THE EQUATION OF STATE
IN THE NAMBU–JONA-LASINIO MODEL

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Talk presented at the
Mini-Workshop BLED 2016: QUARKS, HADRONS, MATTER
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ABSTRACT. We have designed a soluble model similar to the Nambu–Jona-Lasino model, regularized in a box with periodic boundary conditions, and we explore which features of Lattice QCD it can mimic. We present the study of the Equation of State and some new insights we have obtained.
THE EQUATION OF STATE
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OUTLINE

1. THE TWO-LEVEL NAMBU JONA–LASINIO MODEL

2. THE QUASISPIN NJL-like MODEL

3. THE SPECTRUM OF $0^-$ and $0^+$ EXCITATIONS

4. EMERGENCE OF THE $\sigma$ MESON

5. EXTRACTION OF $\pi - \pi$ SCATTERIONG LENGTH

6. THE EQUATION OF STATE AT ZERO BARYON NUMBER

7. THE CHIRAL CONDENSATE

8. THE TWO-FLAVOUR CASE: THE SU(4) SYMMETRY
1 THE TWO-LEVEL NAMBU JONA–LASINIO MODEL


1. We assume a sharp 3-momentum cutoff $0 \leq |\vec{p}_i| \leq \Lambda$;

2. The space is restricted to a box of volume $\mathcal{V}$ with periodic boundary conditions. This gives a finite number of discrete momentum states, $\mathcal{N} = N_h N_c N_f \mathcal{V} \Lambda^3/6\pi^2$ occupied by $N$ quarks.

3. $|\vec{p}_i| \rightarrow P = \frac{3}{4}\Lambda$.

$$
H_{NJL'} = \sum_{i=1}^{N} \left( \gamma_5(i) h(i) P + m_0 \beta(i) \right)
- \frac{2G}{\mathcal{V}} \sum_{i=1}^{N} \sum_{j=1}^{N} \left( \beta(i) \beta(j) + (i\beta(i)\gamma_5(i))(i\beta(j)\gamma_5(j)) \right) \mathcal{P},
$$

$$
\mathcal{P} = \sum_{\vec{p}_i} \sum_{\vec{p}_j} \sum_{\vec{p}_i} \sum_{\vec{p}_j} \delta_{\vec{p}_i' + \vec{p}_j', \vec{p}_i + \vec{p}_j} \langle \vec{p}_i', \vec{p}_j' | \vec{p}_i, \vec{p}_j \rangle
$$

This projector restricts momenta to a sharp cutoff $\Lambda$ and conserves momentum.
2 THE QUASISPIN NJL-like MODEL

(ii) In the NJL model the interaction conserves the sum of momenta of both quarks; in the simplified interaction each quark conserves its momentum.

\[
H = \sum_{k=1}^{N} \left( \gamma_5(k) h(k) P + m_0 \beta(k) \right) \\
- \frac{g}{2} \left( \sum_{k=1}^{N} \beta(k) \sum_{l=1}^{N} \beta(l) + \sum_{k=1}^{N} i \beta(k) \gamma_5(k) \sum_{l=1}^{N} i \beta(l) \gamma_5(l) \right).
\]

Here \( g = 4G/V \).

We introduce the quasispin operators which obey the spin commutation relations

\[
\begin{align*}
j_x &= \frac{1}{2} \beta, \\
j_y &= \frac{1}{2} i \beta \gamma_5, \\
j_z &= \frac{1}{2} \gamma_5,
\end{align*}
\]

\[
R_\alpha = \sum_{k=1}^{N} \frac{1 + h(k)}{2} j_\alpha(k), \\
L_\alpha = \sum_{k=1}^{N} \frac{1 - h(k)}{2} j_\alpha(k), \\
J_\alpha = R_\alpha + L_\alpha = \sum_{k=1}^{N} j_\alpha(k).
\]

The model Hamiltonian can then be written as

\[
H = 2P(R_z - L_z) + 2m_0 J_x - 2g(J_x^2 + J_y^2).
\]
DETERMINING THE MODEL PARAMETERS

\[ M = \sqrt{\left( E_g(N) - E_g(N-1) \right)^2 - P^2} = 335 \text{ MeV} \]

\[ Q = \langle g | \bar{\psi} \psi | g \rangle = \frac{1}{\sqrt{V}} \langle g | \sum_i \beta(i) | g \rangle = \frac{1}{\sqrt{V}} \langle g | J_x | g \rangle = 250^3 \text{ MeV}^3 \]

\[ m_\pi = E_1(N) - E_g(N) = 138 \text{ MeV} \]

Hartree-Fock + RPA gives

\[ M - m_0 = \sqrt{\left( \frac{4}{\pi^2} G \Lambda^3 \right)^2 - \frac{(M - m_0)^2}{M^2} P^2} \]

\[ Q = \frac{\Lambda^3}{\pi^2} \frac{M}{\sqrt{M^2 + P^2}} \]

\[ m_\pi \approx \sqrt{\frac{M^2 + P^2}{M^2}} G \Lambda^3 m_0. \]

\[ \Lambda = 648 \text{ MeV}, \quad G = 40.6 \text{ MeV fm}^3, \quad m_0 = 4.58 \text{ MeV}. \]

These values compare favourably with those of full Nambu-Jona Lasinio

Coimbra: \( \Lambda = 631 \text{ MeV}, \quad G = 40 \text{ MeV fm}^3, \quad m_0 \approx 5 \text{ MeV}, \)

Buballa: \( \Lambda = 664 \text{ MeV}, \quad G = 37.8 \text{ MeV fm}^3, \quad m_0 = 5.0 \text{ MeV}. \)
Figure 1: The $N \rightarrow \infty$ limit
3 THE SPECTRUM OF $0^-$ and $0^+$ EXCITATIONS

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<th>Parity</th>
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<th>$(E - E_0)$ [MeV] $N = 192$</th>
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## 4 EMERGENCE OF THE $\sigma$ MESON

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- 3 $\sigma + \pi (655 + 150)$

+ 56 $\sigma (655)$
# 5 EXTRACTION OF $\pi - \pi$ SCATTERING LENGTH

The average effective pion-pion potential $\bar{V}$ has been extracted from the energy levels of $n$-pion states

$$E_{n\pi} = n m_\pi + \frac{n(n-1)}{2} \bar{V}.$$  

**TEST: $N$-dependence.** In a larger volume, pions are more dilute pions leading to a proportionally smaller $\bar{V}$. In fact, the ratio of $\bar{V}$ for $N = 144$ and $N = 192$ is $10.3/7.1 = 1.45$, close to $192/144 = 1.33$.

In first-order Born approximation ("Lüscher formula")

$$a = \frac{m_\pi/2}{2\pi} \int V(\vec{r}) \, d^3 r = \frac{m_\pi}{4\pi} \bar{V} \mathcal{V}.$$  

For $N = 192$ we have $\bar{V} = -7.1\, \text{MeV}$ and $\mathcal{V} = \pi^2 N/\Lambda^3 = 53\, \text{fm}^3$. This gives $a m_\pi = \frac{m_\pi^2}{4\pi} \bar{V} \mathcal{V} = -0.0768$. to be compared with Lesniak:

-0.034 ("non-uniform fit") or -0.044 ("uniform fit").

The chiral perturbation theory (soft pions) suggests in leading order

$$a_{l=2} m_\pi = -m_\pi^2 / 16\pi f_\pi^2 = -0.0445.$$
We calculate the canonical ensemble for a $N = N'$ quark system as a function of temperature. Physically, this means a system of equal number of quarks and antiquarks.

\[
Z = \sum D(i) \cdot \exp(-E(i)/T),
\]

\[
E = \sum D(i) \cdot \exp(-E(i)/T) \cdot E(i)/Z.
\]

Figure 2: Equation of state in the Quasispin NJL for $N = 96$ and $N = 192$
Figure 3: For comparison: Equation of State in Lattice QCD (also: Petreczky arXiv:1203.5320 )
Figure 4: Equation of state in the Quasispin NJL and in Lattice QCD.
The chiral condensate in the quasispin NJL $Q = \langle g | \bar{\psi} \psi | g \rangle = \frac{1}{V} \langle g | J_x | g \rangle$ is rather constant in the studied range. It drops from $-0.015 \text{ GeV}^3 = (250 \text{ MeV})^3$ at $T = 0$ to $0.009 \text{ GeV}^3$ at $T = 0.3 \text{ GeV}$. The reason is still unclear.

On the other hand, in Lattice QCD the chiral condensate drops to a small value approximately in the temperature range of the phase transition ($T \sim 0.17 \text{ GeV}$).

Figure 5: For comparison: The chiral condensate in Lattice QCD [Petreczky]
In order to proceed to two flavours, a larger group than SU(2) would be needed.

It is the $O(3) \otimes O(3) \subset O(5) \subset O(6)$ group (or equivalently SU(4) group) with fifteen generators

$$\tau^\alpha; \quad \gamma_5 \tau^\alpha; \quad \beta, \quad i\beta \gamma_5 \tau^\alpha; \quad \gamma_5, \quad i\beta \gamma_5, \quad \beta \tau^\alpha,$$

where $\tau^\alpha$ are isospin operators with $\alpha = 1, 2, 3$. With these generators we can express the two-flavour Hamiltonian

$$H = \sum_{k=1}^{N} \left( \gamma_5(k) h(k) P + m_0 \beta(k) \right)$$

$$- \frac{g}{2} \left( \sum_{k=1}^{N} \beta(k) \sum_{l=1}^{N} \beta(l) + \sum_{k=1}^{N} i\beta(k) \gamma_5(k) \vec{\tau}(k) \sum_{l=1}^{N} i\beta(l) \gamma_5(l) \vec{\tau}(l) \right).$$

Work is in progress.
THANKS FOR YOUR ATTENTION!

I SHALL APPRECIATE YOUR CRITICISM AND SUGGESTIONS