# Introduction to supersymmetry

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**Abstract**

This is a pedagogical introduction for graduate students to the (minimal) $N = 1$ supersymmetry in 4 dimensions. It ranges from the supersymmetry algebra, superspace, explicit construction of a general supersymmetric Lagrangian, to the minimal supersymmetric standard model. It discusses various issues like R-parity, the electroweak symmetry breaking, renormalization and spontaneous supersymmetry breaking.

## Contents

# 1 Preface

There are many very good reviews and books on supersymmetry, for example, among many others: [1], [2] and [3] are the classical references, [4] is fastly becoming classical and it is continuously updated, [5] is a very useful introduction (which strongly influenced the present notes) with all computational details, [6] is a clear overview of the main features, [7] and [8] are for those who like more formal approach, [9], [10], [11] and [12] are reviews on susy breaking, [13] is part of the Weinberg's famous course on quantum field theory, [14] is for fans of superspace.

These notes were written for a 10 lectures course of 45 minutes each. At least the basics of field theory are a requisite, as it is the usual course of particle physics, with the standard model.

# 2 Notation and conventions

We will use almost exclusively the 2 component spinor notation (a very detailed review is found for example in [15]). Let us first see its connection with the 4 component notation. The gamma matrices are

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad \gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1}$$

with

$$\sigma^\mu = \left(1, -\sigma^i\right), \quad \bar{\sigma}^\mu = \left(1, \sigma^i\right). \tag{2}$$

so that the usual anticommutation relations are satisfied

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu} \tag{3}$$

with the metric tensor

$$\eta^{\mu\nu} = diag\,(1, -1, -1, -1) \tag{4}$$

As is well known, the Lorentz symmetry algebra SO(1,3) can be equivalently represented by a product SU(2)× SU(2). The spinor indices of the first SU(2) will be denoted by greek letters $\alpha$, $\beta$, ..., while the dotted indices $\dot{\alpha}$, $\dot{\beta}$, ... will denote the second SU(2). The SU(2) indices in (2) will be kept as

$$\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}, \quad \left(\bar{\sigma}^\mu\right)^{\dot{\alpha}\alpha} \tag{5}$$

while a general 4 component Dirac bispinor can be written as

$$\Psi_D = \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \tag{6}$$

The component $\psi_\alpha$ and $\overline{\chi}^{\dot{\alpha}}$ are representations $(1/2, 0)$ and $(0, 1/2)$ respectively under the Lorentz SU(2)× SU(2). Bars on spinors denote usually complex conjugates (an exception is $\bar{\sigma}^\mu$ which is not the complex conjugate of $\sigma^\mu$, but defined explicitly by (5)). During the operation of complex conjugation an undotted index becomes dotted (and the opposite). SU(2) indices can be raised and lowered as

$$\psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta, \quad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta \tag{7}$$

with the 2 dimensional antisymmetric Levi-Civita tensor defined by

$$\epsilon^{12} = 1, \quad \epsilon_{12} = -1 \tag{8}$$

Similarly we have for the antispinors

$$\bar{\psi}_{\dot{\alpha}} = \epsilon_{\dot{\alpha}\dot{\beta}}\bar{\psi}^{\dot{\beta}}, \quad \bar{\psi}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \tag{9}$$

with the 2 dimensional antisymmetric Levi-Civita tensor defined by

$$\epsilon^{\dot{1}\dot{2}} = 1, \quad \epsilon_{\dot{1}\dot{2}} = -1 \tag{10}$$

They enable to express the following useful relations:

$$(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{\alpha\beta}\left(\sigma^\mu\right)_{\beta\dot{\beta}}, \quad (\sigma^\mu)_{\alpha\dot{\alpha}} = \epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}\left(\bar{\sigma}^\mu\right)^{\dot{\beta}\beta} \tag{11}$$

In a 4-component notation, when $\psi = \chi$ we have to do with a Majorana bispinor:

$$\Psi_M = \begin{pmatrix} \psi_\alpha \\ \bar{\psi}^{\dot{\alpha}} \end{pmatrix} \tag{12}$$

for which the usual reality condition applies:

$$\Psi^c_M \equiv C\overline{\Psi}^T_M = \Psi_M, \quad C = i\gamma^2\gamma^0 \tag{13}$$

Lorentz invariant quantities will alway have all SU(2) indices summed (undotted indices with undotted indices, dotted with dotted, this is why it is useful to keep dots), with equal number of upper and lower components. The product convention is

$$\psi\chi \equiv \psi^\alpha\chi_\alpha \tag{14}$$

(the first spinor has always its SU(2) index up), while opposite for antispinors

$$\bar{\psi}\bar{\chi} \equiv \bar{\psi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{15}$$

(the first antispinor has always its SU(2) dotted index down).

The above order convention allows to change order at will in spite of the anticommuting character of the (anti)spinors, i.e.

$$\psi\chi = \chi\psi, \quad \bar{\psi}\bar{\chi} = \bar{\chi}\bar{\psi} \tag{16}$$

The following relations, valid for any spinor $\psi$, $\overline{\psi}$, will turn out to be very useful

$$\psi^\alpha \psi^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\psi\psi \tag{17}$$

$$\overline{\psi}^{\dot{\alpha}}\overline{\psi}^{\dot{\beta}} = +\frac{1}{2}\epsilon^{\dot{\alpha}\dot{\beta}}\overline{\psi\psi} \tag{18}$$

# 3 A supersymmetric free action

Usual symmetries we know are Poincare or internal symmetries like colour or electromagnetism. Generators of such symmetries transform a bosonic field to a bosonic field, and a fermionic field to a fermionic field. So it is natural to think of a diffferent possibility: a symmetry that transforms a bosonic field to a fermionic field and viceversa:

$$BOSON\ (\phi) \quad <-> \quad FERMION\ (\psi) \tag{19}$$

Such symmetries are called supersymmetries. Elements of the group of supersymmetric transformation interchange the bosonic and fermionic components of the supersymmetric multiplet.

We expect that the number of degrees of freedom (d.o.f.) for bosons will be the same as for fermions in a supersymmetric multiplet. This is similar to the zero sum of charges in a multiplet (the sum of all third component spin in a multiplet for example).

Since the Dirac field has 4 d.o.f. on-shell (i.e. after satisfying the equation of motion - the Dirac equation), we need some invariant projection, since in the bosonic sector with spin 0 fields we can have at most a complex field (2 d.o.f.). There are two types of such projections, i.e. the field can be a Weyl (massless) or a Majorana (neutral). We will stick to the two component notation. In the special case below it will be enough to consider the Majorana case, with the limit $m \to 0$ corresponding to the Weyl case.

We first start with free massive fields. In this way we will learn all we need about the supersymmetry algebra. Later on we will generalize the situation for the interacting fields. Let us start then with

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - |m_B|^2 \phi^\dagger \phi + \overline{\psi} i \overline{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m_F \psi\psi - \frac{1}{2} m_F^\dagger \overline{\psi\psi} \quad (20)$$

We want the Lagrangian to be invariant under infinitesimal supersymmetry transformations. What could these transformations be? The ansatz

$$\delta\phi = \epsilon^\alpha \psi_\alpha \quad ; \quad \delta\phi^\dagger = \overline{\psi}_{\dot{\alpha}} \overline{\epsilon}^{\dot{\alpha}} \quad (21)$$

seems reasonable, and tells us, that the mass dimension of the anticommuting Grassman parameter $\epsilon$ is $-1/2$. We then write down for $\delta\psi$ just the most general expansion consistent with linearity in $\epsilon$, Lorentz symmetry and dimensionality:

$$\delta\psi_\alpha = c\,(\sigma^\mu)_{\alpha\dot{\alpha}}\, \overline{\epsilon}^{\dot{\alpha}} \partial_\mu \phi - F \epsilon_\alpha, \quad \delta\overline{\psi}_{\dot{\alpha}} = c^* \epsilon^\alpha\, (\sigma^\mu)_{\alpha\dot{\alpha}}\, \partial_\mu \phi^\dagger - F^\dagger \overline{\epsilon}_{\dot{\alpha}} \quad (22)$$

At this point we don't yet know what are the complex number $c$, and a mass dimension 2 object $F$. To see it, we just plug the above transformations in

$$\begin{aligned} \delta\mathcal{L} &= \partial^\mu \delta\phi^\dagger \partial_\mu \phi + \partial^\mu \phi^\dagger \partial_\mu \delta\phi - |m_B|^2 \left( \delta\phi^\dagger \phi + \phi^\dagger \delta\phi \right) \\ &+ \delta\overline{\psi} i \overline{\sigma}^\mu \partial_\mu \psi + \overline{\psi} i \overline{\sigma}^\mu \partial_\mu \delta\psi - m_F \psi \delta\psi - m_F^\dagger \overline{\psi} \delta\overline{\psi} \end{aligned} \quad (23)$$

A straightforward computation shows that the free Lagrangian is invariant (up to total derivatives) to supersymmetry transformations providing

$$c = -i, \quad m_F = m_B\,(= m), \quad F^\dagger = m\phi \quad (24)$$

In the derivation we used the relation

$$\psi \sigma^\mu \overline{\chi} = -\overline{\chi} \overline{\sigma}^\mu \psi \quad (25)$$

valid for any spinors $\psi$, $\overline{\chi}$.

At this point everything seems ok, except for the fact, that it is a bit strange to have parameters of the Lagrangian (the mass $m$) in the transformation properties. This is connected to the fact that the d.o.f. for a Majorana field and a complex scalar field are the same only on-shell, while off-shell the Majorana field have 4 real d.o.f. It is thus useful to promote the above quantity $F$ to an auxiliary field (2 bosonic d.o.f. off-shell), which equation of motion fixes to (24) (and thus counts zero d.o.f. on-shell). The new Lagrangian can be easily written as

$$\begin{aligned}\mathcal{L} &= \partial^\mu \phi^\dagger \partial_\mu \phi + \overline{\psi} i \overline{\sigma}^\mu \partial_\mu \psi + F^\dagger F \\ &- F m \phi - F^\dagger m^\dagger \phi^\dagger - \frac{1}{2} m \psi\psi - \frac{1}{2} m^\dagger \overline{\psi\psi}\end{aligned} \tag{26}$$

Considering now everything off-shell, we need on top of (21) and (22) also the transformation of $F$:

$$\delta F = -i\partial_\mu \psi^\alpha \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} \overline{\epsilon}^{\dot{\alpha}} \quad ; \quad \delta F^\dagger = i\epsilon^\alpha \left(\sigma^\mu\right)_{\alpha\dot{\alpha}} \partial_\mu \overline{\psi}^{\dot{\alpha}} \tag{27}$$

The Lagrangian (26) is off-shell (i.e. without the use of the equations of motion) invariant (up to total derivatives) under the infinitesimal supersymmetric transformations (21), (22) and (27) collected here below:

$$\delta\phi = \epsilon\psi \quad ; \quad \delta\phi^\dagger = \overline{\psi}\overline{\epsilon} \tag{28}$$

$$\delta\psi = -i\sigma^\mu \overline{\epsilon} \partial_\mu \phi - F\epsilon \quad ; \quad \delta\overline{\psi} = i\epsilon\sigma^\mu \partial_\mu \phi^\dagger - F^\dagger \overline{\epsilon} \tag{29}$$

$$\delta F = -i\partial_\mu \psi \sigma^\mu \overline{\epsilon} \quad ; \quad \delta F^\dagger = i\epsilon\sigma^\mu \partial_\mu \overline{\psi} \tag{30}$$

The equation of motion for the auxilary fields $F$ and $F^\dagger$

$$\frac{\partial \mathcal{L}}{\partial F} = 0, \quad \frac{\partial \mathcal{L}}{\partial F^\dagger} = 0 \tag{31}$$

can be solved explicitly:

$$F^\dagger = m\phi, \quad F = m^\dagger \phi^\dagger \tag{32}$$

giving back the free Lagrangian for a complex boson field and a Majorana fermion field with equal masses:

$$\mathcal{L} = \partial^\mu \phi^\dagger \partial_\mu \phi - |m|^2 |\phi|^2 + \overline{\psi} i \overline{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m \psi\psi - \frac{1}{2} m^\dagger \overline{\psi\psi} \tag{33}$$

# 4 The supersymmetry algebra

The next step is to obtain explicitly the algebra of the generators. To find it out, we can act twice with a general supersymmetry transformation on our fields. Let us show it for the scalar field:

$$\delta_1\delta_2\phi = \delta_1\left(\epsilon_2^\alpha\psi_\alpha\right) = -i\epsilon_2^\alpha\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\overline{\epsilon_1}^{\dot{\alpha}}\partial_\mu\phi - F\epsilon_2^\alpha\epsilon_{1\alpha} \tag{34}$$

which gives

$$\left[\delta_1,\delta_2\right]\phi = -i\left(\epsilon_2\sigma^\mu\overline{\epsilon_1} - \epsilon_1\sigma^\mu\overline{\epsilon_2}\right)\partial_\mu\phi \tag{35}$$

The same exercise can be repeated for the other fields, giving in general

$$\left[\delta_1,\delta_2\right] = -i\left(\epsilon_2\sigma^\mu\overline{\epsilon_1} - \epsilon_1\sigma^\mu\overline{\epsilon_2}\right)\partial_\mu \tag{36}$$

Since this is valid always, i.e. when applied on any field, one can replace

$$\delta \to i\left(\epsilon Q + \overline{Q}\overline{\epsilon}\right) \tag{37}$$

where we introduced a generator for each component of the infinitesimal parameter $\epsilon^\alpha$ ($\overline{\epsilon}^{\dot{\alpha}}$).

Comparing the commutator

$$\begin{aligned}&\left[i\left(\epsilon_1 Q + \overline{Q}\overline{\epsilon_1}\right), i\left(\epsilon_2 Q + \overline{Q}\overline{\epsilon_2}\right)\right] = \\ &\epsilon_1^\alpha\epsilon_2^\beta\left\{Q_\alpha,Q_\beta\right\} + \epsilon_2^\beta\overline{\epsilon_1}^{\dot{\alpha}}\left\{\overline{Q}_{\dot{\alpha}},Q_\beta\right\} - \epsilon_1^\alpha\overline{\epsilon_2}^{\dot{\beta}}\left\{Q_\alpha,\overline{Q}_{\dot{\beta}}\right\} + \overline{\epsilon_1}^{\dot{\alpha}}\overline{\epsilon_2}^{\dot{\beta}}\left\{\overline{Q}_{\dot{\alpha}},\overline{Q}_{\dot{\beta}}\right\}\end{aligned} \tag{38}$$

with (36) one can easily get

$$\left\{Q_\alpha,Q_\beta\right\} = \left\{\overline{Q}_{\dot{\alpha}},\overline{Q}_{\dot{\beta}}\right\} = 0 \tag{39}$$

$$\left\{Q_\alpha,\overline{Q}_{\dot{\alpha}}\right\} = -i\left(\sigma^\mu\right)_{\alpha\dot{\alpha}}\partial_\mu \tag{40}$$

As usual, we assume Poincaré invariance: transformations are generated by the translations

$$P_\mu = -i\partial_\mu \tag{41}$$

and Lorentz transformations, which are the sum of the "angular momentum" and "spin" part

$$M_{\mu\nu} = L_{\mu\nu} + S_{\mu\nu} \tag{42}$$

The "angular momentum" part is

$$L_{\mu\nu} = i\left(x_\mu \partial_\nu - x_\nu \partial_\mu\right) \tag{43}$$

Their algebra is as usual

$$\begin{aligned} [P_\mu, P_\nu] &= 0 & (44)\\ [P_\mu, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho} P_\sigma - \eta_{\mu\sigma} P_\rho\right) & (45)\\ [M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(\eta_{\mu\rho} M_{\nu\sigma} + \eta_{\nu\sigma} M_{\mu\rho} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma}\right) & (46) \end{aligned}$$

How do the supersymetry generators $Q_\alpha$ ($\overline{Q}_{\dot\alpha}$) commute with the generators of the Poincaré algebra $P_\mu$ and $M_{\mu\nu}$?

On the one hand it is difficult to think that a translation can change the effect of a supersymmetric transformation, which reduces to

$$[P_\mu, Q_\alpha] = \left[P_\mu, \overline{Q}_{\dot\alpha}\right] = 0 \tag{47}$$

On the other hand similar commutation relations cannot be true for Lorentz transformations: changing spin with $Q$ or $\overline{Q}$ automatically changes also the Lorentz character. The most general ansatz we can think of is

$$\begin{aligned} [M_{\mu\nu}, Q_\alpha] &= c(\sigma_\mu \overline{\sigma}_\nu - \sigma_\nu \overline{\sigma}_\mu)_\alpha{}^\beta Q_\beta \ , & (48)\\ \left[M_{\mu\nu}, \overline{Q}_{\dot\alpha}\right] &= \bar{c}\overline{Q}_{\dot\beta}(\overline{\sigma}_\mu \sigma_\nu - \overline{\sigma}_\nu \sigma_\mu)^{\dot\beta}{}_{\dot\alpha} & (49) \end{aligned}$$

Using (41) we commute equation (40) with $M_{\mu\nu}$. The righthandside is

$$\left[(\sigma^\rho)_{\alpha\dot\alpha} P_\rho, M_{\mu\nu}\right] = i\,(\sigma^\rho)_{\alpha\dot\alpha}\left(\eta_{\rho\mu} P_\nu - \eta_{\rho\nu} P_\mu\right) \tag{50}$$

while the lefthandside becomes

$$\left[\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\}, M_{\mu\nu}\right] = \left\{Q_\alpha, \left[\overline{Q}_{\dot\alpha}, M_{\mu\nu}\right]\right\} + \left\{\overline{Q}_{\dot\alpha}, [Q_\alpha, M_{\mu\nu}]\right\} \tag{51}$$

Using ($\epsilon^{\mu\nu\rho\chi}$ is a completely antisymmetric Levi-Civita tensor, $\epsilon^{0123} = +1$)

$$\sigma^\mu \overline{\sigma}^\nu \sigma^\rho = \eta^{\mu\nu}\sigma^\rho - \eta^{\mu\rho}\sigma^\nu + \eta^{\nu\rho}\sigma^\mu - i\epsilon^{\mu\nu\rho\chi}\sigma_\chi \tag{52}$$

it is straightforward to obtain $c = -\bar{c} = -i/4$ after comparison with (50). Notice that this has been dictated simply by Lorentz invariance. The complete Poincaré and supersymmetry algebra is thus

$$\begin{aligned}
[P_\mu, P_\nu] &= 0 && (53)\\
[P_\mu, M_{\rho\sigma}] &= i\left(\eta_{\mu\rho}P_\sigma - \eta_{\mu\sigma}P_\rho\right) && (54)\\
[M_{\mu\nu}, M_{\rho\sigma}] &= -i\left(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}\right) && (55)\\
\{Q_\alpha, Q_\beta\} &= \left\{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\right\} = 0 && (56)\\
\left\{Q_\alpha, \overline{Q}_{\dot\alpha}\right\} &= \left(\sigma^\mu\right)_{\alpha\dot\alpha} P_\mu && (57)\\
[P_\mu, Q_\alpha] &= \left[P_\mu, \overline{Q}_{\dot\alpha}\right] = 0 && (58)\\
[M_{\mu\nu}, Q_\alpha] &= -\frac{i}{4}\left(\sigma_\mu\overline{\sigma}_\nu - \sigma_\nu\overline{\sigma}_\mu\right)_\alpha{}^\beta Q_\beta \ , && (59)\\
\left[M_{\mu\nu}, \overline{Q}_{\dot\alpha}\right] &= \frac{i}{4}\overline{Q}_{\dot\beta}\left(\overline{\sigma}_\mu\sigma_\nu - \overline{\sigma}_\nu\sigma_\mu\right)^{\dot\beta}{}_{\dot\alpha} && (60)
\end{aligned}$$

The system is clearly closed. Additional internal (gauge or global) symmetry generators typically commute with the above generators, although there could be exceptions.

# 5 Superspace

It is suggestive that an arbitrary function of translated or Lorentz transformed coordinates can be represented by

$$\begin{aligned}
f(x+a) &= \exp\left(ia^\mu P_\mu\right) f(x) = e^{a^\mu\partial_\mu} f(x) && (61)\\
f(\Lambda x) &= \exp\left(-\frac{i}{2}\theta^{\mu\nu}L_{\mu\nu}\right) f(x) = e^{\frac{1}{2}\theta^{\mu\nu}(x_\mu\partial_\nu - x_\nu\partial_\mu)} f(x) && (62)
\end{aligned}$$

Can something analogue be possible for supersymmetry transformations? In other words, can we generalize space adding new coordinates, such that a supersymmetry transformation will be nothing else than a translation in these new coordinates? The answer is, thanks to Salam and Strathdee, yes, and simplifies a lot the construction of supersymmetry invariant field theories. Since the supersymmetry generators $Q_\alpha$ and $\overline{Q}_{\dot\alpha}$ have fermionic anticommuting (called Grassmann) character, the same will be necessarily true for these new coordinates, $\theta^\alpha$ and $\overline{\theta}^{\dot{\alpha}}$.

In doing that, let us rewrite the generators of supersymmetry transformations with these coordinates and their derivatives. Again, using Lorentz invariance, we can expand

$$Q_\alpha = a\frac{\partial}{\partial\theta^\alpha} + b\left(\sigma^\mu\overline{\theta}\right)_\alpha \partial_\mu \tag{63}$$

$$\overline{Q}_{\dot{\alpha}} = \overline{a}\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + \overline{b}\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \tag{64}$$

with $a$, $b$, $\overline{a}$, $\overline{b}$ complex numbers ($\overline{a}$ and $\overline{b}$ are in general not the complex conjugate of $a$ and $b$). We will try to fix them by requiring the same commutation relations as in (28)-(30).

A generic field of the coordinates $x^\mu$, $\theta^\alpha$ and $\overline{\theta}^{\dot{\alpha}}$ is called a superfield. Due to the limited number ($\theta^1$, $\theta^2$, $\overline{\theta}^{\dot{1}}$, $\overline{\theta}^{\dot{2}}$) and nice properties (a square is always zero) of the Grassmann variables, a superfield can be expanded in a finite series of ordinary space coordinate functions:

$$\begin{aligned} F\left(x,\theta,\overline{\theta}\right) &= f_1(x) + \theta\psi_1(x) + \overline{\theta\psi_2}(x) + \theta\theta f_2(x) + \overline{\theta\theta} f_3(x) \\ &+ \theta\sigma^\mu\overline{\theta} v_\mu(x) + \theta\theta\overline{\theta\psi_3}(x) + \overline{\theta\theta}\theta\psi_4(x) + \theta\theta\overline{\theta\theta} f_4(x) \end{aligned} \tag{65}$$

This is the most general expansion of a function of Grassmann variables, a finite analogue of the Laurent expansion.

The supersymmetry transformation of a superfield is given by

$$\delta F(x,\theta,\bar{\theta}) = i(\epsilon Q + \overline{\epsilon}\overline{Q})F(x,\theta,\overline{\theta}) \tag{66}$$

Such general superfield has however much more components than necessary: off-shell we have 8 complex functions and 4 Majorana spinors, together 16 bosonic and 16 fermionic d.o.f. Fortunately this representation is reducible. In order to do that, we need a constraint that (anti)commutes with the generators (63) and (64), so that it is preserved during the supersymmetry transformations (66). Formally we need such operators $\hat{O}_i$ that reduce the number of d.o.f. (projects any function in superspace into a subsuperspace) through

$$\hat{O}_i F(x,\theta,\overline{\theta}) = 0 \tag{67}$$

and satisfy

$$\left\{\hat{O}_i, Q_\alpha\right\} = \left\{\hat{O}_i, \overline{Q}_{\dot{\alpha}}\right\} = 0 \tag{68}$$

if they have a fermionic character, or

$$\left[\hat{O}_i, Q_\alpha\right] = \left[\hat{O}_i, \overline{Q}_{\dot{\alpha}}\right] = 0 \tag{69}$$

if they have a bosonic character.

## 5.1 Chiral superfields

Operators of the form (63) and (64) with properly chosen constants $a$, $b$, $\overline{a}$, $\overline{b}$ are certainly natural candidates for the operators $\hat{O}_i$ above. It is an easy exercise to find out that operators

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - \left(\frac{\overline{b}}{\overline{a}}\right)\left(\sigma^\mu\overline{\theta}\right)_\alpha \partial_\mu \tag{70}$$

$$\overline{D}_{\dot{\alpha}} = \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - \left(\frac{b}{a}\right)\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \tag{71}$$

do exactly the job we need.

We will not apply both types of operators on the same field, but instead define the chiral superfields those that satisfy

$$\overline{D}_{\dot{\alpha}}\Phi = 0 \tag{72}$$

and antichiral superfields those that satisfy

$$D_\alpha\overline{\Phi} = 0 \tag{73}$$

Let us concentrate on the chiral superfield defined by (72). While a general superfield is a function on $x^\mu$, $\theta^\alpha$ and $\overline{\theta}^{\dot{\alpha}}$, a chiral superfield is a function of only $\theta^\alpha$ and the combination

$$y^\mu = x^\mu - \left(\frac{b}{a}\right)\theta\sigma^\mu\overline{\theta} \tag{74}$$

Due to that, when acting on a chiral superfield, the supersymmetry generators get simplified:

$$Q_\alpha \rightarrow a\frac{\partial}{\partial\theta^\alpha} \tag{75}$$

$$\overline{Q}_{\dot{\alpha}} \rightarrow \frac{\overline{a}b + \overline{b}a}{a}(\theta\sigma^\mu)_{\dot{\alpha}}\partial^y_\mu \tag{76}$$

where the spacetime derivatives $\partial_\mu^y$ act now on $y$ coordinates.

As a check one can see, that any $\Phi = \Phi(y, \theta)$ automatically satisfies (72) and is thus a superfield. It can be expanded as

$$\Phi(y,\theta) = A\phi(y) + B\theta\psi(y) + C\theta\theta F(y) \tag{77}$$

A supersymmetry transformation (66) is then

$$\begin{aligned} & A\delta\phi(y) + B\theta\delta\psi(y) + C\theta\theta\delta F(y) = \\ & i\left(a\epsilon^\alpha \frac{\partial}{\partial\theta^\alpha} + \frac{\overline{a}b + \overline{b}a}{a}(\theta\sigma^\mu\overline{\epsilon})\partial_\mu^y\right)(A\phi(y) + B\theta\psi(y) + C\theta\theta F(y)) \end{aligned} \tag{78}$$

To obtain (28)-(30) the following relations must be satisfied:

$$B = \frac{A}{ia}, \quad C = \frac{A}{2a^2}, \quad \overline{a}b + \overline{b}a = i \tag{79}$$

In doing that we used the relation

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta \tag{80}$$

Similarly an antichiral superfield defined by (73) is a function of only $\overline{\theta}^{\dot{\alpha}}$ and the combination

$$\overline{y}^\mu = x^\mu + \left(\frac{\overline{b}}{\overline{a}}\right)\theta\sigma^\mu\overline{\theta} \tag{81}$$

so that it can be expanded as

$$\overline{\Phi}(\overline{y}, \overline{\theta}) = \overline{A\phi}(\overline{y}) + \overline{B\theta\psi} + \overline{C\theta\theta F}(\overline{y}) \tag{82}$$

When acting on an antichiral superfield, the supersymmetry generators can be written as

$$Q_\alpha \;\to\; \frac{\overline{a}b + \overline{b}a}{\overline{a}}\left(\sigma^\mu\overline{\theta}\right)_\alpha \partial_\mu^{\overline{y}} \tag{83}$$

$$\overline{Q}_{\dot{\alpha}} \;\to\; \overline{a}\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} \tag{84}$$

A supersymmetry transformation (66) on a chiral superfield is

$$\overline{A}\delta\overline{\phi}(\overline{y}) + \overline{B\theta}\delta\overline{\psi}(\overline{y}) + \overline{C\theta\theta}\delta\overline{F}(\overline{y}) = \tag{85}$$
$$i\left(\overline{a}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{\epsilon}^{\dot\alpha} + \frac{\overline{a}b+\overline{b}b}{\overline{a}}\epsilon\sigma^\mu\overline{\theta}\partial^{\overline{y}}_\mu\right)\left(\overline{A\phi}(\overline{y}) + \overline{B\theta\psi}(\overline{y}) + \overline{C\theta\theta F}(\overline{y})\right)$$

and the correct tranasformation properties are obtained only when

$$\overline{B} = -\frac{\overline{A}}{i\overline{a}}, \quad \overline{C} = \frac{\overline{A}}{2\overline{a}^2}, \quad \overline{a}b + \overline{b}a = i \tag{86}$$

which is consistent with (79). Here we used

$$\overline{\theta}^{\dot\alpha}\overline{\theta}^{\dot\beta} = +\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\overline{\theta\theta} \tag{87}$$

Demanding that the hermitian conjugate of the chiral superfield is an antichiral superfield we immediately get also

$$\overline{A} = A^*, \quad \overline{a} = a^* \tag{88}$$

where in this special case the bars on the fields $\phi$, $\psi$, $F$ denote the hermitian conjugation.

The expansion of the chiral and antichiral superfields is then

$$\frac{1}{A}\Phi(y,\theta) = \phi(y) + \frac{1}{ia}\theta\psi(y) + \frac{1}{2a^2}\theta\theta F(y) \tag{89}$$
$$\frac{1}{A^*}\overline{\Phi}(\overline{y},\overline{\theta}) = \overline{\phi}(\overline{y}) - \frac{1}{ia^*}\overline{\theta\psi}(\overline{y}) + \frac{1}{2a^{*2}}\overline{\theta\theta F}(\overline{\theta}) \tag{90}$$

while the transformations of the fields can be simplified by

$$\begin{aligned}\delta\Phi(y,\theta) &= i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)\Phi(y,\theta)\\ &= i\left(a\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} + \frac{i}{a}\theta\sigma^\mu\overline{\epsilon}\partial^y_\mu\right)\Phi(y,\theta)\end{aligned} \tag{91}$$
$$\begin{aligned}\delta\overline{\Phi}(\overline{y},\overline{\theta}) &= i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)\overline{\Phi}(\overline{y},\overline{\theta})\\ &= i\left(a^*\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}}\overline{\epsilon}^{\dot\alpha} + \frac{i}{a^*}\epsilon\sigma^\mu\overline{\theta}\partial^{\overline{y}}_\mu\right)\overline{\Phi}(\overline{y},\overline{\theta})\end{aligned} \tag{92}$$

We will determine the last constants later on from the canonical normalization of fields.

A crucial property of chiral multiplets is that a product of chiral multiplets is again a chiral multiplet. This follows simply from the representation of the covariant derivatives (120), i.e. they are derivatives, and so for any superfields $\Phi_{1,2}$

$$\overline{D}_{\dot{\alpha}}\left(\Phi_1\Phi_2\right) = \left(\overline{D}_{\dot{\alpha}}\Phi_1\right)\Phi_2 + \Phi_1\left(\overline{D}_{\dot{\alpha}}\Phi_2\right) \tag{93}$$

In the special case that $\Phi_{1,2}$ are chiral superfields

$$\overline{D}_{\dot{\alpha}}\Phi_{1,2} = 0 \tag{94}$$

the product is also a chiral superfield since due to (93)

$$\overline{D}_{\dot{\alpha}}\left(\Phi_1\Phi_2\right) = 0 \tag{95}$$

The same is obviously true for antichiral fields.

## 5.2 Vector superfields

Another possible submultiplet of the general superfield is the vector superfield, which is nothing else than a real superfield:

$$\left[V(x,\theta,\overline{\theta})\right]^\dagger = V(x,\theta,\overline{\theta}) \tag{96}$$

In general it can be expanded as

$$\begin{aligned} V(x,\theta,\overline{\theta}) &= C(x) + i\theta\chi(x) - i\overline{\theta}\overline{\chi}(x) + \theta\sigma^\mu\overline{\theta}v_\mu(x) \\ &+ \frac{i}{2}\theta\theta\left[M(x)+iN(x)\right] - \frac{i}{2}\overline{\theta\theta}\left[M(x)-iN(x)\right] \\ &+ i\theta\theta\overline{\theta}\left[\overline{\lambda}(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu\right] - i\overline{\theta\theta}\theta\left[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\overline{\chi}(x)\right] \\ &+ \frac{1}{2}\theta\theta\overline{\theta\theta}\left[D(x) - \frac{1}{2}\partial^\mu\partial_\mu C(x)\right] \end{aligned} \tag{97}$$

We still seem to have a lot of degrees of freedom, 8 bosonic (the real $C$, $M$, $N$, $D$ and $v_\mu$) and 8 fermionic (the two-component complex $\chi$ and $\lambda$). However, if $V$ is a vector multiplet (i.e. real), so is $V+\Phi+\Phi^\dagger$ with $\Phi$ a chiral superfield. It is thus possible to show that the gauge transformation

$$V \rightarrow V_{WZ} = V + \Phi + \Phi^\dagger \tag{98}$$

with properly chosen chiral multiplet $\Phi$ can bring the vector multiplet into a simple form

$$V_{WZ}(x,\theta,\overline{\theta}) = \theta\sigma^\mu\overline{\theta}v_\mu(x) + i\theta\theta\overline{\theta\lambda}(x) - i\overline{\theta\theta}\theta\lambda(x) + \frac{1}{2}\theta\theta\overline{\theta\theta}D(x) \quad (99)$$

This Wess-Zumino gauge is useful because it explicitly reduces the number of degrees of freedom, but one should keep in mind, that a supersymmetry transformation does not preserve it, i.e. $\delta V_{WZ}$ cannot be written anymore in the form (99).

## 5.3 Supersymmetry invariants

We want now see how to obtain supersymmetric invariants in the Lagrangian. The idea is very simple, and can be seen from (30). The highest component ($F$) of a chiral superfield (89) transforms under supersymmetry translation as a total derivative, so its spacetime integral is a supersymmetry invariant. This property is due to the dimensionality of the auxiliary field $F$: since from (26) the mass dimensions of the fields are

$$[\phi] = 1, \quad [\psi] = 3/2, \quad [F] = 2 \quad (100)$$

and that of the transformation parameters

$$[\epsilon] = [\overline{\epsilon}] = -1/2 \quad (101)$$

the field with the largest dimension ($F$ in this case) can transform only through a spacetime derivative.

So the $\theta\theta$ component of the chiral superfield is a supersymmetry invariant (up to total derivatives) and thus a possible invariant term in the Lagrangian:

$$[\Phi]_{\theta\theta} \quad (102)$$

Due to the known properties of chiral superfields, any combination

$$\left[\prod_{i=1}^{n}\Phi_i^{n_i}\right]_{\theta\theta} \quad (103)$$

is also a candidate for a supersymmetric invariant term in the Lagrangian.

Similarly the $\overline{\theta\theta}$ component of any antichiral superfield is an invariant

$$[\overline{\Phi}]_{\overline{\theta}\overline{\theta}} \tag{104}$$

which means that any combination

$$\left[\prod_{i=1}^{n} \overline{\Phi}_i^{n_i}\right]_{\overline{\theta}\overline{\theta}} \tag{105}$$

is also a possible supersymmetry invariant combination in the Lagrangian. Notice that above only pure holomorphic (103) or antiholomorphic (105) products of chiral or antichiral fields are allowed.

For mixed products one should resort to the vector multiplets. As for chiral superfields, also the highest component of the vector multiplet (the auxiliary field $D$) transforms as a total derivative. So the term

$$[V]_{\theta\theta\overline{\theta}\overline{\theta}} \tag{106}$$

is again a supersymmetric invariant (up to total derivatives) and thus a good candidate for a supersymmetry invariant Lagrangian. Notice that it does not matter in which gauge the vector multiplet is written down: the difference is again proportional to a total spacetime derivative, and so irrelevant for the dynamics.

A general supersymmetric invariant Lagrangian can be thus written as

$$\mathcal{L} = \left[K(\Phi, \Phi^\dagger)\right]_{\theta\theta\overline{\theta}\overline{\theta}} + [W(\Phi)]_{\theta\theta} + \left[W^\dagger(\Phi^\dagger)\right]_{\overline{\theta}\overline{\theta}} \tag{107}$$

where the real function $K(\Phi, \Phi^\dagger)$ (which is thus a vector superfield) is called the Kähler potential, while the holomorphic function $W(\Phi)$ is the superpotential. Notice that we have chosen $\overline{W} = W^\dagger$ in order to satisfy the hermiticity (reality) condition for the Lagrangian.

## 5.4 The free Lagrangian again

At this point we have all the ingredients to derive explicitly the free Lagrangian of section 3. We choose for the Kähler potential and superpotential

$$K(\Phi, \Phi^\dagger) = \Phi^\dagger \Phi, \quad W(\Phi) = \frac{m}{2}\Phi^2 \tag{108}$$

Then we expand the chiral superfield using (89). Of course the spacetime coordinates are $x^\mu$, not $y^\mu$ (74), so we have to expand further the single functions as

$$\begin{aligned}
\phi(y) &= \phi(x) - \left(\frac{b}{a}\right)\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) + \frac{1}{4}\left(\frac{b}{a}\right)^2\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi(x) \\
\theta\psi(y) &= \theta\psi(x) + \frac{1}{2}\left(\frac{b}{a}\right)\theta\theta\left(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}\right) \\
\theta\theta F(y) &= \theta\theta F(x)
\end{aligned} \tag{109}$$

where we used

$$(\sigma^\mu)_{\alpha\dot{\alpha}}(\overline{\sigma}^\nu)^{\dot{\alpha}\alpha} = 2\eta^{\mu\nu} \tag{110}$$

Similarly we expand the antichiral superfield $\Phi^\dagger$ as in (90) and then using (81)

$$\begin{aligned}
\phi^*(\overline{y}) &= \phi^*(x) + \left(\frac{\overline{b}}{\overline{a}}\right)\theta\sigma^\mu\overline{\theta}\partial_\mu\phi^*(x) + \frac{1}{4}\left(\frac{\overline{b}}{\overline{a}}\right)^2\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi^*(x) \\
\overline{\theta\psi}(\overline{y}) &= \overline{\theta\psi}(x) - \frac{1}{2}\left(\frac{\overline{b}}{\overline{a}}\right)\overline{\theta\theta}\left(\theta\sigma^\mu\partial_\mu\overline{\psi}(x)\right) \\
\overline{\theta\theta}F^*(\overline{y}) &= \overline{\theta\theta}F^*(x)
\end{aligned} \tag{111}$$

The comparison between (109) and (111) gives

$$\frac{\overline{b}}{\overline{a}} = -\left(\frac{b}{a}\right)^* \tag{112}$$

and due to (79) (or (86)) and (88) we find

$$\frac{b}{a} + \frac{\overline{b}}{\overline{a}} = \frac{i}{|a|^2} \tag{113}$$

Up to total derivatives we get

$$\left[\Phi^\dagger\Phi\right]_{\theta\theta\overline{\theta\theta}} = \left|\frac{A}{2a^2}\right|^2\left(\partial^\mu\phi^*\partial_\mu\phi + \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi + F^*F\right) \tag{114}$$

At this point we can choose

$$A = \overline{A} = 1, \quad a = -\overline{a} = -\frac{i}{\sqrt{2}}, \quad b = -\overline{b} = \frac{1}{\sqrt{2}} \tag{115}$$

In a similar way we can calculate the second part, i.e.

$$\left[\frac{m}{2}\Phi^2\right]_{\theta\theta} = -m\phi F - \frac{m}{2}\psi\psi \tag{116}$$

$$\left[\left(\frac{m}{2}\Phi^2\right)^\dagger\right]_{\overline{\theta\theta}} = -m^*\phi^* F^* - \frac{m^*}{2}\overline{\psi\psi} \tag{117}$$

so that the total Lagrangian (107) with the choice (108) coincides with the free (supersymmetry invariant) Lagrangian (26).

## 5.5 Explicit formulae

At this point, after having fixed all the different constants, it is maybe useful to rewrite some of the above formulae.

The supersymmetry generators are

$$Q_\alpha = -\frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\theta^\alpha} + i\left(\sigma^\mu\overline{\theta}\right)_\alpha \partial_\mu\right) \tag{118}$$

$$\overline{Q}_{\dot{\alpha}} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} + i\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu\right) \tag{119}$$

and the covariant derivatives

$$D_\alpha = \frac{\partial}{\partial\theta^\alpha} - i\left(\sigma^\mu\overline{\theta}\right)_\alpha \partial_\mu \tag{120}$$

$$\overline{D}_{\dot{\alpha}} = \frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}} - i\left(\theta\sigma^\mu\right)_{\dot{\alpha}} \partial_\mu \tag{121}$$

A chiral superfield is a function of $\theta^\alpha$ and

$$y^\mu = x^\mu - i\theta\sigma^\mu\overline{\theta} \tag{122}$$

and can be expaded as

$$\Phi(y,\theta) = \phi(y) + \sqrt{2}\theta\psi(y) - \theta\theta F(y) \tag{123}$$

When acting on a chiral superfield the supersymmetry generators becomes

$$
\begin{aligned}
Q_\alpha &\rightarrow -\frac{i}{\sqrt{2}}\frac{\partial}{\partial\theta^\alpha} &(124)\\
\overline{Q}_{\dot\alpha} &\rightarrow -\sqrt{2}(\theta\sigma^\mu)_{\dot\alpha}\partial^y_\mu &(125)
\end{aligned}
$$

so that a general supersymmetry transformation is

$$
\delta\Phi(y,\theta) = i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)\Phi(y,\theta) = \frac{1}{\sqrt{2}}\left(\epsilon^\alpha\frac{\partial}{\partial\theta^\alpha} - 2i\theta\sigma^\mu\overline{\epsilon}\partial^y_\mu\right)\Phi(y,\theta) \tag{126}
$$

The single components of the chiral superfield are

$$
\begin{aligned}
\phi(y) &= \phi(x) - i\theta\sigma^\mu\overline{\theta}\partial_\mu\phi(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi(x) &(127)\\
\theta\psi(y) &= \theta\psi(x) + \frac{i}{2}\theta\theta\left(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}\right) &(128)\\
\theta\theta F(y) &= \theta\theta F(x) &(129)
\end{aligned}
$$

Putting all together, a chiral superfield gets expanded in general as

$$
\begin{aligned}
\Phi(x,\theta,\overline{\theta}) &= \phi(x) + \sqrt{2}\theta\psi(x) - \theta\theta F(x) - i\left(\theta\sigma^\mu\overline{\theta}\right)\partial_\mu\phi(x)\\
&+ \frac{i}{\sqrt{2}}\theta\theta\left(\partial_\mu\psi(x)\sigma^\mu\overline{\theta}\right) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^\mu\partial_\mu\phi(x) &(130)
\end{aligned}
$$

In a similar way the antichiral superfield is a function of $\overline{\theta}^{\dot\alpha}$ and

$$
\overline{y}^\mu = x^\mu + i\theta\sigma^\mu\overline{\theta} \tag{131}
$$

and can be expanded as

$$
\overline{\Phi}(\overline{y},\overline{\theta}) = \overline{\phi}(\overline{y}) + \sqrt{2}\overline{\theta\psi}(\overline{y}) - \overline{\theta\theta F}(\overline{y}) \tag{132}
$$

When acting on an antichiral superfield the supersymmetry generators becomes

$$
\begin{aligned}
Q_\alpha &\rightarrow \sqrt{2}\left(\sigma^\mu\overline{\theta}\right)_\alpha\partial^{\overline{y}}_\mu &(133)\\
\overline{Q}_{\dot\alpha} &\rightarrow \frac{i}{\sqrt{2}}\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} &(134)
\end{aligned}
$$

so that a general supersymmetry transformation is

$$\delta\overline{\Phi}(\overline{y},\overline{\theta}) = i\left(\epsilon Q + \overline{\epsilon}\overline{Q}\right)\overline{\Phi}(\overline{y},\overline{\theta}) = -\frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\overline{\theta}^{\dot{\alpha}}}\overline{\epsilon}^{\dot{\alpha}} - 2i\epsilon\sigma^{\mu}\overline{\theta}\partial_{\mu}^{\overline{y}}\right)\overline{\Phi}(\overline{y},\overline{\theta}) \tag{135}$$

The single components of the chiral superfield are

$$\overline{\phi}(\overline{y}) = \overline{\phi}(x) + i\theta\sigma^{\mu}\overline{\theta}\partial_{\mu}\overline{\phi}(x) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^{\mu}\partial_{\mu}\overline{\phi}(x) \tag{136}$$

$$\overline{\theta\psi}(\overline{y}) = \overline{\theta\psi}(x) - \frac{i}{2}\overline{\theta\theta}\left(\theta\sigma^{\mu}\partial_{\mu}\overline{\psi}(x)\right) \tag{137}$$

$$\overline{\theta\theta F}(\overline{y}) = \overline{\theta\theta F}(x) \tag{138}$$

Putting all together, an antichiral superfield gets expanded in general as

$$\begin{aligned}\overline{\Phi}(x,\theta,\overline{\theta}) &= \overline{\phi}(x) + \sqrt{2}\overline{\theta\psi}(x) - \overline{\theta\theta F}(x) + i\left(\theta\sigma^{\mu}\overline{\theta}\right)\partial_{\mu}\overline{\phi}(x) \\ &- \frac{i}{\sqrt{2}}\overline{\theta\theta}\left(\theta\sigma^{\mu}\partial_{\mu}\overline{\psi}(x)\right) - \frac{1}{4}\theta\theta\overline{\theta\theta}\partial^{\mu}\partial_{\mu}\overline{\phi}(x)\end{aligned} \tag{139}$$

# 6 The Wess-Zumino model

We can now generalize the free Lagrangian for an arbitrary interacting case. In this section we will consider only theories without gauge interactions, called Wess-Zumino models.

It is not difficult to derive the $\theta\theta$ component for a general superpotential of a single chiral superfield $W(\Phi)$. All we need is just to expand the superpotential around the bosonic component $\phi$:

$$W(\Phi) = W(\phi) + \frac{\partial W}{\partial\phi}(\phi)\left(\Phi - \phi\right) + \ldots \tag{140}$$

which gives

$$\left[W(\Phi)\right]_{\theta\theta} = -\frac{\partial W}{\partial\phi}(\phi)F - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}(\phi)\psi\psi \tag{141}$$

For one chiral superfield, the total Lagrangian for the canonical Kähler

$$K(\Phi,\Phi^{\dagger}) = \Phi^{\dagger}\Phi \tag{142}$$

and a general superpotential $W(\Phi)$ gives

$$\begin{aligned}\mathcal{L} &= \partial^\mu\phi^*\partial_\mu\phi + \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi + F^*F \\ &- \left[\frac{\partial W}{\partial\phi}(\phi)F + \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}(\phi)\psi\psi + h.c.\right]\end{aligned} \tag{143}$$

The equation of motion for the auxiliary field is enough to determine it:

$$F^* = \frac{\partial W}{\partial\phi} \tag{144}$$

and the single field Wess-Zumino Lagrangian with the canonical Kähler potential looks like

$$\mathcal{L} = \partial^\mu\phi^*\partial_\mu\phi - \left|\frac{\partial W}{\partial\phi}\right|^2 + \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}\frac{\partial^2 W}{\partial\phi^2}\psi\psi - \frac{1}{2}\frac{\partial^2 W^*}{\partial\phi^{*2}}\overline{\psi\psi} \tag{145}$$

Needless to say, once we get rid of the auxiliary fields, we are left with just a field theory, although a bit peculiar. In principle we don't need even to know that the theory is supersymmetric. We could in principle use just the usual field theory method, although using the power of supersymmetry makes any analysis simpler and it is thus worth to make it explicit.

Notice that the potential is

$$V = |F|^2 = \left|\frac{\partial W}{\partial\phi}\right|^2 \tag{146}$$

and the energy always positive.

From the mass dimensions

$$[K] = 2, \quad [W] = 3 \tag{147}$$

it can easily be seen that any noncanonical Kähler potential with cubic or higher powers of the chiral and antichiral superfields and any superpotential with quartic or higher powers of chiral superfields is nonrenormalizable.

So the most general renormalizable single field Wess-Zumino model is for

$$W(\Phi) = a\Phi + \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \tag{148}$$

giving for the auxiliary field

$$F^* = a + m\phi + \lambda\phi^2 \tag{149}$$

The explicit form of the Lagrangian is

$$\begin{aligned} \mathcal{L} &= \partial^\mu\phi^*\partial_\mu\phi - \left|a + m\phi + \lambda\phi^2\right|^2 \\ &+ \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}\left(m + 2\lambda\phi\right)\psi\psi - \frac{1}{2}\left(m^* + 2\lambda^*\phi^*\right)\overline{\psi\psi} \end{aligned} \tag{150}$$

At first glance the Lagrangian does not look supersymmetric. For example, it is not clear whether the bosonic and fermionic masses are equal. But the mass is a concept defined only in the minimum of the potential. This is seen explicitly if we expand as usual the bosonic field

$$\phi = v + \varphi \tag{151}$$

with

$$a + mv + \lambda v^2 = 0, \quad \langle\varphi\rangle = 0 \tag{152}$$

The above Lagrangian (150) becomes

$$\begin{aligned} \mathcal{L} &= \partial^\mu\varphi^*\partial_\mu\varphi - \left|\mu\varphi + \lambda\varphi^2\right|^2 \\ &+ \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi - \frac{1}{2}\left(\mu + 2\lambda\varphi\right)\psi\psi - \frac{1}{2}\left(\mu^* + 2\lambda^*\varphi^*\right)\overline{\psi\psi} \end{aligned} \tag{153}$$

with

$$\mu = m + 2\lambda v \tag{154}$$

the common bosonic and fermionic mass. The Lagrangian above has most of the terms a renormalizable (also nonsupersymmetric) field theory would have. What is special in the supersymmetric Lagrangian (153) is that there are relations between different parameters. For example, only two parameters $\mu$ and $\lambda$ describe the mass, the trilinear and quadrilinear terms of the boson field, as well as the fermion mass and Yukawa couplings. This is special with supersymmetry. These relations get maintained by radiative corrections, which would not be true in nonsupersymmetric models, even if we imposed them at tree order.

There are different ways in which the above results can be generalized. We leave it for the exercise.

# 7 Gauge theories

We have already introduced the vector multiplet. It contains a vector field $v_\mu$ which seems a good candidate for a gauge boson. The problem is that the $\theta\theta\overline{\theta\theta}$ of just powers of the vector superfield will not give enough structure. For example, we have in the Wess-Zumino gauge

$$[V_{WZ}]_{\theta\theta\overline{\theta\theta}} = \frac{1}{2}D, \quad [V_{WZ}^2]_{\theta\theta\overline{\theta\theta}} = \frac{1}{2}v^\mu v_\mu, \quad [V_{WZ}^n]_{\theta\theta\overline{\theta\theta}} = 0 \quad \text{for } n \geq 3 \tag{155}$$

This is due to the fact that the fields in $V_{WZ}$ are multiplied by already some powers of $\theta$ and/or $\overline{\theta}$. In practice, what we need, is to reduce their number. We can obtain that by applying the covariant derivative on the vector superfield. This will also introduce automatically the spacetime derivatives, needed for the kinetic terms. Let us concentrate first on the abelian case, i.e. a U(1) gauge theory.

## 7.1 The abelian case

A natural first possibility would be to consider fields like $D_\alpha V$ and $\overline{D}_{\dot\alpha} V$. This makes things just complicated, because these fields are no more real on one side, i.e. $(D_\alpha V)^\dagger \neq D_\alpha V$, and not gauge invariant on the other, i.e. $D_\alpha(\Phi + \Phi^\dagger) \neq 0$ for any chiral supermultiplet $\Phi$. We can solve both problems by introducing a chiral superfield

$$W_\alpha = -\frac{1}{4}\left(\overline{DD}\right) D_\alpha V \tag{156}$$

The field is chiral, because $\overline{D}^3 = 0$ as usual (the operator $\overline{D}$ is a spinor operator) and so

$$\overline{D}_{\dot\alpha} W_\alpha = 0 \tag{157}$$

and it is gauge invariant because

$$\left(\overline{DD}\right) D_\alpha \left(\Phi + \Phi^\dagger\right) = \epsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}\overline{D}_{\dot\beta} D_\alpha \Phi = -2i\epsilon^{\dot\alpha\dot\beta}\overline{D}_{\dot\alpha}\left(\sigma^\mu\right)_{\alpha\dot\beta}\partial_\mu \Phi = 0 \tag{158}$$

where we used the explicit representations (120) and (121).

From the general definition (97) we can now evaluate the chiral superfield

$$W_\alpha(y,\theta) = -i\lambda_\alpha(y) + \theta_\alpha D(y) + \frac{i}{2}\left(\theta\sigma^\mu\overline{\sigma}^\nu\right)_\alpha F_{\mu\nu}(y) - \theta\theta\left(\sigma^\mu\partial_\mu\overline{\lambda}(y)\right)_\alpha \tag{159}$$

with the usual gauge field strength

$$F_{\mu\nu} = \partial_\mu v_\nu - \partial_\nu v_\mu \tag{160}$$

The kinetic term is obtained from the Lorentz invariant product of two such chiral superfields. Up to a total derivative we get

$$[W^\alpha W_\alpha]_{\theta\theta} = 2\overline{\lambda}i\overline{\sigma}^\mu\partial_\mu\lambda + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{161}$$

In a completely analogous way we expand

$$\overline{W}_{\dot\alpha}(\overline{y},\overline{\theta}) = i\overline{\lambda}_{\dot\alpha}(\overline{y}) + \overline{\theta}_{\dot\alpha}D(\overline{y}) - \frac{i}{2}\left(\sigma^\mu\overline{\sigma}^\nu\overline{\theta}\right)_{\dot\alpha} F_{\mu\nu}(\overline{y}) - \overline{\theta\theta}\left(\partial_\mu\lambda(\overline{y})\sigma^\mu\right)_{\dot\alpha} \tag{162}$$

and get

$$\left[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right]_{\overline{\theta\theta}} = 2\overline{\lambda}i\overline{\sigma}^\mu\partial_\mu\lambda + D^2 - \frac{1}{2}F^{\mu\nu}F_{\mu\nu} + \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{163}$$

The correct supersymmetric kinetic term for a U(1) gauge field $v_\mu$ and its partner gaugino $\lambda_\alpha$ is thus

$$\mathcal{L} = \frac{1}{4}[W^\alpha W_\alpha]_{\theta\theta} + \frac{1}{4}\left[\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}\right]_{\overline{\theta\theta}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\lambda}i\overline{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \tag{164}$$

The theory has four bosonic d.o.f. off-shell (3 from $v_\mu$ and one real $D$) and 4 fermionic ($\lambda_\alpha$), which reduce to 2 d.o.f. each when equations of motion are applied.

How do we couple the U(1) vector superfield to a charged chiral superfield? One is tempted to assume a gauge transformation for the whole superfield, since all the single components presumably carry the same U(1) charge. But the transformed superfield

$$\Phi' = e^{i\Lambda(x)}\Phi(y,\theta) \tag{165}$$

is not a chiral superfield[1] anymore since $\overline{D}_{\dot{\alpha}}\Lambda(x) \neq 0$ . This difficulty is immediately solved if $\Lambda(x)$ gets promoted to a chiral superfield $\Lambda(y,\theta)$, so that the transformed superfield remains chiral:

$$\Phi'(y,\theta) = e^{i\Lambda(y,\theta)}\Phi(y,\theta) \tag{166}$$

But in this case $\Lambda^\dagger \neq \Lambda$ and so the usual canonical $\Phi^\dagger\Phi$ in the Kähler potential is not gauge invariant anymore. This is expected, since we need the equivalent of gauge covariant derivatives. We know already the invariance of the gauge kinetic terms under the transformation (98). We called it a gauge transformation and we can now understand the reason. In fact, assuming

$$V \to V - i\left(\Lambda - \Lambda^\dagger\right) \tag{167}$$

as equivalent to the gauge tansformation, we find immediately an invariant Kähler:

$$\left(\Phi^\dagger e^V \Phi\right)' = \Phi^\dagger e^{-i\Lambda^\dagger} e^{V - i\left(\Lambda - \Lambda^\dagger\right)} e^{i\Lambda}\Phi = \Phi^\dagger e^V \Phi \tag{168}$$

since $\Lambda$, $\Lambda^\dagger$ and $V$ all commute. Let us see explicitly what this term is. In doing that we expand the exponential and find, using the Wess-Zumino gauge (a different gauge would give the same result, since the Kähler is gauge invariant)

$$\begin{aligned}
\left[\Phi^\dagger\Phi\right]_{\theta\theta\overline{\theta}\overline{\theta}} &= \partial^\mu\phi^*\partial_\mu\phi + \overline{\psi}i\overline{\sigma}^\mu\partial_\mu\psi + F^*F & (169)\\
\left[\Phi^\dagger V\Phi\right]_{\theta\theta\overline{\theta}\overline{\theta}} &= \frac{i}{2}\left(\partial_\mu\phi^* v^\mu\phi - \phi^* v^\mu\partial_\mu\phi\right) - \frac{1}{2}\overline{\psi}\overline{\sigma}^\mu v_\mu\psi \\
&+ \frac{D}{2}|\phi|^2 + \frac{i}{\sqrt{2}}\left(\phi^*\lambda\psi - \overline{\psi}\overline{\lambda}\phi\right) & (170)\\
\frac{1}{2}\left[\Phi^\dagger V^2\Phi\right]_{\theta\theta\overline{\theta}\overline{\theta}} &= \frac{1}{4}v^\mu v_\mu|\phi|^2 & (171)
\end{aligned}$$

To get the usual normalization, we redefine $V \to 2gV$ in the Kähler potential (and in the transformation rule), with $g$ the gauge coupling:

$$\begin{aligned}
\left[\Phi^\dagger e^{2gV}\Phi\right]_{\theta\theta\overline{\theta}\overline{\theta}} &= \left(D^\mu\phi\right)^* D_\mu\phi + \overline{\psi}i\overline{\sigma}^\mu D_\mu\psi + F^*F \\
&+ gD|\phi|^2 + ig\sqrt{2}\phi^*\lambda\psi - ig\sqrt{2}\overline{\psi}\overline{\lambda}\phi & (172)
\end{aligned}$$

[1]Notice that this problem does not arise for global symmetries, when $\Lambda$ is a constant and thus automatically a chiral superfield.

where the covariant derivatives (not to be confused with the supersymmetry covariant derivatives $D_\alpha$, $\overline{D}_{\dot{\alpha}}$) are defined as usual

$$D_\mu \phi = (\partial_\mu + igv_\mu)\,\phi, \quad D_\mu \psi = (\partial_\mu + igv_\mu)\,\psi \tag{173}$$

We would have guessed the first three terms, i.e. they are obtained with the usual replacement $\partial_\mu \to D_\mu$, but the last three terms are of purely supersymmetric origin.

Another possible supersymmetry and gauge symmetry invariant term is the so-called Fayet-Iliopoulos term, which is nonzero only in the U(1) case:

$$[\xi 2gV]_{\theta\theta\overline{\theta}\overline{\theta}} = g\xi D \tag{174}$$

where $\xi$ is an arbitrary real number.

For more chiral superfields, the Lagrangian, invariant under a U(1) gauge symmetry and supersymmetry, can be written as

$$\mathcal{L} = \left[\sum_i \Phi_i^\dagger e^{2gq_iV}\Phi_i + \xi 2gV\right]_{\theta\theta\overline{\theta}\overline{\theta}} + \left(\left[W(\Phi_i) + \frac{1}{4}W^\alpha W_\alpha\right]_{\theta\theta} + h.c.\right) \tag{175}$$

The U(1) gauge invariance can be explicitly seen (up to total derivatives or field independent terms) from

$$V \to V - \frac{i}{2g}\left(\Lambda - \Lambda^\dagger\right), \quad \Phi_i \to e^{iq_i\Lambda}\Phi_i \tag{176}$$

where $q_i$ is the U(1) charge of $\Phi_i$. Notice that the superpotential $W(\Phi_i)$ must also be invariant under the U(1) guage transformation (176).

Let us now rewrite (175) with component fields. Using (172), (174), (141) and (164) we get

$$\begin{aligned}
\mathcal{L} &= (D^\mu\phi_i)^* D_\mu\phi_i + \overline{\psi}_i i\overline{\sigma}^\mu D_\mu\psi_i + |F_i|^2 \\
&- \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\lambda} i\overline{\sigma}^\mu\partial_\mu\lambda + \frac{1}{2}D^2 \\
&- \left(\frac{\partial W}{\partial\phi_i}F_i + \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(\phi)\psi_i\psi_j + h.c.\right) \\
&+ gDq_i|\phi_i|^2 + ig\sqrt{2}q_i\phi_i^*\lambda\psi_i - ig\sqrt{2}q_i\overline{\psi}_i\overline{\lambda}\phi_i + g\xi D
\end{aligned} \tag{177}$$

with the covariant derivatives

$$D_\mu(\phi,\psi)_i = (\partial_\mu + iq_i g v_\mu)(\phi,\psi)_i \tag{178}$$

One can easily integrate out the auxiliary fields $F_i$ and $D$:

$$F_i^* = \frac{\partial W}{\partial \phi_i}, \quad D = -g\left(q_i|\phi_i|^2 + \xi\right) \tag{179}$$

and finally obtain

$$\begin{aligned}\mathcal{L} &= (D^\mu\phi_i)^* D_\mu\phi_i + \overline{\psi}_i i\overline{\sigma}^\mu D_\mu\psi_i - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\lambda}i\overline{\sigma}^\mu\partial_\mu\lambda \\ &- \left(\frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(\phi)\psi_i\psi_j - ig\sqrt{2}q_i\phi_i^*\lambda\psi_i + h.c.\right) - V\end{aligned} \tag{180}$$

where the potential $V = V(\phi_i,\phi_i^*)$ is the sum of the F-term

$$V_F = \sum_i |F_i|^2 = \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2 \tag{181}$$

and D-term

$$V_D = \frac{1}{2}D^2 = \frac{g^2}{2}\left(\sum_i q_i|\phi_i|^2 + \xi\right)^2 \tag{182}$$

i.e.

$$V(\phi_i,\phi_i^*) = \sum_i \left|\frac{\partial W}{\partial\phi_i}\right|^2 + \frac{g^2}{2}\left(\sum_i q_i|\phi_i|^2 + \xi\right)^2 \tag{183}$$

## 7.2 The nonabelian case

Finally, the whole procedure gets a bit complicated if non-abelian theories are considered. This amounts in changing the formulae above as

$$V \to V^a T^a, \quad \Lambda \to \Lambda^a T^a \tag{184}$$

where the generators of gauge transformation satisfy as usual

$$\left[T^a, T^b\right] = if_{abc}T^c \tag{185}$$

The problem is, that now $\Lambda$, $\Lambda^\dagger$ and $V$ do not commute, and the combination $\Phi^\dagger e^V \Phi$ is not invariant anymore, see eq. (168). We thus generalize the gauge transformation of the vector superfield as

$$\Phi' = e^{i\Lambda}\Phi, \quad \left(e^{2gV}\right)' = e^{i\Lambda^\dagger} e^{2gV} e^{-i\Lambda} \tag{186}$$

which reduces to (176) in the U(1) case. Such a change would make $W_\alpha$ from (156) gauge noninvariant, so we have to generalize it by

$$W_\alpha = -\frac{1}{4}\left(\overline{DD}\right) e^{-2gV} D_\alpha e^{2gV} \tag{187}$$

The function of the new terms with respect to (156) is simply to introduce the covariant derivative on a gaugino $\lambda$ and generalize the expression for the gauge field strength:

$$W_\alpha = T^a W_\alpha^a, \quad Tr\left(T^a T^b\right) = C\delta^{ab} \tag{188}$$

Due to that we have to properly normalize the gauge kinetic term

$$\frac{1}{16g^2C}\left[Tr\left(W^\alpha W_\alpha\right)\right]_{\theta\theta} + h.c. \tag{189}$$

A second difference with respect to the abelian case, is that there is no Fayet-Iliopoulos term (174) now. The reason is simple: it would be either gauge nonivariant (if left as it is) or zero (if taken with a trace).

The Lagrangian for a supersymmetric nonabelian gauge theory can thus be written first as

$$\mathcal{L} = \left[\Phi_i^\dagger\left(e^{2gV}\right)_{ij}\Phi_j\right]_{\theta\theta\overline{\theta\theta}} + \left(\left[W(\Phi_i) + \frac{1}{16g^2}W^{a\alpha}W_\alpha^a\right]_{\theta\theta} + h.c.\right) \tag{190}$$

Then (177) generalizes to

$$\begin{aligned}
\mathcal{L} &= \left(D^\mu\phi_i\right)^* D_\mu\phi_i + \overline{\psi}_i i\overline{\sigma}^\mu D_\mu\psi_i + |F_i|^2 \\
&- \frac{1}{4}F_{\mu\nu}^a F^{a\mu\nu} + \overline{\lambda}^a i\overline{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2}D^a D^a \\
&- \left(\frac{\partial W}{\partial\phi_i}F_i + \frac{1}{2}\frac{\partial^2 W}{\partial\phi_i\partial\phi_j}(\phi)\psi_i\psi_j + h.c.\right) \\
&+ gD^a\phi_i^*\left(T^a\right)_{ij}\phi_j + ig\sqrt{2}\phi_i^*\lambda^a\left(T^a\right)_{ij}\psi_j - ig\sqrt{2}\overline{\psi}_i\overline{\lambda}^a\left(T^a\right)_{ij}\phi_j
\end{aligned} \tag{191}$$

with

$$D_\mu \phi_i = \partial_\mu \phi_i + igv^a_\mu \left(T^a\right)_{ij} \phi_j \tag{192}$$

$$D_\mu \psi_i = \partial_\mu \psi_i + igv^a_\mu \left(T^a\right)_{ij} \psi_j \tag{193}$$

$$D_\mu \lambda^a = \partial_\mu \lambda^a - g f_{abc} v^b_\mu \lambda^c \tag{194}$$

$$F^a_{\mu\nu} = \partial_\mu v^a_\nu - \partial_\nu v^a_\mu - g f_{abc} v^b_\mu v^c_\nu \tag{195}$$

Integrating the auxiliary fields

$$F^*_i = \frac{\partial W}{\partial \phi_i}, \quad D^a = -g\phi^* \left(T^a\right)_{ij} \phi_j \tag{196}$$

leads to the final result

$$\begin{aligned} \mathcal{L} &= \left(D^\mu \phi_i\right)^* D_\mu \phi_i + \overline{\psi}_i i \overline{\sigma}^\mu D_\mu \psi_i - \frac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \overline{\lambda}^a i \overline{\sigma}^\mu D_\mu \lambda^a \\ &- \left( \frac{1}{2} \frac{\partial^2 W}{\partial \phi_i \partial \phi_j} \psi_i \psi_j - ig\sqrt{2} \phi^*_i \lambda^a \left(T^a\right)_{ij} \psi_j + h.c. \right) - V \end{aligned} \tag{197}$$

where the potential $V = V(\phi_i, \phi^*_i)$ is again a sum of the F-terms

$$V_F = \sum_i |F_i|^2 = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \tag{198}$$

and D-terms

$$V_D = \sum_a \frac{1}{2} D^a D^a = \frac{g^2}{2} \left( \phi^*_i \left(T^a\right)_{ij} \phi_j \right)^2 \tag{199}$$

giving

$$V(\phi_i, \phi^*_i) = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 + \frac{g^2}{2} \left( \phi^*_i \left(T^a\right)_{ij} \phi_j \right)^2 \tag{200}$$

# 8 Renormalization

Although in principle a supersymmetric theory is just a field theory and so one can use for renormalization the usual rules, there are some particularities that is good to keep in mind.

First, to keep supersymmetry explicit, one needs the same number of degrees of freedom for bosons and fermions. This means that in using dimensional regularization, one should always run the space-time index $\mu$ of gauge bosons from 0 to 3. Such a scheme is called dimensional reduction (DRED) instead of dimensional regularization (DREG). Since at one loop order they both lead to same results, we will not make the above statements more explicit.

Second, as known, in many processes radiative contributions of fermion loops have a minus sign with respect to boson contributions in the loops. Since in supersymmetry couplings are related, sometimes these cancellations are exact, and true at any order of perturbation theory. This is the case of the superpotential, for which it has been shown, that, due to its holomorphicity, it does not get renormalized at any order in perturbation theory (the famous non-renormalization theorem). Since we are considering only renormalizable models, only the renormalization of the (canonical) Kähler is possible: there is nothing more than wave-function renormalization:

$$\Phi = Z_\Phi^{1/2}\Phi_B \tag{201}$$

where $\Phi$ is the renormalized field and $\Phi_B$ the bare one. Of course the other parameters in the superpotential gets renormalized by induction. Take for example the simplest Wess-Zumino model

$$W = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \tag{202}$$

The nonrenormalization theorem tells us that the superpotential looks the same written in original ($\Phi_B$, $m_B$, $\lambda_B$) or renormalized ($\Phi$, $m$, $\lambda$) quantities, i.e.

$$\frac{m_B}{2}\Phi_B^2 + \frac{\lambda_B}{3}\Phi_B^3 = \frac{m}{2}\Phi^2 + \frac{\lambda}{3}\Phi^3 \tag{203}$$

This means that

$$m = Z_\Phi^{-1} m_B, \quad \lambda = Z_\Phi^{-3/2}\lambda_B \tag{204}$$

In other words, once we know the wave-function $Z_\Phi$, we know everything needed about renormalization.

So for example when we are looking for the renormalization group equation for the coupling constant, we get for $t = \log\mu$ (remember that the bare parameters are independent of $\mu$)

$$\frac{d\lambda}{dt} = \lambda_B \frac{d}{dt} Z_\Phi^{-3/2} = -\frac{3}{2}\lambda \frac{d}{dt}\log Z_\Phi \tag{205}$$

In a similar way the mass is also multiplicatively renormalizable:

$$\frac{dm}{dt} = -m\frac{d}{dt}\log Z_\Phi \tag{206}$$

On top of that, in gauge theories, the gauge couplings also undergo the process of renormalization. In an ordinary theory, the one loop result for the running of the gauge coupling constant is given by

$$\frac{dg}{dt} = -\frac{b}{(4\pi)^2}g^3 \tag{207}$$

where the 1-loop $\beta$ function coefficient is

$$b = \frac{11}{3}C(G) - \frac{2}{3}T_F - \frac{1}{3}T_B\ . \tag{208}$$

The Dynkin index

$$T(R)\delta_{ab} = Tr\,(T_a(R)T_b(R)) \tag{209}$$

and the second Casimir

$$C(R)\delta^{ij} = \sum_a (T_a(R)T_a(R))^{ij} \tag{210}$$

depend on the choice of the gauge group and on the representation involved. The indices $a$, $b$ run over the generators of the group ($N^2-1$ in SU(N)), while $i$, $j$ run from 1 to the dimension of the representation. The normalization usually chosen is $T = 1/2$ for the fundamental representation (quarks, leptons). Then one has in the SU(N) group for the fundamental representation $C = (N^2-1)/(2N)$, and for the adjoint $T = C = N$. The dimension of the representation is $N$ for fundamentals and $N^2-1$ for adjoint. To remember also that in SU(2) the generators in the fundamental are the Pauli matrices $T_a^{ij} = \tau_a^{ij}/2$, while in the adjoint representations are the Levi-Civita antisymmetric tensor $T_a^{ij} = -i\epsilon_{aij}$.

For supersymmetric theories we know that for each fermion (boson) there is a boson (fermion) in the same group representation, so (208) can be written more compactly as

$$b = 3C(G) - T\ . \tag{211}$$

Now let us consider a specific example from [16]

$$W(\Phi) = \lambda\Phi_1\Phi_2\Phi_3 \tag{212}$$

The RG equation for the wave-functions are

$$\frac{d}{dt}\log Z_i = \frac{1}{\pi}\left(C_i\alpha - \frac{D_i}{2}\alpha_\lambda\right) \tag{213}$$

where as usual $\alpha = g^2/4\pi$, $\alpha_\lambda = \lambda^2/4\pi$. $C_i$ is the quadratic Casimir of $\Phi_i$ and $D_i$ is the number of internal fields involved in the Yukawa loop of the $\Phi_i$ propagator. The RGE for the coupling constant can be derived in a similar way as (205) giving ($C = \sum_i C_i$, $D = \sum_i D_i$ and the sum goes over all the fields in (212))

$$\frac{d}{dt}\alpha_\lambda = \frac{\alpha_\lambda}{\pi}\left(\frac{D}{2}\alpha_\lambda - C\alpha\right) \tag{214}$$

which together with (207)

$$\frac{d}{dt}\alpha^{-1} = \frac{b}{2\pi} \tag{215}$$

form a closed system of differential equations.

All this simplifies the calculation with respect to ordinary, non supersymmetric, theories. On top of that, all parameters (massive or dimensionless) are multiplicatively renormalized. This means that its radiative correction is proportional to the tree order value. We have always been used to that for example for fermion and gauge boson masses, because the zero mass limit represented a new symmetry (chiral or gauge). But in ordinary theories this was not true for the mass of a spin 0 field: a tree order massless spin 0 boson could get through radiative corrections in principle arbitrary contributions to its mass. Supersymmetry prevents that, since it links the spin 0 boson mass to the spin 1/2 fermion mass, which is protected by chiral symmetry. In the technical language this is a consequence of the non-renormalization theorem of the superpotential.

In practice the above results are connected with the absence of quadratic divergences in supersymmetric field theories. What happens is that loops with internal fermions cancel the quadratic divergent piece of loops with internal bosons, leaving in the final result at most logarithmic divergences.

# 9 The minimal supersymmetric standard model (MSSM)

Now we have enough knowledge to supersymmetrize a realistic model, i.e. the standard model. Is it enough to promote each field of the standard model to a (chiral or vector) superfield? It turns out that two are the things we have to worry about.

First, as is clear from the above, the superpotential $W$ is a function of chiral superfields. Superfields contain the spinor $\psi$, which is nothing else than the left-handed field in the Dirac notation. On the contrary its hermitian conjugate $W^*$ can contain only right-handed spinors $\overline{\psi}$. But the standard model needs both left-handed fields $Q_L$, $L_L$, and right-handed fields $u_R$, $d_R$, $e_R$. This apparent problem is fortunately only a problem in notation: defining the conjugated field as

$$\psi^c_\alpha \equiv i\left(\sigma^2\right)_{\alpha\dot{\alpha}} \overline{\psi}^{\dot{\alpha}} \tag{216}$$

we see that $\psi^c$ is a left-handed field (has Lorentz index $\alpha$), although the original field $\overline{\psi}$ was a right-handed field (dotted index $\dot{\alpha}$). Of course this is nothing else than the 2-component analogue of the 4-component

$$\Psi^c = C\overline{\Psi}^T \tag{217}$$

Since $C = i\gamma^2\gamma^0$ it automatically follows that

$$\left(\frac{1-\gamma^5}{2}\right)\Psi = \Psi \quad \rightarrow \quad \left(\frac{1+\gamma^5}{2}\right)\Psi^c = \Psi^c \tag{218}$$

i.e. if $\Psi$ is right-handed, $\Psi^c$ is left-handed.

So the fermion fields of the standard model (and its bosonic partners) will be described by the following chiral superfields:

$$\begin{array}{rcl} Q & \sim & (3, 2, 1/6) \\ L & \sim & (1, 2, -1/2) \\ u^c & \sim & (\bar{3}, 1, -2/3) \\ d^c & \sim & (\bar{3}, 1, 1/3) \\ e^c & \sim & (1, 1, 1) \end{array} \tag{219}$$

with the $SU(3)_C \times SU(2)_L \times U(1)_{Y/2}$ numbers explicitly specified. Notice the reversed representations or opposite quantum numbers of the conjugated fields.

The second complication comes from the fact that transforming the Higgs boson into a chiral superfield introduces a new fermionic field (the spin 1/2 Higgsino), so that the anomaly constraints of the standard model gets automatically changed. This would spoil the anomaly cancellation conditions and thus the renormalizability. It is thus mandatory to add another Higgs superfield with the opposite hypercharge. Together we need two chiral superfields (the notation will become clear in the next subsection):

$$\begin{aligned} H_u &\sim (1, 2, 1/2) \\ H_d &\sim (1, 2, -1/2) \end{aligned} \tag{220}$$

Finally we supersymmetrize the gauge bosons, introducing the vector superfields

$$\begin{aligned} V_B &\sim (1, 1, 0) \\ V_W &\sim (1, 3, 0) \\ V_G &\sim (8, 1, 0) \end{aligned} \tag{221}$$

## 9.1 The supersymmetric Lagrangian

Let us concentrate first on the superpotential. To mimic as much as possible the usual Yukawa terms, see (197) and assuming a non-zero ($\mu$) mass for the Higgs superfields we write the following gauge invariant terms:

$$W = \mu H_u H_d + Y_U^{ij} H_u Q_i u_j^c + Y_D^{ij} H_d Q_i d_j^c + Y_E^{ij} H_d L_i e_j^c \tag{222}$$

where $i, j$ are generation indices. The above compact notation should read for example

$$Y_U^{ij} H_u Q_i u_j^c \to Y_U^{ij} (H_u)^m (i\tau^2)_{mn} (Q_i)^{\alpha n a} (u_j^c)_{\alpha a} \tag{223}$$

where $\alpha = 1, 2$ is the Lorentz (left) SU(2) index, $m, n = 1, 2$ the gauge SU(2) indices, and $a = 1, 2, 3$ the gauge SU(3) index .

It is now clear that the notation $H_{u,d}$ tells us, to which right-handed quark each Higgs is coupled in the superpotential. Also, we see that there is another reason why it is not enough to introduce one single Higgs in supersymmetry: either the up quarks or the down quarks would have to be massless, since terms with $H^*_{u,d}$ cannot be introduced due to the holomorphicity of the superpotential.

The Kähler potential is the canonical one:

$$
\begin{aligned}
K_{MSSM} &= Q^\dagger \exp(2g_s V_G) \exp(2gV_W) \exp\left(g' V_B/3\right) Q \\
&+ (u^c)^\dagger \exp(-2g_s V_G) \exp\left(-4g' V_B/3\right) u^c \\
&+ (d^c)^\dagger \exp(-2g_s V_G) \exp\left(2g' V_B/3\right) d^c \\
&+ L^\dagger \exp(2gV_W) \exp\left(-g' V_B\right) L \\
&+ (e^c)^\dagger \exp\left(2g' V_B\right) e^c \\
&+ (H_u)^\dagger \exp(2gV_W) \exp\left(g' V_B\right) H_u \\
&+ (H_d)^\dagger \exp(2gV_W) \exp\left(-g' V_B\right) H_d
\end{aligned} \tag{224}
$$

Finally the gauge kinetic terms are found from

$$
\left[\sum_{a=1}^{8} \frac{W^{a\alpha}(V_G) W^a_\alpha(V_G)}{16 g_s^2} + \sum_{a=1}^{3} \frac{W^{a\alpha}(V_W) W^a_\alpha(V_W)}{16 g^2} + \frac{W^{\alpha}(V_B) W_\alpha(V_B)}{16 g'^2}\right]_{\theta\theta} + h.c. \tag{225}
$$

where

$$
\begin{aligned}
\sum_{a=1}^{8} W^a_\alpha(V_G) \frac{\lambda^a}{2} &= -\frac{1}{4}\left(\overline{DD}\right) e^{-2g_s V_G} D_\alpha e^{2g_s V_G} \\
\sum_{a=1}^{3} W^a_\alpha(V_W) \frac{\tau^a}{2} &= -\frac{1}{4}\left(\overline{DD}\right) e^{-2g V_W} D_\alpha e^{2g V_W} \\
W_\alpha(V_B) &= -\frac{1}{4}\left(\overline{DD}\right) D_\alpha \left(2g' V_B\right)
\end{aligned} \tag{226}
$$

where $\lambda^a$ and $\tau^a$ are the Gell-Mann and Pauli matrices respectively.

The particles of the MSSM are the one from the SM, quarks ($Q$, $u^c$, $d^c$), leptons ($L$, $e^c$), Higgses ($H_u$, $H_d$), gauge bosons ($g$, $W$, $B$), and their supersymmetric partners, squarks ($\tilde{Q}$, $\tilde{u}^c$, $\tilde{d}^c$), sleptons ($\tilde{L}^c$, $\tilde{e}^c$), Higgsinos ($\tilde{H}_u$, $\tilde{H}_d$), gauginos ($\tilde{g}$, $\tilde{W}$, $\tilde{B}$). There are twice as much particles than in the standard model, plus one extra Higgs supermultiplet.

## 9.2 R-parity and other symmetries

The above superpotential (222) has 7 type of fields: $Q$, $L$, $u^c$, $d^c$, $e^c$, $H_u$, $H_d$. Assuming only generation independent symmetries, we have 4 terms, so there can be $7 - 4 = 3$ independent U(1) symmetries. These are the gauge hypercharge, the baryon and the lepton numbers.

In the limit $\mu \to 0$ there is another symmetry, under which the matter fields ($Q$, $u^c$, $d^c$, $L$, $e^c$) have charge $+1$, while the Higgses ($H_u$, $H_d$) have charge $-2$. This is called the Peccei-Quinn symmetry, since it transforms in the opposite way left-handed and right-handed fermions.

This would be all if we were not in supersymmetry. In fact there are other fermions in the theory, gauginos, that could in principle rotate by a (axial) phase. One could think that any gaugino could have its own independent rotation, but a quick glance to (99) is enough to convince us of the contrary. Since any gauge boson is a real field, it cannot have any phase redefinition, and so the U(1) rotation of the gaugino $\lambda$ must be neutralized by an equal rotation of the spinor coordinate $\theta$. The same spinor $\theta$ appears in all vector and chiral superfields, so all gauginos have the same rotation, which is also equal to the difference between the rotation of the boson $\phi$ and fermion $\psi$ in any chiral multiplet.

This is confirmed by an explicit check in (197). We can thus summarize the charges of this so called U(1) R-symmetry:

$$\begin{aligned} R(\lambda) = R(\phi_i) - R(\psi_i) = R(\theta) = 1, \\ R(W) = R(W^\alpha) = R(\phi) - R(F_i) = 2 \end{aligned} \tag{227}$$

For example, MSSM is invariant under the following R-symmetry:

$$\begin{aligned} R(H_u) = 2, \quad R(H_d) = 0, \quad R(u^c) = -1 \\ R(Q) = R(d^c) = R(L) = R(e^c) = R(\lambda) = R(\theta) = 1 \end{aligned} \tag{228}$$

The mentioned symmetries are not necessary anomaly free, although some linear combinations are (for example the hypercharge, or the baryon minus lepton number).

At this point we must however realize, that the situation is different from the standard model. In fact, in the non-supersymmetric standard model gauge symmetry was enough to forbid any renormalizable baryon or lepton number violating term. In some sense, the (approximate) baryon and lepton conservation is a success of the standard model. Any violation of them must automatically include a new scale and it is suppressed by it. This is not true in the supersymmetric version of the standard model. In fact, although (222) is baryon and lepton conserving, not all gauge invariant renormalizable operators have been included. One could as well add

$$\delta W = \epsilon_i H_u L_i + \lambda_{ijk} L_i L_j e^c_k + \lambda'_{ijk} L_i Q_j d^c_k + \lambda''_{ijk} u^c_i d^c_j d^c_k \tag{229}$$

The total superpotential with these terms included would in general be invariant only under the gauge symmetries. Such terms would violate baryon and lepton numbers, and thus mediate proton decay. Due to strong constraints the new baryon and/lepton number violating parameters in (229) should typically be small. It is thus tempting to regulate this smallness with a symmetry. An ideal candidate is a $Z_2$ subgroup of the U(1) R-symmetry (228). If we write the above symmetry transformation as

$$\Phi'_i = e^{iR(\Phi_i)\alpha}\Phi_i \tag{230}$$

then it is enough to keep the element denoted by $\alpha = \pi$. We call this discrete symmetry R-parity. All the fields of the SM have a positive parity, while all the superpartners have negative R-parity. It can be written as

$$R = (-1)^{3(B-L)+2S} \tag{231}$$

where $B$, $L$ and $S$ are baryon number, lepton number, and spin. Clearly, this parity, as all R-symmetries, does not commute with supersymmetry (bosons and fermions in the same multiplet have different R-charges).

Most works in the MSSM assume this parity. It must be kept in mind however, that there is really no good reason to believe it exact. In fact, all we know is that constraints from nucleon lifetime (and others) must be satisfied. But this still allows at least some of the baryon and/or lepton number violating parameters nonzero, and in some cases also large. Not only, with for example just lepton number violating terms (and without introducing any other degree of freedom like for example gauge singlet right-handed neutrinos) we can describe the nonzero neutrino masses, without entering into conflict with any experimental result. So, although we will assume in the following for simplicity that this R-symmetry is exact, we must remember that this is just an assumption, which will have to be experimentally verified sooner or later.

## 9.3 The supersymmetry breaking soft terms

As we repeatedly said, supersymmetry implies exact degeneracy among members of the same supermultiplet. This means that for example a spin 0 particle with the same mass and quantum numbers of the electron must exist. Since this is experimentally ruled out, supersymmetry, if it exists, must be broken. In this section we will not be interested in how supersymmetry is broken, i.e. in the dynamics, which study we postpone. All we need now is the parametrization of this breaking, applied to the (R-parity conserving) MSSM.

A strongly broken symmetry is not a symmetry even as an approximation, so we will limit ourselves to soft breaking. Technically this means that only operators with mass dimension less or equal 3 are allowed, which in turn implies in theories without gauge singlets (like the MSSM) that no quadratic divergences will be generated.

Let's now classify them (we limit ourselves to the R-parity conserving ones):

- Gaugino masses: $m_\lambda \lambda\lambda + h.c.$

$$m_{\tilde{B}}\tilde{B}\tilde{B} + m_{\tilde{W}}\tilde{W}\tilde{W} + m_{\tilde{G}}\tilde{G}\tilde{G} + h.c. \tag{232}$$

- Boson masses: $\left(m^2_{\tilde{\phi}}\right)_{ij}\tilde{\phi}_i\tilde{\phi}^*_j$

$$\begin{aligned} &\left(m^2_{\tilde{Q}}\right)_{ij}\tilde{Q}_i\tilde{Q}^*_j + \left(m^2_{\tilde{u}^c}\right)_{ij}\tilde{u}^c_i\tilde{u^c}^*_j + \left(m^2_{\tilde{d}^c}\right)_{ij}\tilde{d}^c_i\tilde{d^c}^*_j \qquad (233)\\ + \quad &\left(m^2_{\tilde{L}}\right)_{ij}\tilde{L}_i\tilde{L}^*_j + \left(m^2_{\tilde{e}^c}\right)_{ij}\tilde{e}^c_i\tilde{e^c}^*_j + m^2_{H_u}|H_u|^2 + m^2_{H_d}|H_d|^2 \end{aligned}$$

- bilinear terms ($B$ terms): $B_{ij}\phi_i\phi_j + h.c.$

$$BH_uH_d + h.c. \tag{234}$$

- trilinear terms ($A$ terms): $A_{ijk}\phi_i\phi_j\phi_k + h.c.$

$$A^{ij}_U H_u\tilde{Q}_i\tilde{u}^c_j + A^{ij}_D H_d\tilde{Q}_i\tilde{d}^c_j + A^{ij}_E H_d\tilde{L}_i\tilde{e}^c_j + h.c, \tag{235}$$

- non-holomorphic trilinear terms ($A'$ terms): $A'_{ijk}\phi_i^*\phi_j\phi_k + h.c.$

$$A_U^{\prime ij} H_d^\dagger \tilde{Q}_i \tilde{u}_j^c + A_D^{\prime ij} H_u^\dagger \tilde{Q}_i \tilde{d}_j^c + A_E^{\prime ij} H_u^\dagger \tilde{L}_i \tilde{e}_j^c + h.c. \tag{236}$$

Notice that there is no need to add the fermionic mass terms, since with proper redefinition of the parameters in the superpotential and the soft terms of the form above one can always get rid of them.

What are the experimental constraints on these parameters? First of all, let's remind the reader, that the parameters in the superpotential are (with the exception of $\mu$) the same as in the standard model. The soft terms however describe interactions of yet unfound particles, like Higgses, sfermions or gauginos. So the first constraints come already from direct searches at LEP, which typically give for the lower limit for masses something around 100 GeV or so. Another way of constraining them are through rare processes, for example flavour changing neutral currents (FCNC) mediated $K - \overline{K}$ mixing, $b \to s\gamma$, $\mu \to e\gamma$, $\mu \to 3e$, $\mu - e$ conversion, etc. Light sfermions could in principle contribute hugely to these processes. This limits the above soft parameters, although due to the large number of them, not uniquely. For example, the easiest way to get rid of such unwanted contribution is to substantially increase the masses of the spartners, since all the above rare processes are inversely proportional to these masses. Masses of order 10 to 100 TeV would suffice, but this would be sad for our perspectives at LHC. Other possibilities in the huge MSSM parameter space are however good as well. For example, if the off-diagonal $\tilde{m}^2$, $A$ and $A'$ terms are very small (or the $A$ and $A'$ have a similar flavour structure as their corresponding Yukawa matrices), then the FCNC contributions will be similarly small as in the SM. A detailed analysis of these processes and constraints is beyond the scope of this introduction.

## 9.4 Electroweak symmetry breaking

There is another experimental constraint on the MSSM parameters: this is the spontaneous electroweak symmetry breaking. It is easy to see, that without soft terms and nonzero $\mu$ the Higgses cannot get nonzero vacuum expectation values. So here the soft contributions are crucial. Let us study this in more detail. The Higgs potential is

$$
\begin{aligned}
V(H_u, H_d) &= \left(|\mu|^2 + m_{H_u}^2\right)|H_u|^2 + \left(|\mu|^2 + m_{H_d}^2\right)|H_d|^2 + (BH_uH_d + h.c.) \\
&+ \frac{g^2}{8}\left(H_u^\dagger \tau^a H_u + H_d^\dagger \tau^a H_d\right)^2 + \frac{g'^2}{8}\left(H_u^\dagger H_u - H_d^\dagger H_d\right)^2 \qquad (237)
\end{aligned}
$$

The terms proportional to $|\mu|^2$ come from the superpotential, the last two are the D-terms for SU(2) ($\propto g^2$) and U(1) ($\propto g'^2$), while the other are supersymmetry breaking soft terms.

In terms of the $U(1)_{EM}$ and CP (i.e. $v_{u,d}$ real) preserving vevs

$$
\langle H_u \rangle = \begin{pmatrix} 0 \\ v_u \end{pmatrix}, \quad \langle H_d \rangle = \begin{pmatrix} v_d \\ 0 \end{pmatrix} \qquad (238)
$$

the potential becomes (with a rotation of $H_{u,d}$ we can always make $B$ real and positive, so $v_{u,d}$ will also be positive)

$$
V(v_u, v_d) = m_u^2 v_u^2 + m_d^2 v_d^2 - 2Bv_uv_d + \frac{g^2 + g'^2}{8}\left(v_u^2 - v_d^2\right)^2 \qquad (239)
$$

where we redefined

$$
m_{u,d}^2 = |\mu|^2 + m_{H_{u,d}}^2 \qquad (240)
$$

By inspecting the potential in the $v_u = v_d$ direction we can immediately see that it is bounded from below only if

$$
m_u^2 + m_d^2 - 2B \geq 0 \qquad (241)
$$

The equations of motion are

$$
\begin{aligned}
\frac{\partial V}{\partial v_u} &= 2m_u^2 v_u - 2Bv_d + \frac{g^2 + g'^2}{2} v_u \left(v_u^2 - v_d^2\right) = 0 \qquad (242) \\
\frac{\partial V}{\partial v_d} &= 2m_d^2 v_d - 2Bv_u - \frac{g^2 + g'^2}{2} v_d \left(v_u^2 - v_d^2\right) = 0 \qquad (243)
\end{aligned}
$$

From

$$
0 = v_u \frac{\partial V}{\partial v_u} + v_d \frac{\partial V}{\partial v_d} = 2m_u^2 v_u^2 + 2m_d^2 v_d^2 - 4Bv_uv_d + \frac{g^2 + g'^2}{2}\left(v_u^2 - v_d^2\right)^2 \qquad (244)
$$

we immediately find out that in the minimum

$$V = -\frac{g^2 + g'^2}{8} \left( v_u^2 - v_d^2 \right)^2 \tag{245}$$

i.e. any minimum will be lower than the one at $v_u = v_d$.

Now we write (experimentally $v = 174$ GeV)

$$v_u^2 + v_d^2 = v^2, \quad \left( g^2 + g'^2 \right) v^2 = 2M_Z^2 \tag{246}$$

and introduce

$$\tan\beta = \frac{v_u}{v_d} \tag{247}$$

On the one side, from

$$0 = v_d \frac{\partial V}{\partial v_u} + v_u \frac{\partial V}{\partial v_d} = 2 \left( m_u^2 + m_d^2 \right) v_u v_d - 2B \left( v_u^2 + v_d^2 \right) \tag{248}$$

we get

$$\sin\beta \cos\beta = \frac{B}{m_u^2 + m_d^2} \tag{249}$$

On the other side, from

$$0 = v_u \frac{\partial V}{\partial v_u} - v_d \frac{\partial V}{\partial v_d} = 2 \left( m_u^2 + M_Z^2/2 \right) v_u^2 - 2 \left( m_d^2 + M_Z^2/2 \right) v_d^2 \tag{250}$$

it follows

$$\tan^2\beta = \frac{m_d^2 + M_Z^2/2}{m_u^2 + M_Z^2/2} \tag{251}$$

Comparing now (249) and (251) we obtain a nontrivial relation between the parameters

$$\left( m_u^2 + M_Z^2/2 \right) \left( m_d^2 + M_Z^2/2 \right) = B^2 \left( 1 + \frac{M_Z^2}{m_u^2 + m_d^2} \right)^2 \tag{252}$$

Since the sum of the two factors on the left-handside is positive (see (241) and use the positivity of $B$) each factor separately must also be positive.

A little algebra gives

$$\begin{aligned}
& \left(m_u^2 + m_d^2 + M_Z^2\right)^2 \left(B^2 - m_u^2 m_d^2\right) \\
= & \left(m_u^2 - m_d^2\right)^2 \left(m_u^2 + m_d^2 + M_Z^2/2\right) M_Z^2/2 \qquad (253)
\end{aligned}$$

from which it follows that

$$B^2 \geq m_u^2 m_d^2 \qquad (254)$$

This inequality means that there is a tachyonic state at $v_u = v_d = 0$, and so not even a local minumum is possible at the origin.

What would change if we added the Fayet-Iliopoulos term to the MSSM? Effectively this would amount in an extra factor (183)

$$\delta_{FI} V(H_u, H_d) = \frac{g'^2}{2} \xi \left(|H_u|^2 - |H_d|^2\right) \qquad (255)$$

The same results as above would still be valid, providing now

$$m_u^2 = |\mu|^2 + m_{H_u}^2 + g'^2 \xi/2, \quad m_d^2 = |\mu|^2 + m_{H_d}^2 - g'^2 \xi/2 \qquad (256)$$

instead of (240).

To summarize, the constraints (252) and (241) must be satisfied by the MSSM parameters.

# 10 The mass spectrum in MSSM

As we already saw, there are not many hints on where should the spartners lie. I spite of this we will see that there are some relations among different masses that can represent a test of the MSSM. Let us discuss the different options.

## 10.1 Higgs mass

This can be easily found out from

$$\begin{aligned}
V(H_u, H_d) = & \; m_u^2 |H_u|^2 + m_d^2 |H_d|^2 + 2B \left(H_u H_d + h.c.\right) \qquad (257) \\
+ & \; \frac{g^2 + g'^2}{8} \left(|H_u|^2 - |H_d|^2\right)^2 + \frac{g^2}{2} \left(H_u^\dagger H_d\right) \left(H_d^\dagger H_u\right)
\end{aligned}$$

Then we first explicitly expand in terms of

$$H_u = \begin{pmatrix} \phi_u^+ \\ v_u + \phi_u^0 \end{pmatrix}, \quad H_d = \begin{pmatrix} v_d + \phi_d^{0*} \\ \phi_d^- \end{pmatrix} \tag{258}$$

Using the obvious notation

$$\phi_{u,d}^{-*} = \phi_{u,d}^+, \quad \phi_{u,d}^0 = \frac{1}{\sqrt{2}} \left( R_{u,d} + i I_{u,d} \right) \tag{259}$$

we can rewrite the Higgs quadratic terms in the form

$$\frac{1}{2} \left( R_u \,, R_d \right) M_R^2 \begin{pmatrix} R_u \\ R_d \end{pmatrix} + \frac{1}{2} \left( I_u \,, I_d \right) M_I^2 \begin{pmatrix} I_u \\ I_d \end{pmatrix} + \frac{1}{2} \left( \phi_u^+ \,, \phi_d^+ \right) M_\pm^2 \begin{pmatrix} \phi_u^- \\ \phi_d^- \end{pmatrix} \tag{260}$$

The matrices $M_I^2$ and $M_\pm^2$ have one zero eigenvalue each, i.e. due the would-be Goldstones eaten by the $Z$ and $W^\pm$. The eigenvectors of $M_R^2$ are the two CP even neutral scalars $h^0$ (the lighter) and $H^0$ (the heavier), while the physical remaining eigenvalues of $M_I^2$ and $M_\pm^2$ are the CP odd neutral scalar $A^0$ and the charged $H^\pm$.

The masses of $h^0$, $H^0$, $A^0$ and $H^\pm$ cannot be arbitrary. There are experimental lower limit constraints as well as theoretical relations among. The most interesting is the theoretical upper bound for the mass of $h^0$. In fact, one finds at tree level

$$m_{h^0} < M_Z \left| \cos (2\beta) \right| \tag{261}$$

This upper limit is unexpected (nothing of this kind happens in the standard model for example), but it gets corrected at 1-loop. The full result is quite complicated but in the large stop mass $m_{\tilde{t}} \gg m_t, A_t$ (from the soft term $A_t H_u \tilde{t} \tilde{t}^c$) and large $\tan\beta$ limit it reduces to [17]

$$m_{h^0}^2 \lesssim M_Z^2 + \frac{3 g^2 m_t^4}{8 \pi^2 M_W^2} \log \left( \frac{m_{\tilde{t}}^2}{m_t^2} \right) \tag{262}$$

The bound now depends on the value of the unknown stop mass, although only logarithmically. It can be further relaxed by a larger $A_t$ value. For not too large stop mass $m_{\tilde{t}} \lesssim 100$ TeV), the lightest Higgs boson is still quite light, less that 200 GeV or so. For low $\tan\beta$ the experimental bound on the Higgs mass can be reached and constitutes a constraint on $\tan\beta$.

It is often claimed that MSSM predicts for the lightest Higgs to be lighter than 130 GeV or so. Strictly speaking, since the other

MSSM parameters ($m_{\tilde{t}}$, $A_t$, etc) are not known, such a statement is not correct, and these type of constraints should actually be interpreted as relations among the (unknown) supersymmetric parameters.

## 10.2 Chargino mass

We have originally four charged massless Weyl spinors, $\tilde{W}^+$, $\tilde{W}^-$, $\tilde{H}_u^+$, $\tilde{H}_d^-$, in the supersymmetric Lagrangian with $\mu = 0$ and unbroken electroweak symmetry. Notice that their charged conjugated spinors are $\overline{\tilde{W}^+}$, $\overline{\tilde{W}^-}$, $\overline{\tilde{H}_u^+}$, $\overline{\tilde{H}_d^-}$, which are not $\tilde{W}^-$, $\tilde{W}^+$, $\tilde{H}_u^-$, $\tilde{H}_d^+$. In fact $\tilde{W}^+$ and $\tilde{W}^-$ are two independent spinors, not connected by conjugation, while $\tilde{H}_u^-$ and $\tilde{H}_d^+$ do not even exist. The charges on a spinor denote a different state, not only a charge. It is more like writing $\tilde{\psi}^\pm = a \pm ib$, with $a$ and $b$ complex.

Once $\mu$, supersymmetry and electroweak symmetry breaking are introduced, the four spinors have a $4 \times 4$ mass matrix

$$\frac{1}{2}\left(\tilde{W}^+ , \overline{\tilde{W}^-} , \tilde{H}_u^+ , \overline{\tilde{H}_d^-}\right) M_{\tilde{C}} \begin{pmatrix} \overline{\tilde{W}^+} \\ \tilde{W}^- \\ \overline{\tilde{H}_u^+} \\ \tilde{H}_d^- \end{pmatrix} + h.c. \tag{263}$$

There are only two different eigenvalues, as expected: a charged spinor can be either massless (not our case) or of a Dirac type (i.e. two spinors have the same mass and thus form a massive Dirac field). Only neutral spinors can have Majorana masses.

The eigenvectors are called chargini and are conventionally denoted by $\tilde{C}_i^\pm$, with $i = 1$ for the lighter, $i = 2$ for the heavier.

## 10.3 Neutralino mass

The four neutral fermionic spartners $\tilde{B}$, $\tilde{W}^0$, $\tilde{H}_u^0$, $\tilde{H}_d^0$, have a $4 \times 4$ Majorana mass matrix

$$\frac{1}{2}\left(\tilde{B} , \tilde{W}^0 , \tilde{H}_u^0 , \tilde{H}_d^0\right) M_{\tilde{N}} \begin{pmatrix} \tilde{B} \\ \tilde{W}^0 \\ \tilde{H}_u^0 \\ \tilde{H}_d^0 \end{pmatrix} + h.c. \tag{264}$$

The four linear combinations (mass eigenstates) of the original fields are called neutralini. They are denoted by $\tilde{N}_i$, $i$ =1 for the lightest, $i = 4$ for the heaviest. In a generic point of the MSSM parameter space they are massive Majorana particles, although neither a Weyl (one massless eigenvalue) or a Dirac (two equal eigenvalues) possibility is excluded.

## 10.4 Sfermion mass

Finally we have the sfermions, i.e. the spin 0 partners of the SM fermions. They are three generations of $\tilde{q} = \tilde{u}$ or $\tilde{d}$, charged leptons $\tilde{l}$ and neutral leptons (sneutrinos) $\tilde{\nu}$. Every charged sfermion has both the left ($\tilde{f}$) and right component ($\tilde{f}^c$) and so its mass is a $6 \times 6$ matrix. Since there is no right-handed neutrino superfield ($\tilde{\nu}^c$) in the MSSM the mass matrix for the neutral $\tilde{\nu}$ is only $3 \times 3$.

$$
\begin{aligned}
& \left(\tilde{u}^* \,, \tilde{u}^c\right) M_{\tilde{u}}^2 \begin{pmatrix} \tilde{u} \\ \tilde{u}^{c*} \end{pmatrix} + \left(\tilde{d}^* \,, \tilde{d}^c\right) M_{\tilde{d}}^2 \begin{pmatrix} \tilde{d} \\ \tilde{d}^{c*} \end{pmatrix} \\
+ \quad & \left(\tilde{e}^* \,, \tilde{e}^c\right) M_{\tilde{e}}^2 \begin{pmatrix} \tilde{e} \\ \tilde{e}^{c*} \end{pmatrix} + \tilde{\nu}^* M_{\tilde{\nu}}^2 \tilde{\nu}
\end{aligned} \tag{265}
$$

Notice that the above masses are already Hermitian, so that the eigenvectors are complex fields, as they must, being charged massive bosons.

## 10.5 Gluino mass

There is not much to say here. Gluino cannot mix with any other field due to the unbroken colour SU(3). The Majorana mass term is simply

$$
\frac{1}{2}\tilde{g} M_{\tilde{g}} \tilde{g} \tag{266}
$$

with $M_{\tilde{g}}$ a complex number (not a matrix).

# 11 Possible experimental signatures from supersymmetry

As we have seen the MSSM has more than 100 parameters. Although some combinations of them are actually excluded already by current data, almost all of the parameter space is still available. So predictions are very model dependent. Essentially there are two ways of constraining the parameter space: one is through rare processes, the other through direct collider searches. This has led so far to various lower limits for the sparticle masses (from LEP II and Tevatron they must be $\gtrsim 100$ GeV or so, although exceptions are possible) and small mixings in the sfermion mass matrices. We will assume in the following that the spectrum is low enough to be detectable at the LHC. However, nothing is known about which sparticle is lighter which is heavier (with few exceptions, like the CP even Higgs boson mentioned before), so that a precise prediciton of the processes that will dominate and the decays involved is very difficult. In other words, the supersymmetric spectrum is practically arbitrary. Most of the analysis are done having some particular model in mind, and so the expectations should be taken with care.

Anyway, generically sparticles are generated in colliders mostly through the following interactions

$$\tilde{V}\tilde{V}V, \quad \tilde{V}\tilde{f}\bar{f}, \quad \tilde{f}\tilde{f}^{*}V, \quad \tilde{f}\tilde{f}^{*}VV \tag{267}$$

where $V$ is a gauge boson and $f$ a fermion.

Things simplify if R-parity is assumed. Then sparticles are always produced in pairs. The lightest supersymmetric partner (LSP) is stable and escapes the detector (it behaves like a massive neutrino). So large missing energy is a common prediction of R-parity conserving low energy supersymmetry models.

If the squarks and gluinos are not too heavy ($\lesssim 1$ TeV), the Large Hadron Collider (pp collider at 14 (10) TeV), will produce them in pairs mainly through strong interactions with $gg \to \tilde{g}\tilde{g}, \tilde{q}\tilde{q}^{*}$, presumably the dominant modes through s-channel gluon and t-channel $\tilde{g}, \tilde{q}$ exchange, or $gq \to \tilde{g}\tilde{q}$ through t-channel $\tilde{g}$ exchange. Other possibilities are productions of chargini and/or neutralini through electroweak processes. Of course all these final state sparticles decay into SM particles plus lighter sparticles again, until only jets, leptons and two LSPs are left.

In most cases such processes have large SM backgrounds that can hide the signal. This makes some processes that have small SM background particularly interesting. For example, in the SM missing energy comes from neutrini. Since they are mostly produced in combination with charged leptons through $W$, missing energy without charged leptons is predicted relatively small in the SM, and it is considered one of the golden plates of supersymmetry. On top of the choice of the channel, particular care must be taken to put kinematical cuts, as usual.

Notice that due to the non-detectability of the lightest neutralino, finding the masses of the sparticles produced is not so easy. It turns out that careful analysis (through kinematical edges) could give some information on combinations of masses, mainly differences. In any way, even if LHC finds some spartilcles, this will probably be just the beginning of a long search, and other, more precise colliders will have to be used.

# 12 What is supersymmetry good for?

We have seen that supersymmetry is an interesting theoretical construction that could give us a hint of what is the physics beyond the standard model. If MSSM is really realized in nature, then there is a very rich phenomenology waiting us in future. Unfortunately there is no hint of where supersymmetry should be broken, i.e. what are the values of the soft terms. This seems to make our possibility to find supersymmetry in near future colliders wishful thinking, since in principle any scale between $M_Z$ and $M_{Planck}$ looks equally good. We will see in this section that there are actually various arguments of why at least some of the spartners should lie in the TeV region, possibly accessible at the LHC.

## 12.1 Gauge coupling unification

The first hint of why this should happen comes form the possibility of gauge coupling unification. This idea postulates that at some large energy the three gauge interactions of the standard model are described by just one type of gauge interaction, that embodies the known three SM ones. The SM is thus just the low energy approximations of such a grand unified theory (GUT), that unifies not only the gauge interactions, but also the matter fields (leptons and quarks typically behave similarly, although details depend on the choice of the grand unified group and representations).

As we know, in the standard model the gauge couplings do not unify, and new physics is needed to change at some point the beta functions and restore running to unification. Such new physics could be supersymmetry. In fact, using (211) it is a simple exercise to show that the beta coefficients in MSSM are $b_i = (-33/5, -1, 3)$ instead of $b_i = (-41/10, 19/6, 7)$ in the SM (positive coefficients here mean asymptotic freedom). One knows the experimental values of $g_i$ at $M_Z$ and can evolve them towards larger scales $\mu$ using the three equations (215) for the three gauge couplings. Assuming that all the superpartners lie at the same scale $\tilde{m}$, we can find that they unify to a single point at $\mu = M_{GUT} \approx 10^{16}$ GeV providing $\tilde{m} \approx 1$ TeV.

I believe this is the best argument for low energy supersymmetry. It has to be stressed however that the solution is not unique. If for example only gauginos and higgsinos lied at TeV, while the sfermions and second Higgs were heavier (such a solution is called split supersymmetry), nothing would really change. But more split supersymmetry is also possibile: a light (TeV) wino, intermediate ($\approx 10^8$ GeV) gluino and heavy rest would also unify. The conclusion is however always similar: at least some partner should be pretty light, although not always in the reach of the LHC.

## 12.2 Dark matter

Another good reason to believe in supersymmetry is the dark matter problem in the universe. As we know, there is no dark matter candidate in the SM (a SM with added three right-handed neutrinos to describe the neutrino mass and mixings have a dark matter candidate [18]). What is missing is essentially a discrete symmetry to make a particle stable. This is provided in MSSM with R-parity. The lightest supersymmetric partner (LSP) is stable due to R-parity. If the spectrum is such that the LSP is neutralino (charged dark matter is not allowed by observation), then it feels weak interactions and is thus a weakly interacting massive particle (WIMP), which is known to be an ideal dar matter candidate. It is important to realize here that strictly speaking supersymmetry is not responsible for that. What is crucial is the R-symmetry, but its conservation has nothing to do with supersymmetry. Also it is not important whether all sparticles are light (low energy supersymmetry) or we have a kind of split supersymetry: what is important is just a light (TeV) neutralino.

It is interesting that supersymmetry provides another candidate on top of the neutralino. As we will se later, the generalization of supersymmetry to gravity introduces a spin 3/2 partner of the graviton, called gravitino. It turns out that gravitino can also be a dark matter candidate.

## 12.3 The hierarchy problem

Although there are many different ways of describing this problem, one can approxiamtely say, that the issue here is the stability (or better, instability) of the electroweak scale under quantum corrections. This is visible only in the quantum corrections of some lower dimensional operators, more precisely, in the coefficient (mass squared) of the bilinear product of scalar fields. The problem is easily seen using a sharp cutoff regularization. This regularization is however not often used in field theory, on top of this it can give also wrong results, especially because it explicitly breaks Lorentz and gauge invariance. On the contrary, dimensional regularization is the correct tool in the regularization and renormalization procedure, and so we will use it here.

To have a feeling of what the hierarchy problems is about, let us consider a simple toy model with a charged scalar field $H$, which I will call the Higgs boson for future use, another charged boson field $\phi$ and a Dirac fermion field $\psi$. The tree order masses of the three fields will be denoted by $m_{H0}$, $m_{B0}$ and $m_{F0}$, while the interaction terms relevant for our discussion are

$$\mathcal{L}_{int} = -\lambda \left|H\right|^2 \left|\phi\right|^2 - yH\bar{\psi}\psi + h.c. \, . \tag{268}$$

Now let us calculate the 1-loop correction to the Higgs mass: there will be one Feynman diagram with the boson $\phi$ and one with the fermion $\psi$, giving after dimensional regularization and $\overline{MS}$ renormalization

$$m_H^2 = m_{H0}^2 - c_B \frac{\lambda}{16\pi^2} m_{B0}^2 + c_F \frac{y^2}{16\pi^2} m_{F0}^2 + \dots \, , \tag{269}$$

where the dots denote the eventual log terms coming after the integration and are not relevant for our discussion. What is relevant is the fact that the second and third terms on the righthandside of (269) are corrections to the mass not proportional to itself, i.e. they correct the mass $m_{H0}$, but are not dependent on $m_{H0}$. In particular, they can give a nonzero correction, even if the original mass $m_{H0}$ were zero. In other words, there is no multiplicative renormalization for the scalar (Higgs) mass.

So we have the following two possibilities:

(1) $m_{H0} \approx 100$ GeV (electroweak scale); if there are heavy particles, $m_{B0,F0} \gg m_{H0}$, it follows that $m_H \gg m_{H0}$. The Higgs mass got destabilized in this case by quantum corrections.
(2) $m_H \approx 100$ GeV (this is almost an experimental fact, although, admittedly, the Higgs boson has not been found yet). In this case, if $m_{B0,F0} \gg m_H$, one needs also a very large tree approximation $m_{H0} \gg m_H$ to cancel the large contributions due to $m_{B0,F0}$, again a strange and somewhat unnatural situation.

Of course, strictly speaking, in the standard model alone there is no problem at all. In fact:
- there is no extra boson;
- $m_{F0} \approx y < H >= \mathcal{O}(m_{H0})$.

However, as soon as we have some new physics (at $M_{GUT}$, $M_{Planck}$), eq. (269) becomes

$$\mathcal{O}(M_{GUT}^2 \text{ or } M_{Planck}^2) + m_{H0}^2 \approx m_H^2 = \mathcal{O}((100\text{GeV})^2) \ , \qquad (270)$$

which gives

$$\mathcal{O}((10^{16}\text{GeV})^2 \text{ or } (10^{19}\text{GeV})^2) + m_{H0}^2 \approx m_H^2 = \mathcal{O}((100\text{GeV})^2) \ . \qquad (271)$$

But now we clearly need to fine-tune $m_{H0}$! And not only approximately, but all the 16 or so decimal points! This is very unnatural and this is what we call the hierarchy problem.

This is typical for scalar (spin=0) fields. In fact for fermions (spin=1/2) one can easily find out that the one-loop correction to the mass is

$$m_F = m_{F0} + c\frac{y^2}{16\pi^2} m_{F0} + \ldots \ . \qquad (272)$$

So the correction is proportional to the tree order value, i.e. the mass renormalization for fermi fileds is multiplicative. If for example the tree order mass is zero, then no perturbation will ever generate a nonzero mass (this is not necessary for nonperturbative corrections, but these are out of the scope of these lectures and I will not consider them here), i.e.

$$m_{F0} = 0 \rightarrow m_F = 0 \ . \qquad (273)$$

There is a symmetry here that forbids the appearence of a mass term. If $m_{F0} = 0$ then the Lagrangian has a $U(1)^2$ chiral symmetry and it is invariant under $\psi_{L,R} \to \exp(i\alpha_{L,R})\psi_{L,R}$, which can not be broken at any perturbative order.

Similarly, in the case of spin 1 particles (for example gluons), it is the gauge symmetry that makes the mass renormalization multiplicative.

One of the possible solutions to the hierarchy problem is supersymmetry. Essentially what supersymmetry does is to connect the Higgs self-coupling $\lambda$ and the Yukawa coupling $y$ by $\lambda = y^2$ , as well as the boson and fermion masses by $m_{B0} = m_{F0}$, so that the particular combination in (269) cancels out. This is equivalent to say that there are no quadratic divergences in supersymmetric models.

Of course supersymmetry must be broken. In fact there is no experimental evidence for example for the scalar partner of the electron with the same mass. Although we still need all the superpartners to exist, we will put their mass safely high enough. In doing this we must however be careful not to reintroduce the hierarchy problem. If the masses were too high, we could integrate them out and thus get the standard model back with all its hierarchy problems. In other words, the 1-loop correction to the Higgs mass in a model with $\lambda_i = y_i^2$ is

$$\delta m_H^2 = m_H^2 - m_{H0}^2 \approx \sum_i y_i^2 \frac{(m_{Bi}^2 - m_{Fi}^2)}{16\pi^2} \ . \tag{274}$$

Naturalness constraint (no fine-tuning) requires that this square mass difference should not exceed too much the Higgs mass itself (the expected one), i.e. $(100\text{GeV})^2$, from which it follows that

$$y_i^2 \left| m_{Bi}^2 - m_{Fi}^2 \right| \lesssim (1\text{TeV})^2 \ . \tag{275}$$

If we now take the plausible assumption (at least in nature we expect this to hold) that $m_{Bi} >> m_{Fi}$, it follows that

$$m_{Bi} \lesssim \frac{1\text{TeV}}{y_i} \tag{276}$$

(but bigger than the experimental lower limit which is approximately 100 GeV). The biggest Yukawa is the top one with $y_t \approx 1$. This means that the most constrained is the stop $\tilde{t}$. Also, since we do not want the supersymmetry breaking terms to break the weak SU(2), the sbottom

$\tilde{b}$, superpartner of the lefthanded bottom quark, must also have the same mass as its SU(2) partner $\tilde{t}$. So this gives approximately

$$m_{\tilde{t},\tilde{b}} \lesssim 1\text{TeV} , \tag{277}$$

while for the other sfermions the constraints are milder (i.e. they can be heavier).

One comment at the end. Although all these requirements for no fine-tuning are amusing and bring into the game some constraints to the otherwise almost completely free parameters of the theory, one should not take them too seriously. After all, who can say which exactly is an admissible fine-tuning and which not? Is a 40% correction to the Higgs mass ok? Is a 7 times bigger tree order mass compared to the 1-loop mass a sign of fine-tuning? Nobody can give a firm answer, so one should have a common sense in judging these issues. Otherwise there is a danger to end in a paradoxical situation.

# 13 Spontaneous supersymmetry breaking

As we can see, the bad side of the MSSM, even assuming R-parity) is a big proliferation of unknown parameters, something like 100 or so. The desire to simplify things adding a new symmetry (supersymmetry) clashed with the facts of nature, forcing us to add the new terms (232)-(236). Theoretically it would be thus much better to have a model of calculating or deriving the unknown coefficients. This will be studied in detail in the remaining part of this section. We will present two different scenarios.

## 13.1 MSSM is not enough

The best option would be to break supersymmetry spontaneously, as we do for example in gauge theories. We will see that this is not easy to do in two examples.

In supersymmetric theories the hamiltonian is proportional to the sum of the squares of the generators of supersymmetry algebra. So the energy is zero iff susy is preserved, and positive if susy is spontaneously broken. We saw in the previous lecture that the potential (energy) is made from two terms, the F-term (198) and the D-term (199). We will consider here only the so called F-term breaking:

$$\frac{\partial W}{\partial \phi_i} \neq 0 \quad \text{for at least some } i \ . \tag{278}$$

Notice that such a requirement automatically implies a massless fermion. In fact, from the minimization of (198), the second derivative of the superpotential in the field space direction defined by (278) must vanish. This is expected from the Goldstone theorem. In the case of spontaneously broken internal symmetry a massless Goldstone boson appears. Here the spontaneously broken supersymmetry generator is a Grassmann object, thus a fermionic zero mode follows. This object is called the goldstino.

Although $W$ is a gauge singlet, in the SM $\phi_i$ are not. Thus the only field that could break supersymmetry without breaking some unwanted extra gauge symmetry is one of the two Higgses $H_u$ or $H_d$. Imagine we do it with $H_d$. Then suppose that we are able to properly change the superpotential so that

$$F^*_{H_d} \equiv \left\langle \frac{\partial W}{\partial H_d} \right\rangle \neq 0 \ . \tag{279}$$

To see why this cannot work, let us concentrate on the mass of a down squark or charged slepton $\tilde{f}$:

$$W = y_f f h^0_d f^c + ... \tag{280}$$

From (198) and (279) the mass terms for the sfermion in the potential are

$$V = \begin{pmatrix} \tilde{f}^* & \tilde{f}^c \end{pmatrix} \begin{pmatrix} y_f^2 v_d^2 & y_f F^*_{H_d} \\ y_f F_{H_d} & y_f^2 v_d^2 \end{pmatrix} \begin{pmatrix} \tilde{f} \\ \tilde{f}^{c*} \end{pmatrix} \ , \tag{281}$$

where $v_d$ is the vev of $H_d$. The eigenvalues satisfy the mass sum rule

$$m^2_{\tilde{f}} + m^2_{\tilde{f}^c} - 2m^2_f = 0 \ . \tag{282}$$

This is in contradiction with what we know from low energy physics: there is for example no scalar with quantum numbers of the electron and a smaller (or equal) mass.

It is possible to show that any spontaneously broken global supersymmetric model has similar unacceptable mass sum rules at tree order. There are two possible ways out: either one goes local, i.e. to supergravity, or one transmits the information of susy breaking not at tree order, but at one loop. In both cases the mass sum rules change and unwanted constraints get relaxed.

Either way, the mechanism should roughly look as follows: a sector with no interaction with the SM fields (and thus called the hidden sector) is responsible for the spontaneous breaking of supersymmetry. The information that susy is broken in this hidden sector gets thus transmitted to our SM sector either by $1/M_{Pl}$ suppressed higher dimensional terms (as in supergravity) or through an intermediate (messenger) field, that couples to both SM and hidden sector fields. In this scenario loops with external SM fields and internal messenger fields (with susy breaking couplings and/or masses) transmit the information on susy breaking to our sector.

Let us see now in more details the above mechanisms of susy breaking mediation.

## 13.2 Gravity mediation

In the case of local supersymmetry (supergravity), one needs to introduce the gravity multiplet, which essentially means the spin 2 graviton (2 d.o.f. on shell) plus the spin 3/2 gravitino (also 2 d.o.f. on shell). Analogous to the case of spontaneously breaking of a local symmetry, where the would-be goldstone boson gets eaten by the longitudinal component of the vector boson, here the gravitino eats the goldstino, acquiring a nonzero mass, $m_{3/2}$. This mass turns out to be the typical scale, and soft masses will be proportional to it.

The potential in the supergravity case becomes ($\varphi_i$ denote both SM fields $\phi_i$ and hidden sector fields $X$)

$$V_F = e^{K/M_*^2} \left[ \left( \frac{\partial W}{\partial \varphi^i} + \frac{\partial K}{\partial \varphi^i} \frac{W}{M_*^2} \right) \left(K^{-1}\right)^i_j \left( \frac{\partial W^*}{\partial \varphi^*_j} + \frac{\partial K}{\partial \varphi^*_j} \frac{W^*}{M_*^2} \right) - 3 \frac{|W|^2}{M_*^2} \right] , \tag{283}$$

where $M_* = M_{Pl}/\sqrt{8\pi} \approx 2 \times 10^{18}$ GeV is the so called reduced Planck mass and $\left(K^{-1}\right)^i_j$ is the inverse matrix of $\partial^2 K/\partial\varphi^i \partial\varphi^*_j$. The Kähler potential $K(\varphi, \varphi^*)$ is often assumed canonical, $K_{can} = \sum_i \varphi^*_i \varphi_i$, but since supergravity is anyway nonrenormalizable, it can be actually modified with higher dimensional polinomials.

Assuming that it is a singlet field $X$ that breaks susy, the order parameter in supergravity is defined as

$$F_X^* = \frac{\partial W}{\partial X} + \frac{\partial K}{\partial X}\frac{W}{M_*^2} \ . \tag{284}$$

It must be nonzero, so to break supersymmetry. In flat spacetime one can parametrize this breaking by the gravitino mass:

$$F_X = \sqrt{3} m_{3/2} M_* \ . \tag{285}$$

As we said before, this field $X$ should have small enough couplings with the SM fields $\phi_i$. This is most easily obtained, assuming

$$W(X,\phi) = W(X) + W_{SM}(\phi) \ . \tag{286}$$

Then all the couplings between the two sectors are through $1/M_{Pl}$ suppressed operators.

One then needs the following requirements at $X = \langle X \rangle$:

$$W = m_{3/2} M_*^2 \qquad \text{(zero cosmological constant)} \ , \tag{287}$$

$$\frac{\partial V}{\partial X} = 0 \qquad \text{(extremum of the potential)} \ , \tag{288}$$

$$\left|\frac{\partial^2 V}{\partial X^2}\right| \leq \frac{\partial^2 V}{\partial X \partial X^*} \qquad \text{(no tachyons)} \ . \tag{289}$$

As an example take the so-called Polonyi superpotential:

$$W(X) = aX + b \tag{290}$$

and a canonical Kähler. With properly chosen constants $a$ and $b$, so to satisfy (284), (287)-(289), it is possible to determine all the soft parameters in (232)-(236) in terms of few parameters at the Planck scale, which are at the moment still compatible with any experimental constraint.

We were thus able to reduce the 100 and so parameters to very few of them. It has to be kept in mind however, that this is no more than just a reparamentrization of the original soft terms. In fact, there is absolutely no real reason to believe that the superpotential should look like (290) and even less that the Kähler potential is canonical. In fact, taking for example

$$K = X^* X + \phi_i^* \phi_i + c_{ij} \frac{X^* X \phi_i^* \phi_j}{M_{Pl}^2}, \tag{291}$$

gives the sfermion soft mass squares

$$\left(m^2_{\tilde{f}}\right)_{ij} = (\delta_{ij} + c_{ij}) \frac{|F_X|^2}{M^2_{Pl}} \ , \tag{292}$$

which not only introduces new couplings ($c$'s), but may very easily be in contradiction with experiment due to possible large contribution to the flavour changing neutral currents. Although there are some ways of making these $c$'s small in the infrared via running, this requires extra physics, and is certainly not a minimalist's approach. It is safe to conclude that supergravity cannot explain the structure, less the particular values of the soft parameters.

## 13.3 Gauge mediation

The problem with the previous example was, that gravity is not actually flavour blind (the masses of fermions of different generations are very different!), which can be parametrized by a nontrivial matrix $c$ above. This led people to use as mediators of susy breaking gauge interactions, which are known to be flavour blind

As before, we have to forbid that supersymmetry breaking is transmitted to the SM particles at tree order, so no SM field can satisfy (278). Thus imagine that we have a gauge singlet field $X$ with a non zero F-term

$$F^*_X = \left\langle \frac{\partial W}{\partial X} \right\rangle \neq 0 \ . \tag{293}$$

Can it be coupled to any of the standard model fields? Allowing only renormalizable couplings the only possibility is to transform the dimension 2 term (222) to a dimension 3 term [19]

$$W_\mu = \lambda_X X H_u H_d \ . \tag{294}$$

The higgsino mass appears from the vev of $X$,

$$\mu = \lambda_X \langle X \rangle \ . \tag{295}$$

The sum rule (282) applied to the Higgs supermultiplets is not dangerous in this case, since no mass of this multiplet has been measured yet. On the other side, the sfermions do not couple directly to $X$, so they are not influenced by it at tree order and no mass sum rule thus applies. Unfortunately the one loop correction to the masses of the sfermions gets negative and proportional to the relevant Yukawas [20]. This would destabilize the stop, breaking SU(3).

In other words, one cannot couple only the Higgs to the $X$ field, but must use another pair of multiplets, call them $\Phi$ and $\bar{\Phi}$:

$$W = W(X) + \lambda_X X\bar{\Phi}\Phi + W_{SM} \ . \tag{296}$$

These new multiplets are not gauge singlets, so they can interchange gauge bosons (and gauginos) with the SM fields. Thus at leading order in a small $F_X/M_\Phi^2$ ratio, where $M_\Phi = \lambda_X \langle X \rangle$ is the susy preserving $\Phi$ mass, the gaugino mass gets a 1-loop contribution

$$m_\lambda = c_i \frac{\alpha_i}{4\pi} \frac{\lambda_X F_X}{M_\Phi} \ , \tag{297}$$

while the sfermions get a two-loop contribution

$$m_{\tilde{f}}^2 = d_i \left(\frac{\alpha_i}{4\pi}\right)^2 \frac{|\lambda_X F_X|^2}{M_\Phi^2} \ , \tag{298}$$

with $c_i$, $d_i$ ($i = 1, 2, 3$) depending on the representation under the SM gauge group SU(3)×SU(2)×U(1). Similar relations are possible also for the other soft parameters. Although this seems to solve the flavour problem mentioned above, one must be careful. In fact, there is no gaurantee that the $\Phi$ and $\bar{\Phi}$ do not couple or mix with the SM fields. Typically these mediators have the quantum numbers of the sfermions. This mixing can in principle spoil the flavour blindness of the mediation [20].

A very useful way of getting the soft terms in gauge mediation is to solve the renormalization group equations in the presence of the messengers of supersymmetry breaking. Then one promotes the messenger mass $M = \lambda_X X$ to a chiral superfield, with $X = \langle X \rangle + \theta\theta F_X$. Finally, using the formulae ($F = \lambda_X F_X$)

$$m_{\lambda_a} = -\frac{1}{2} \frac{\partial \log \alpha_a^{-1}(X)}{\partial \log X}\bigg|_{X=M} \frac{F}{M} \tag{299}$$

$$m_{\tilde{f}}^2 = - \frac{\partial \log Z_f(X, X^\dagger)}{\partial \log X \partial \log X^\dagger}\bigg|_{X=M} \frac{FF^\dagger}{MM^\dagger} \tag{300}$$

$$A_{ijk} = \frac{\partial}{\partial \log X} \log\left(Z_i(X) Z_j(X) Z_k(X)\right)\bigg|_{X=M} \frac{F}{M} \tag{301}$$

valid at the messenger scale $M$. For details see [16].

# 14 Exercises

Solve (at least !) one exercise from group 1 and one from group 2.

**Group 1**

1. Derive (36) when applied on $\psi_\alpha$ and $F$. In doing that use the Fierz identity (valid for arbitrary spinors $\psi$, $\xi$, $\eta$)

$$\psi_\alpha\,(\xi\eta) + \xi_\alpha\,(\eta\psi) + \eta_\alpha\,(\psi\xi) = 0 \tag{302}$$

which you derive from the well known Jacobi identity

$$\epsilon_{\alpha\beta}\epsilon_{\gamma\delta} + \epsilon_{\alpha\gamma}\epsilon_{\delta\beta} + \epsilon_{\alpha\delta}\epsilon_{\beta\gamma} = 0\,. \tag{303}$$

This in practice means that

$$(\psi_1\psi_2)\,(\psi_3\psi_4) + (\psi_1\psi_3)\,(\psi_2\psi_4) + (\psi_1\psi_4)\,(\psi_2\psi_3) = 0 \tag{304}$$

Notice that only pairs are important, never the order ($\eta\chi = \chi\eta$). Then derive eq. (52) using the known relations for the $\gamma$ matrices and eq. (79). Finally show explicitly that (98) can bring (97) into the form (99).

2. Write in components (in terms of $\phi$ and $\psi$ only) the most general renormalizable multifield Wess-Zumino model, i.e. generalize (145). Write in components the one chiral superfield Wess-Zumino model for the most general form of $K(\Phi, \Phi^\dagger) = \Phi^\dagger\Phi +$ higher terms, and $W(\Phi)$. Hint: expand $K$ and $W$ around $\Phi = \phi$.

3. Consider the simplest possible supersymmetric theory with U(1) gauge invariance. Such a theory is called supersymmetric QED. Argue why such a theory cannot be consistent with one single chiral superfield. Write down the simplest consistent and renormalizable quantum field theory. Find the minima of the potential and compute the spectrum of the theory as a function of the model parameters, $\xi$ included.

4. Consider a SU(N) gauge model with the superpotential

$$W = y\overline{\Phi}\Sigma\Phi \tag{305}$$

with $\overline{\Phi}$, $\Sigma$ and $\Phi$ the antifundamental, adjoint and fundamental representations of SU(N). Compute the unknown parameters in the 1-loop RGE equations and solve them analytically.

**Group 2**

1. Derive all the components of the mass matrices $M_R^2$, $M_I^2$, $M_\pm^2$, $M_{\tilde{C}}$, $M_{\tilde{N}}$, $M_{\tilde{u}}^2$, $M_{\tilde{d}}^2$, $M_{\tilde{e}}^2$, $M_{\tilde{\nu}}^2$, $M_{\tilde{g}}$ from the MSSM Lagrangian. Show explicitly that (a) $M_R^2$ has non-negative eigenvalues, (b) the zero eigenvalues predicted by the Nambu-Goldstone theorem and (c) that there are only two different eigenvalues in the chargino mass matrix.

2. Calculate explicilty the sfermion soft masses and $A$ terms in the Polonyi supergravity model with canonical Kähler potential. How could gaugini get a mass term (i.e. which operator has for its $\theta\theta\overline{\theta\theta}$ component the gaugino mass)?

3. Calculate the soft terms of the MSSM assuming as messengers $\Phi$ ($\overline{\Phi}$) to have the same quantum numbers as the quark $Q$ (conjugate of $Q$). Suppose the scale $M$ is at $10^6$ GeV. What is the minimal value of $F$ allowed by data? Is unification of couplings still obtained?

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