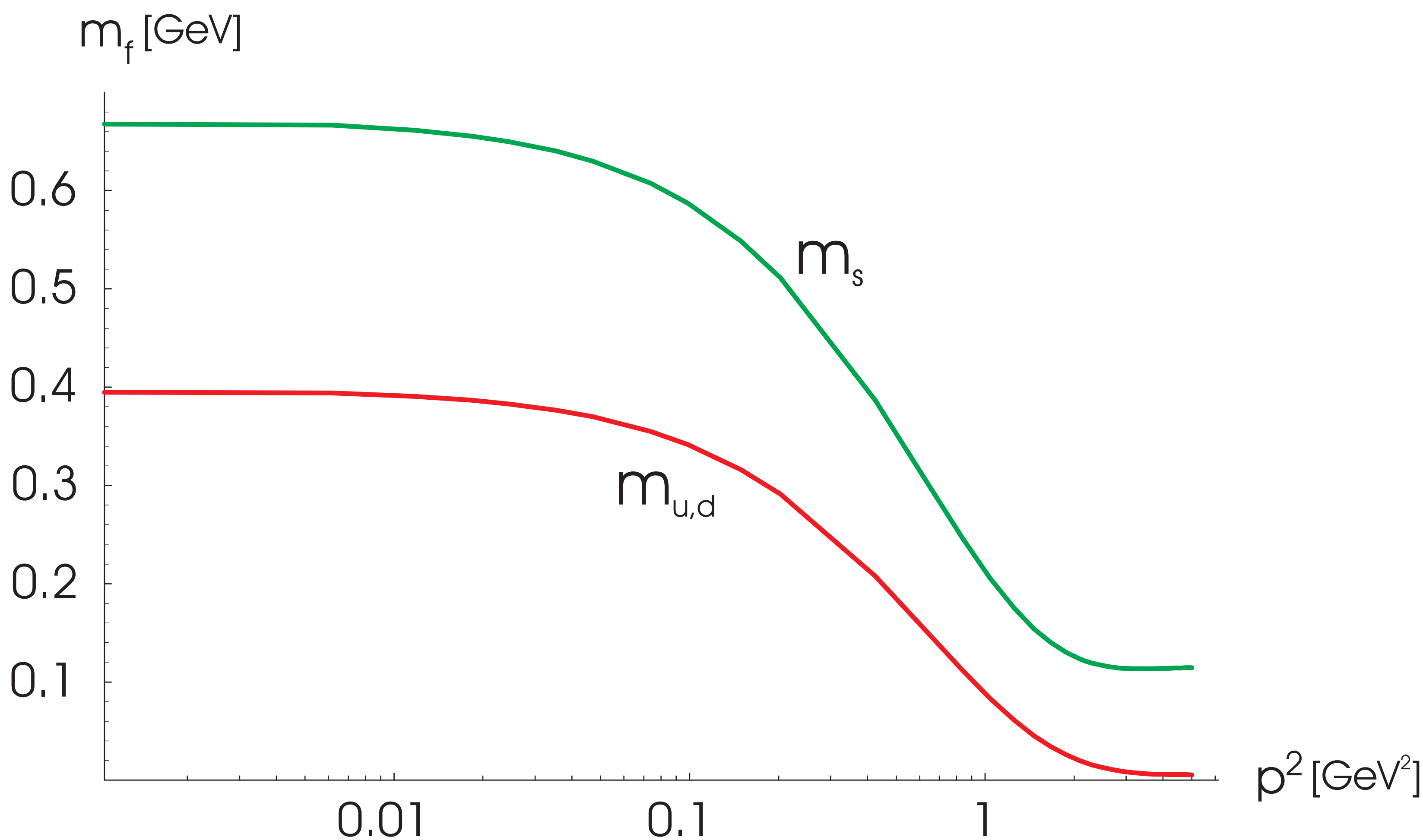
$\longrightarrow$ Dynamical Chiral Symmetry Breaking (DChSB) gives

$B_{u,d}$ and B_s yielding $g_0^2/g_{24}^2 \sim 1.5$ similar to GBE RCQM



DChSB = nonperturb. generation of large quark masses ...

- ... even in the chiral limit ($\widetilde{m}_f \to 0$), where the octet pseudoscalar mesons are Goldstone bosons of DChSB!



Good DS results for PSeudoscalar mesons π, K and "unphysical" $s\bar{s}$:

- Separable model parameter values reproducing experimental data: $\widetilde{m}_{u,d} = 5.5$ MeV, $\Lambda_0 = 758$ MeV, $\Lambda_1 = 961$ MeV, $p_0 = 600$ MeV, $D_0\Lambda_0^2 = 219$, $D_1\Lambda_1^4 = 40$ fixed by fitting M_π, f_π, M_ρ, $g_{\rho\pi^+\pi^-}$, $g_{\rho e^+e^-}$ $\longrightarrow$ pertinent predictions $a_{u,d} = 0.672$, $b_{u,d} = 660$ MeV, i.e., $m_{u,d}(p^2)$, $\langle\bar{u}u\rangle$
- $\widetilde{m}_s = 115$ MeV (fixed by fitting $M_K \rightarrow$ predictions $a_s = 0.657$, $b_s = 998$ MeV, i.e., $m_s(p^2)$, $\langle\bar{s}s\rangle$, $M_{s\bar{s}}$, f_K, $f_{s\bar{s}}$)
- Summary of results (all in GeV) for $q = u, d, s$ and pseudoscalar mesons without the influence of gluon anomaly:

PS	M_{PS}	M_{PS}^{exp}	f_{PS}	f_{PS}^{exp}	$m_q(0)$	$-\langle q\bar{q}\rangle_0^{1/3}$
π	0.140	0.1396	0.092	0.0924 ± 0.0003	0.398	0.217
K	0.495	0.4937	0.110	0.1130 ± 0.0010		
$s\bar{s}$	0.685		0.119		0.672	

Using these $f_\pi = f_{u\bar{u}} = f_{d\bar{d}}$ and $f_{s\bar{s}}$ with $B_{u,d}$ and B_s shown before, yields $g_0^2/g_{24}^2 \sim 1.5$ similar to GBE RCQM of Plessas et al.

Summary

- Dyson-Schwinger approach to the pseudoscalar meson nonet provides an explanation for couplings in GBE RCQM
- DChSB leads to nonperturbatively dressed quarks. Their propagator funct'ns $A(q^2)$ and $B(q^2)$ yield dressed masses $\mathcal{M}(q^2)$ explaining the notion of constituent quarks & their various relationships with current quarks for all flavors in spite of very different current masses.
- Thanks to $B(q^2)_s/f_{s\bar{s}} > B(q^2)_{u,d}/f_\pi$, Dyson-Schwinger approach explains qualitatively the value $g_0^2/g_{24}^2 > 1$ in GBE RCQM.
- More specifically, assuming non-anomalous pseudoscalar nonet, we get $g_0^2/g_{24}^2 \sim 1.5$, in agreement with Plessas, Day & Choi fitting GBE RCQM to phenomenology. Therefore, we propose to try GBE RCQM where: 1.) octet would have π, K but $\eta_{NS} = (u\bar{u} + d\bar{d})/\sqrt{2}$ [degenerate with π!] instead of η_8, and 2.) instead of the singlet η_0, the "unphysical" $\eta_S = \eta_{s\bar{s}} = s\bar{s}$.
  [Note that off-shell particles need not be mass eigenstates anyway!]

Appendix on η-η'

For easier understanding how we estimated g_0^2/g_{24}^2, and what exactly are $\eta_{NS} = (u\bar{u} + d\bar{d})/\sqrt{2}$ and $\eta_S = \eta_{s\bar{s}} = s\bar{s}$ which we propose to try in GBE RCQM, we add this Appendix with additional slides explaining our treatment of the η-η' complex based on the references

- D. Klabučar and D. Kekez, Phys. Rev. D **58** (1998) 096003 [hep-ph/9710206].
- D. Kekez, D. Klabučar and M. D. Scadron, J. Phys. G **26** (2000) 1335 [hep-ph/0003234].
- D. Kekez and D. Klabucar, Phys. Rev. D **73** (2006) 036002 [hep-ph/0512064].

Anomaly and mixing in η-η' complex

- Dyson-Schwinger approach yields mass^2 eigenvalues $M^2_{u\bar{d}} = M^2_{\pi^+}, M^2_{u\bar{s}} = M^2_K, ..., \hat{M}^2_{NA} = \mathrm{diag}(M^2_{u\bar{u}}, M^2_{d\bar{d}}, M^2_{s\bar{s}})$
- $|u\bar{d}\rangle = |\pi^+\rangle, |u\bar{s}\rangle = |K^+\rangle$, ... but $|u\bar{u}\rangle$, $|d\bar{d}\rangle$ and $|s\bar{s}\rangle$ do not correspond to any physical particles (at $T = 0$ at least!), although in the isospin limit (adopted from now on) $M_{u\bar{u}} = M_{d\bar{d}} = M_{u\bar{d}} = M_\pi$. I is a good quantum number!
- $\Rightarrow$ recouple into "more physical" $I_3 = 0$ octet-singlet basis

$$
\begin{aligned}
I = 1 \qquad |\pi^0\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle - |d\bar{d}\rangle) \ , \\
I = 0 \qquad |\eta_8\rangle &= \frac{1}{\sqrt{6}}(|u\bar{u}\rangle + |d\bar{d}\rangle - 2|s\bar{s}\rangle) \ , \\
I = 0 \qquad |\eta_0\rangle &= \frac{1}{\sqrt{3}}(|u\bar{u}\rangle + |d\bar{d}\rangle + |s\bar{s}\rangle) \ .
\end{aligned}
$$

Anomaly and mixing in η-η' complex

- the "non-anomalous" (chiral-limit-vanishing!) part of the mass-squared matrix of π^0 and η's is (in π^0-η_8-η_0 basis)

$$\hat{M}^2_{NA} = \begin{pmatrix} M_\pi^2 & 0 & 0 \\ 0 & M_{88}^2 & M_{80}^2 \\ 0 & M_{08}^2 & M_{00}^2 \end{pmatrix}$$

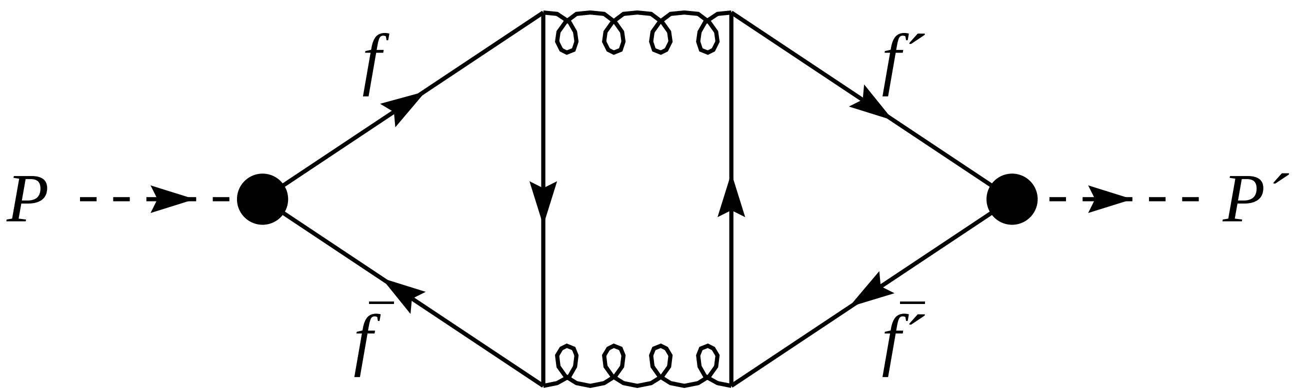
$$M_{88}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_8 \rangle = \frac{2}{3}(M_{s\bar{s}}^2 + \frac{1}{2} M_\pi^2),$$

$$M_{80}^2 \equiv \langle \eta_8 | \hat{M}^2_{NA} | \eta_0 \rangle = M_{08}^2 = \frac{\sqrt{2}}{3}(M_\pi^2 - M_{s\bar{s}}^2)$$

$$M_{00}^2 \equiv \langle \eta_0 | \hat{M}^2_{NA} | \eta_0 \rangle = \frac{2}{3}(\frac{1}{2} M_{s\bar{s}}^2 + M_\pi^2),$$

- Not enough! In order to avoid the $U_A(1)$ problem, one must break the $U_A(1)$ symmetry (as it is destroyed by the gluon anomaly) at least at the level of the masses.

Gluon anomaly is not accessible to ladder approximation!



- **Diamond graph**: an example of a transition $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ ($q, q' = u, d, s[...]$), contributing to the anomalous masses in the η-η' complex, but not included in the interaction kernel in the ladder approximation.

Anomaly and mixing in η-η' complex

- All masses in $\hat{M}^2_{NA}$ are calculated in the ladder approx., which cannot include the gluon anomaly contributions.
- Large N_c: the gluon anomaly suppressed as $1/N_c$! $\to$ Include its effect just at the level of masses: break the $U_A(1)$ symmetry and avoid the $U_A(1)$ problem by shifting the η_0 (squared) mass by anomalous contribution 3β.
- complete mass matrix is then $\hat{M}^2 = \hat{M}^2_{NA} + \hat{M}^2_A$ where

$$\hat{M}^2_A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3\beta \end{pmatrix} \qquad \text{does not vanish in the chiral limit.}$$

$3\beta = \Delta M^2_{\eta_0}$ = the anomalous mass^2 of η_0 [in SU(3) limit incl. ChLim] is related to the YM topological susceptibility. Fixed by phenomenology or (here) taken from the lattice.

Anomaly and mixing in η-η' complex

- we can also rewrite $\hat{M}_A^2$ in the $q\bar{q}$ basis $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, $|s\bar{s}\rangle$

$$\hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow[\text{breaking}]{\text{flavor}} \hat{M}_A^2 = \beta \begin{pmatrix} 1 & 1 & X \\ 1 & 1 & X \\ X & X & X^2 \end{pmatrix}$$

- We introduced the effects of the flavor breaking on the anomaly-induced transitions $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$ $(q, q' = u, d, s)$. $s\bar{s}$ transition suppression estimated by $X \approx f_\pi / f_{s\bar{s}}$.
- Then, $\hat{M}_A^2$ in the octet-singlet basis is modified to

$$\hat{M}_A^2 = \beta \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{2}{3}(1-X)^2 & \frac{\sqrt{2}}{3}(2-X-X^2) \\ 0 & \frac{\sqrt{2}}{3}(2-X-X^2) & \frac{1}{3}(2+X)^2 \end{pmatrix}$$

- $\rightarrow$ In the isospin limit, one can always restrict to 2×2 submatrix of etas $(I=0)$, as π^0 $(I=1)$ is decoupled then.

Anomaly and mixing in η-η' complex

- nonstrange (*NS*) – strange (*S*) basis

$$
\begin{aligned}
|\eta_{NS}\rangle &= \frac{1}{\sqrt{2}}(|u\bar{u}\rangle + |d\bar{d}\rangle) = \frac{1}{\sqrt{3}}|\eta_8\rangle + \sqrt{\frac{2}{3}}|\eta_0\rangle \ , \\
|\eta_S\rangle &= |s\bar{s}\rangle = -\sqrt{\frac{2}{3}}|\eta_8\rangle + \frac{1}{\sqrt{3}}|\eta_0\rangle \ .
\end{aligned}
$$

- the η–η' matrix in this basis is

$$
\hat{M}^2 = \begin{pmatrix} M^2_{\eta_{NS}} & M^2_{\eta_S \eta_{NS}} \\ M^2_{\eta_{NS}\eta_S} & M^2_{\eta_S} \end{pmatrix} = \begin{pmatrix} M^2_{u\bar{u}} + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & M^2_{s\bar{s}} + \beta X^2 \end{pmatrix} \xrightarrow{\phi} \begin{pmatrix} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{pmatrix}
$$

- *NS*–*S* mixing relations

$$
|\eta\rangle = \cos\phi|\eta_{NS}\rangle - \sin\phi|\eta_S\rangle \ , \qquad |\eta'\rangle = \sin\phi|\eta_{NS}\rangle + \cos\phi|\eta_S\rangle \ .
$$

$$
\theta = \phi - \arctan\sqrt{2}
$$

Anomaly and mixing in η-η' complex

- Let lowercase m_M's denote the empirical mass of meson M. From our calculated, model mass matrix in *NS*–*S* basis, we form its empirical counterpart $\hat{m}^2_{\rm exp}$ by
- *i)* obvious substitutions $M_{u\bar{u}} \equiv M_\pi \to m_\pi$, $M_{s\bar{s}} \to m_{s\bar{s}}$
- *ii)* by noting that $m_{s\bar{s}}$, the “empirical" mass of the unphysical $s\bar{s}$ pseudoscalar bound state, is given in terms of masses of physical particles as $m^2_{s\bar{s}} \approx 2m^2_K - m^2_\pi$. Then,

$$\hat{m}^2_{\rm exp} = \left[\begin{array}{cc} m^2_\pi + 2\beta & \sqrt{2}\beta X \\ \sqrt{2}\beta X & 2m^2_K - m^2_\pi + \beta X^2 \end{array} \right] \xrightarrow[\phi_{\rm exp}]{} \left[\begin{array}{cc} m^2_\eta & 0 \\ 0 & m^2_{\eta'} \end{array} \right].$$

Finally, fix anomalous contribution to η-η':

- the trace of the empirical $\hat{m}^2_{\rm exp}$ demands the 1^{st} equality in

$$\beta(2+X^2) = m_\eta^2 + m_{\eta'}^2 - 2m_K^2 = \frac{2N_f}{f_\pi^2}\chi_{\rm YM} \quad (2^{\rm nd}\text{equality} = \text{WV relation})$$

- requiring that the experimental trace $(m_\eta^2 + m_{\eta'}^2)_{exp} \approx 1.22$ GeV^2 be reproduced by the theoretical $\hat{M}^2$, yields $\beta_{\rm fit} = \frac{1}{2+X^2}[(m_\eta^2 + m_{\eta'}^2)_{exp} - (M_{u\bar{u}}^2 + M_{s\bar{s}}^2)]$
- Or, get β from lattice $\chi_{\rm YM}$! Then no free parameters!
- then, can calculate the NS-S mixing angle ϕ

$$\tan 2\phi = \frac{2\, M^2_{\eta_S \eta_{NS}}}{M^2_{\eta_S} - M^2_{\eta_{NS}}} = \frac{2\sqrt{2}\beta X}{M^2_{\eta_S} - M^2_{\eta_{NS}}} \qquad \text{and}$$

$$M^2_{\eta_{NS}} = M^2_{u\bar{u}} + 2\beta = M^2_\pi + 2\beta, \quad M^2_{\eta_S} = M^2_{s\bar{s}} + \beta X^2 = M^2_{s\bar{s}} + \beta\,\frac{f_\pi^2}{f_{s\bar{s}}^2}$$

Anomaly and mixing in η-η' complex

- The diagonalization of the NS-S mass matrix then finally gives us the *calculated* η and η' masses:

$$
\begin{aligned}
M_\eta^2 &= \cos^2\phi\, M_{\eta_{NS}}^2 - \sqrt{2}\beta X \sin 2\phi + \sin^2\phi\, M_{\eta_S}^2 \\
M_{\eta'}^2 &= \sin^2\phi\, M_{\eta_{NS}}^2 + \sqrt{2}\beta X \sin 2\phi + \cos^2\phi\, M_{\eta_S}^2
\end{aligned}
$$

- Equivalently, from the secular determinant,

$$
\begin{aligned}
M_\eta^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 - \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) - \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right] \\
M_{\eta'}^2 &= \frac{1}{2}\left[M_{\eta_{NS}}^2 + M_{\eta_S}^2 + \sqrt{(M_{\eta_{NS}}^2 - M_{\eta_S}^2)^2 + 8\beta^2 X^2} \right] \\
&= \frac{1}{2}\left[M_\pi^2 + M_{s\bar{s}}^2 + \beta(2+X^2) + \sqrt{(M_\pi^2 + 2\beta - M_{s\bar{s}}^2 - \beta X^2)^2 + 8\beta^2 X^2} \right]
\end{aligned}
$$

Separable model results on η and η' mesons (at $T = 0$)

	$\beta_{\rm fit}$	$\beta_{\rm latt.}$	Exp.
θ	-12.22°	-13.92°	
M_η	548.9	543.1	547.75
$M_{\eta'}$	958.5	932.5	957.78
X	0.772	0.772	
3β	0.845	0.781	

- masses are in units of MeV, 3β in units of GeV^2 and the mixing angles are dimensionless.
- $\beta_{\rm latt.}$ was obtained from $\chi_{\rm YM}(T = 0) = (175.7\ {\rm MeV})^4$
- $X = f_\pi / f_{s\bar{s}}$ as well as the whole $\hat{M}^2_{NA}$ (consisting of M_π and $M_{s\bar{s}}$) are calculated model quantities.

For three DS models: summary of $T = 0$ results from WV

from Ref.	J-M&WV	A^2 &WV	separab&WV	orig. Shore	Experiment
M_π	137.3	135.0	140.0		$(138.0)^{isospin}_{average}$
M_K	495.7	494.9	495.0		$(495.7)^{isospin}_{average}$
$M_{s\bar{s}}$	700.7	722.1	684.8		
f_π	93.1	92.9	92.0		92.4 ± 0.3
f_K	113.4	111.5	110.1		113.0 ± 1.0
$f_{s\bar{s}}$	135.0	132.9	119.1		
M_η	568.2	577.1	542.3		547.75 ± 0.12
$M_{\eta'}$	920.4	932.0	932.6		957.78 ± 0.14
ϕ	41.42^o	39.56^o	40.75^o	38.24^o	
θ	-13.32^o	-15.18^o	-13.98^o	-16.5^o	
θ_0	-2.86^o	-5.12^o	-6.80^o	-12.3^o	
θ_8	-22.59^o	-24.14^o	-20.58^o	-20.1^o	
f_0	108.8	107.9	101.8	106.6	
f_8	122.6	121.1	110.7	104.8	
f_η^0	5.4	9.6	12.1	22.8	
$f_{\eta'}^0$	108.7	107.5	101.1	104.2	
f_η^8	113.2	110.5	103.7	98.4	
$f_{\eta'}^8$	-47.1	-49.5	-38.9	-36.1	