ν PHYSICS AND GUTs

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1 LECTURE 1: Neutrino from low energy effective theories (1h30m)

1.1 Basic facts about neutrinos

We know how neutrinos interact from collider experiments. They are electrically neutral but interact under weak interactions, and thus live in a weak doublet. On the other hand, we know already for long time, that their mass is so tiny to be undetectable in collider physics. For that we need to measure neutrino oscillations among different flavors. At the moment we know about 3 flavors, two mixing angles and two mass differences [1][1]:

$$m_2^2 - m_1^2 = (7.6 \pm 0.2) 10^{-5}\,\mathrm{eV}^2 \quad , \quad \left| m_3^2 - m_2^2 \right| = (2.4 \pm 0.2) 10^{-3} \tag{1}$$

$$\tan^2\theta_{12} = 0.48 \pm 0.05 \quad , \quad \sin^2 2\theta_{23} = 1.02 \pm 0.04 \tag{2}$$

If $m_3^2 - m_2^2 > (<) 0$ we say we have a normal (inverse) hierarchy, but at the moment the sign is unknown. On top of that we know from other ground-based or cosmological and astrophysical observations that the sum of the neutrino masses cannot exceed few tenths of eV, and that θ_{13} is less than approximately 13^o. At the moment we do not have any information on the CP violating phase(s).

1.2 Neutrinos in the SM, how to give mass

Strictly speaking, the standard model (SM) cannot account for a nonzero neutrino mass at the renormalizable level. The reason is gauge symmetry

[1] I would like to stress that throughout this paper citations are completely incomplete and given here just for eventual further reading. This is obviously not a review on the subject, but just some modest lecture notes.

and the absence of a right-handed neutrino. In fact, the neutrino spinor (in the 1-generation standard model) is just a component of the left-handed lepton doublet

$$L = \begin{pmatrix} \nu \\ e \end{pmatrix} = (1, 2, -1) \tag{3}$$

where the three numbers on the right-hand-side denote the field dimensionality under $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$.

Although it is in principle possible to write down a Majorana mass term for a neutral Weyl fermion $(1/2, 0)$ or $(0, 1/2)$ of the Lorentz group $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ at the renormalizable level, i.e. (σ_2 is the 2×2 Pauli matrix with either $\mathrm{SU}(2)_L$ or $\mathrm{SU}(2)_R$ Lorentz indices)

$$\mathcal{L}_{\text{Majorana mass}} = -\frac{1}{2} m \psi^T i\sigma_2 \psi + h.c. \tag{4}$$

this cannot be used for the neutrino, since it is not a SM singlet.

Exercise: Show that (4) is Lorentz invariant.

So, without introducing a new degree of freedom, i.e. the right-handed neutrino, we can get the neutrino mass term in the SM only from a non-renormalizable Lagrangian, the so-called dimension 5 Weinberg operator

$$\mathcal{L}_{d=5} = -y^M \frac{(L^T i\tau_2 H) i\sigma_2 (H^T i\tau_2 L)}{M} \tag{5}$$

Here $i\tau_2$ takes care of the $\mathrm{SU}(2)_W$ indices, i.e. makes $L^T i\tau_2 H$ gauge invariant, while $i\sigma_2$ does the same for the Lorentz invariance of spinor bilinears.

Exercise: With the fields H and L one can write two other dimension 5 invariant operators by connecting the $\mathrm{SU}(2)_W$ indices differently. Show that these other two operators boil down to the same form (5), so that only one is actually independent.

Once the Higgs doublet vev spontaneously breaks the electroweak gauge symmetry,

$$\langle H \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} \quad , \quad v = 246\,\mathrm{GeV} \tag{6}$$

the neutrino gets a Majorana mass term (the M of y^M reminds us of the Majorana nature)

$$\mathcal{L}_{d=5}(v) = -y^M \frac{v^2}{2M} \nu^T i\sigma_2 \nu + h.c. \quad \rightarrow \quad m_\nu = y^M \frac{v^2}{M} \tag{7}$$

We do not know better now what the mass is, but at least we can fit it. In this respect it is similar to the masses and Yukawas for charged fermions in the SM: we cannot predict the number, but we can connect the interaction Yukawa coupling with the value of the mass.

Notice that we do not actually need two parameters, y^M and M, but only one, the ratio y^M/M. This way of writing suggests however the energy scale of new physics, M.

For a 3-generation SM the operator (5) generalizes to

$$\mathcal{L}_{d=5} = -y_{ij}^M \frac{(L_i^T i\tau_2 H) i\sigma_2 (H^T i\tau_2 L_j)}{M} \tag{8}$$

The energy scale M suggests the validity of the low energy effective theory, i.e. it is the cutoff of the SM. On the other side the six complex Yukawa couplings $y_{ij}^M = y_{ji}^M$ have all the flavor information of the neutrino world. The neutrino mass matrix is

$$(m_\nu)_{ij} = y_{ij}^M \frac{v^2}{M} \tag{9}$$

gets diagonalized with a unitary rotation from the flavor to the mass basis

$$m_\nu = V^* m_\nu^{diag} V^\dagger \tag{10}$$

which is obtained by simply rotating the neutrinos from the flavor diagonal basis ν_f into the mass diagonal basis ν_m

$$\nu_f \rightarrow V \nu_m \tag{11}$$

In the basis of diagonal charged lepton mass matrix, this unitary matrix V is called the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix and is the lepton analogue of the Cabibbo-Kobayashi-Maskawa (CKM) matrix of the quark sector. It gets usually parametrized as

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \mathrm{diag}(1, e^{i\alpha}, e^{i\beta}) \tag{12}$$

where $c_{ij} = \cos\theta_{ij}$, $s_{ij} = \sin\theta_{ij}$. Notice that on top of the usual matrix, completely similar to the CKM one, we have here also a diagonal matrix with two extra CP-violating phases, α and β. This is the consequence of the Majorana nature of the term (8).

Does all this mean that the neutrino must have a Majorana mass? Obviously this is not true. Neutrinos could be Dirac particles as well. It is enough to introduce the right-handed neutrino and write the usual Yukawa term

$$\mathcal{L}_{Yukawa} = y^D_{ij}\overline{\nu}_{Ri} H^T L_j + h.c. \tag{13}$$

which gives after electroweak symmetry breaking

$$\mathcal{L}_{Yukawa}(v) = y^D_{ij}\frac{v}{\sqrt{2}}\overline{\nu}_R \nu_L + h.c. \tag{14}$$

The Dirac mass is thus (the D of y^D stands for Dirac)

$$(m_\nu) = y^D_{ij}\frac{v}{\sqrt{2}} \tag{15}$$

and gets diagonalized with

$$m_\nu = V_R m_\nu^{diag} V^\dagger \tag{16}$$

i.e. by rotating the left and right neutrinos as

$$\nu_L \to V\nu_L \quad , \quad \nu_R \to V_R \nu_R \tag{17}$$

but now with (12) without the extra Majorana phases α and β. The reason for it is that these extra phases can be absorbed in the Dirac case by V_R, which does not enter the weak interaction, being the right-handed neutrino a gauge singlet. In the Majorana case on the other side it is V that must absorb such extra phases, so that they remain in the weak interaction term (in the mass eigenbasis).

From the measurement of V and m^{diag} only it is hard to tell the nature of the neutrino mass, i.e. whether it is Majorana or Dirac. As we said, in the Majorana case there are two CP-violating phases more than in the Dirac case, but all this is very hard to measure in practice.

The most promising channel is actually due to non-conservation of the lepton number in the Majorana case. In fact, from (8) and (9) it is clear that the lepton number is not conserved and that a nonzero Majorana mass breaks lepton number by two units. So the idea is to look for a process that goes through neutrino mass and violate lepton number by two. If we see such a process, then neutrino is a Majorana fermion. This brings us to the neutrino-less double beta decay, which is an atomic decay of the type $(A, Z) \to (A, Z+2) + 2e^-$. The diagram of the neutrino-less double beta decay is depicted in fig. 1.

The Lagrangian we are using is the usual weak interaction

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}} \bar{e} \gamma^\mu P_L \nu_M W_\mu^- + h.c. \tag{18}$$

where ν_M is a Majorana neutrino and the projector $P_L = (1+\gamma^5)/2$ lets only the left-handed components interact.

A naive estimate of the diagram would give

$$\frac{1}{M_W^2} \left(\gamma^\mu P_L\right) \frac{1}{\not{p} - m} \left(\gamma^\nu P_L\right) \frac{1}{M_W^2} \tag{19}$$

which is clearly wrong since it does not go through the neutrino mass (the diagram violates lepton number by two units, so it must be proportional to the Majorana neutrino mass). We have not been careful, since we applied the Feynman rules for a Dirac fermion, while we have to do with a Majorana fermion, so its propagator is not the one used above. We can try to correct things either working with a 2-component notation, or, as we will do now, rewrite everything consistently with a 4-component Majorana.

The definition of a Majorana fermion in four Lorentz components is (this equation is the fermionic analogue of the reality condition $\phi^* = \phi$ for a bosonic field ϕ)

$$\nu_M^c \equiv C \overline{\nu}_M^T = \nu_M \tag{20}$$

where we introduced the charge conjugation matrix

$$C = i\gamma_2\gamma_0 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix} \tag{21}$$

The solution of (20) is, written in terms of 2-component spinors ν

$$\nu_M = \begin{pmatrix} \nu \\ i\sigma_2 \nu^* \end{pmatrix} \tag{22}$$

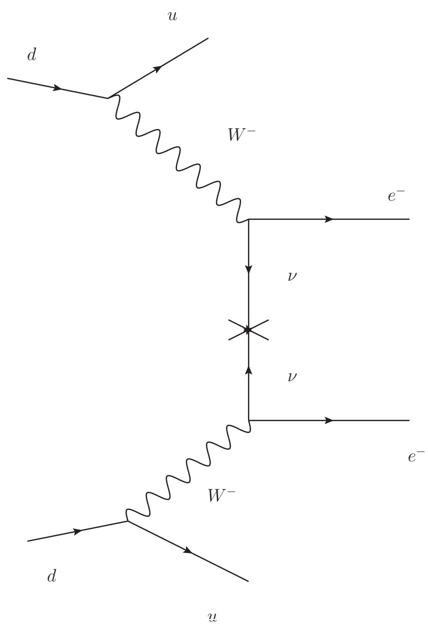

Figure 1: The neutrino-less double beta decay

The effective Majorana mass term (7) after the electroweak symmetry breaking can now be written in four component notation as

$$-\frac{m_M}{2}\left(\nu_M^T C \nu_M + h.c.\right) \tag{23}$$

Using the general formula

$$\begin{aligned}
\bar{f}_1 \left(\gamma_1\gamma_2\dots\gamma_n\right) f_2 &= f_1^{cT} C \left(\gamma_1\gamma_2\dots\gamma_n\right) C \overline{f_2^c}^T \\
&= -\overline{f_2^c} C^T \left(\gamma_1\gamma_2\dots\gamma_n\right)^T C^T f_1^c \\
&= (-1)^n \, \overline{f_2^c} \left(\gamma_n\dots\gamma_2\gamma_1\right) f_1^c
\end{aligned} \tag{24}$$

(γ_a is any matrix γ^{μ_a}) and the known relations

$$\bar{f} = f^{cT} C \quad f = C\overline{f^c}^T \quad C\gamma^\mu = -\gamma^{\mu T} C \quad C^T = -C \quad C^2 = -1 \tag{25}$$

half of the term in (18) can be changed to get

$$\begin{aligned}
\mathcal{L}_{int} &= \frac{g}{2\sqrt{2}} \left(\bar{e}\gamma^\mu P_L \nu_M - \overline{\nu_M^c}\gamma^\mu P_R e^c\right) W_\mu^- + h.c. \\
&= \frac{g}{2\sqrt{2}} \left(\bar{e}\gamma^\mu P_L \nu_M - \bar{\nu}_M\gamma^\mu P_R e^c\right) W_\mu^- + h.c.
\end{aligned} \tag{26}$$

where passing to the second line we used (20). Notice that now we have P_R in the second term. This gives the result

$$\frac{1}{M_W^2}\left(\gamma^\mu P_L\right)\frac{1}{\not{p}-m_\nu}\left(\gamma^\nu P_R\right)\frac{1}{M_W^2}\propto\frac{1}{M_W^4}\frac{m_\nu}{p^2} \tag{27}$$

for the amplitude if the neutrino is of Majorana type (and zero if it is Dirac). So a non-zero rate for the neutrino-less double beta decay is a smoking gun for the Majorana nature of neutrino. On the other side, it is harder to prove anything if nothing is seen. Since electrons are produced in this process, the mass m_ν in (27) must be replaced by the appropriate matrix element, i.e. $(m_\nu)_{11}$ (the above derivation is valid also for a matrix m_ν). Using (10), (12) and the experimental constraints in (1), it turns out, that the Majorana nature of the neutrino can be excluded only in the case of inverse hierarchy.

Exercise: Show it. Compare the result for example with fig. (8.5c) on page 106 of [1].

1.3 The seesaw mechanism (type I, II, II)

The form of the Weinberg operator (7) and its generalization (8) tells us that the smallness of the neutrino mass can be simply due to a large cutoff M. This apparently simple observation is actually at the core of the so called see-saw mechanism. But, to have a physical meaning, one should derive the non-renormalizable Weinberg's operator from a renormalizable UV completion. Of course this means introducing new degrees of freedom close to the scale M. If we introduce just one type of new fields, we have three possibilities, that define the three types of see-saw.

1.3.1 Type I see-saw [2]

Imagine to have a fermionic weak singlet (the right-handed neutrino) ν_R. Then we can write with it the Yukawa term

$$\mathcal{L}_{Yukawa}=\overline{\nu}_R y_D^T H^T L+\overline{L}y_D^* H^*\nu_R \tag{28}$$

but since the new field is a gauge singlet, also the Majorana mass term is allowed (for more generations M is in general a symmetric complex matrix)

$$\mathcal{L}_M=-\frac{1}{2}\nu_R^T M^*\nu_R-\frac{1}{2}\overline{\nu}_R M^T\overline{\nu}_R^T \tag{29}$$

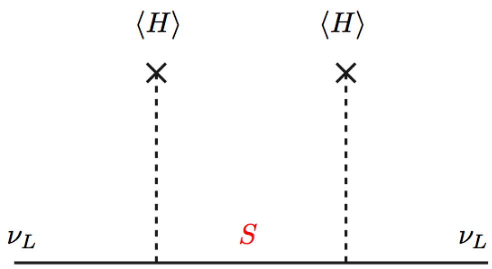

Figure 2: The type I see-saw contribution of the fermionic singlet $S(1,1,0)$ to the neutrino mass. Type III is the same diagram, but with the singlet $S(1,1,0)$ replaced by a weak fermionic triplet $T(1,3,0)$.

Now imagine the situation in which $M >> m_D = y_D v/\sqrt{2}$, i.e. the Majorana mass dominates over the Dirac one. We can integrate out the heavy field to get an effective mass for the light one

$$\frac{\partial \mathcal{L}}{\partial \overline{\nu}_R} = 0 \to -M^T \overline{\nu}_R^T + m_D^T \nu = 0 \tag{30}$$

which brings us to the final expression

$$\mathcal{L} = \frac{1}{2} \nu^T m_D M^{-1} m_D^T \nu + h.c. \tag{31}$$

The result is valid also for n_L left-handed and n_R right-handed neutrinos, being now m_D a $(n_L \times n_R)$ and M a $(n_R \times n_R)$ matrix. The light neutrino Majorana mass matrix is

$$m_\nu = m_D . \frac{1}{M} . m_D^T \tag{32}$$

Its rank is at most n_R for $n_L > n_R$ and at most n_L otherwise, which means that there are at least $n_L - n_R$ massless neutrinos for $n_L > n_R$. To give two non-vanishing mass differences in the 3-generation SM one thus need $n_R = 2$ (the lightest neutrino is massless) or 3 (all neutrinos can have mass).

This, type I, see-saw contribution is depicted on Fig. 2.

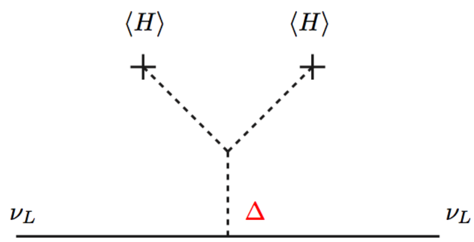

Figure 3: The type II see-saw contribution of the bosonic triplet $\Delta(1,3,2)$ to the neutrino mass.

1.3.2 Type II see-saw [3]

The mediator here is a bosonic triplet Δ with SM quantum number (1,3,2). The diagram contributing to the neutrino mass is shown on Fig. 3.

The relevant part of the Lagrangian is

$$\mathcal{L} = -M_\Delta^2|\Delta|^2 + \frac{1}{2}Y_{ij}^\Delta L_i^T \Delta L_j + \mu H^T \Delta^\dagger H + h.c. \tag{33}$$

The equations of motion give a non-vanishing triplet vev once the Higgs doublet spontaneously breaks the electroweak symmetry:

$$\langle \Delta \rangle = \frac{\mu v^2}{2M_\Delta^2} \tag{34}$$

which triggers the neutrino mass

$$m_\nu = \langle \Delta \rangle Y^\Delta \tag{35}$$

A spectacular signal at a collider would be the detection of a doubly charge component of this triplet, the Δ^{++}. If light enough, it could be produced through a Drell-Yan process ($u\bar{d} \to W^+ \to \Delta^{++}\Delta^-$). Its leptonic decay

$$\Gamma(\Delta^{++} \to l_i^+ l_j^+) \approx M_\Delta |Y_{ij}^\Delta|^2 \tag{36}$$

would directly measure the Yukawa couplings and thus test the neutrino mass matrix. To be able to do that one needs (on top of Δ being light enough) that the other decay channel

$$\Gamma(\Delta^{++} \to W^+ W^+) \approx \langle\Delta\rangle^2 / M_\Delta \tag{37}$$

is smaller, which is true for $\langle\Delta\rangle \lesssim 10^{-4}$ GeV.

1.3.3 Type III see-saw

This is very similar to the type I case. The only difference is that instead of the singlet $S(1,1,0)$ one has as a mediator a fermionic triplet $T(1,3,0)$. If light, it has however a much better chances to get noticed at the LHC than the singlet, because it can be produced (and decay) through gauge interactions.

1.4 Neutrinos in MSSM

One could think that the issue of neutrino mass in MSSM is similar to the SM. But this is not the case. The reason is, that although the Lagrangian at tree order does not allow a neutrino mass, essentially for the same reasons (gauge symmetry) as in the SM, the lepton number is not automatic anymore in MSSM at tree order. In fact, the issue of the lepton number conservation (as the baryon number too) is peculiar to SM: it is simply impossible in the SM to write down a lepton number violating term at tree level. We say that lepton (and baryon) number is an accidental symmetry of the SM, it does not need to be imposed, but it follows from the field content and the requirement of gauge and Lorentz invariance. Thus, apart from anomalies (that give however a far too small contribution, proportional to $\exp(-4\pi/\alpha_2) \approx 10^{-150}$) lepton number remains conserved, and thus loops cannot generate a nonzero neutrino mass. This is no longer true in MSSM.

To see it, let's write down the most general renormalizable superpotential in MSSM:

$$\begin{aligned} W &= (y_U)_{ij} Q_i H_u u^c_j + (y_D)_{ij} Q_i H_d d^c_j + (y_E)_{ij} L_i H_d e^c_j + \mu H_u H_d \\ &+ \frac{1}{2}\lambda_{ijk} L_i L_j e^c_k + \lambda'_{ijk} Q_i L_j d^c_k + \frac{1}{2}\lambda''_{ijk} u^c_i d^c_j d^c_k + \mu'_i L_i H_u \end{aligned} \tag{38}$$

The first four terms represent the usual Yukawa terms of the SM properly supersymmetrized (except for the presence of the second Higgs, which is mandatory in MSSM), and are lepton (L) and baryon (B) number conserved.

The model without the last four terms has a discrete Z_2 symmetry, called R-parity, which is defined for any field as

$$R = (-1)^{3(B-L)+2S} \tag{39}$$

where S is the spin. It does not commute with supersymmetric transformations (due to the explicit presence of the spin) and is even for SM fields and odd for their superpartners. If conserved, it has an interesting prediction for the SM: the existence of a stable non-SM particle (the lightest supersymmetric partner - LSP), which, if neutral, can be a perfect dark matter (DM) candidate.

The last four terms however violate this R-symmetry, as well as either lepton or baryon number, but are nevertheless allowed by gauge symmetry. Of course they cannot be arbitrary, since this would lead for example to a spectacularly fast proton decay. In fact, exactly from proton decay constraints one could derive that roughly

$$\lambda'\lambda'' \lesssim 10^{-27} \left(\frac{m_{\widetilde{d}}}{300\mathrm{GeV}}\right)^2 \tag{40}$$

while from neutron-antineutron oscillation constraint it follows that

$$\lambda'' \lesssim (10^{-6} - 10^{-7}) \left(\frac{m_{\widetilde{d}}}{300\mathrm{GeV}}\right)^2 \left(\frac{m_{\widetilde{\chi}^0}}{100\mathrm{GeV}}\right)^{1/2} \tag{41}$$

where $m_{\widetilde{d}}$ and $m_{\widetilde{\chi}^0}$ are the lightest down squark and neutralino masses.

All we need is just a small enough λ'', i.e. a very good approximate baryon number conservation, but we definitely can have nonzero λ, λ' and μ'. But once we have a nonzero λ, we can generate at one loop the Weinberg operator and thus a nonvanishing neutrino mass through the left diagram of fig. 4, which gives (assuming the slepton mass $m_{\widetilde{\ell}}$ to be the largest mass in the loop)

$$m_\nu \simeq \frac{\lambda^2 (m^2_{\widetilde{\ell}})_{LR} m_\tau}{16\pi^2 m^2_{\widetilde{\ell}}} \qquad , \qquad (m^2_{\widetilde{\ell}})_{LR} = A_\ell v_d - \mu^* y_\tau v_u \tag{42}$$

There are other possibilities though. For example, if $\lambda' \neq 0$, a similar diagram with internal sleptons and leptons replaced by proper squarks and quarks give rise to a very similar contribution. There is a possibility also of a tree order mass. It is shown in the right diagram of fig. 4, due to a nonzero superpotential parameter μ'. Notice that this diagram is essentially a

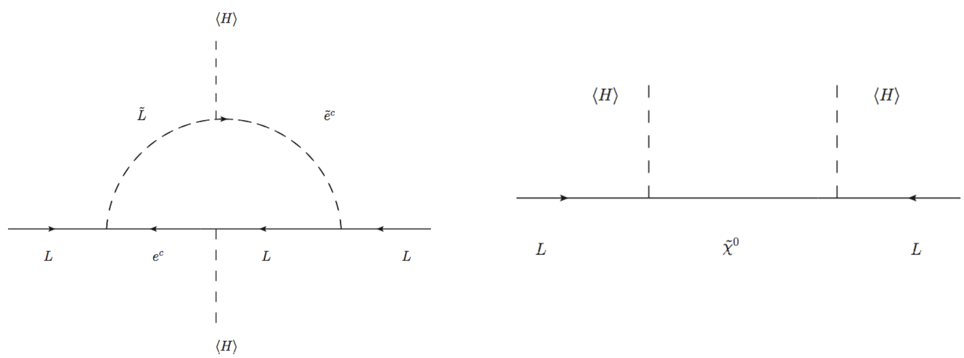

Figure 4: The leading contributions to neutrino mass in MSSM: the left 1-loop diagram is proportional to λ (a similar diagram with internal (s)leptons replaced by (s)quarks is proportional to λ'), the right one tree-level is proportional to μ'.

type I see-saw with neutralino replacing the right-handed neutrino. Finally, soft supersymmetry breaking terms can also violate R-parity and lead to non-vanishing contributions to the neutrino mass.

One last comment. If R-parity is broken to give a realistic mass to neutrinos, the LSP decays fast and cannot be a DM candidate. Then, assuming gravity has also to be supersymmetrized (in this case we are talking about local supersymmetric transformation, i.e. about supergravity), the only remained DM candidate is the gravitino, the spin 3/2 superpartner of the graviton. It still decays, but the extremely tiny gravitation coupling makes it almost stable on the timescale of the universe.

2 LECTURE 2: Examples of theories of neutrino mass: GUT (1h30m)

The (MS)SM has 3 gauge interactions and 5 different matter representations for each generation. The idea of grand unification theories (GUT) is to reduce these numbers to one single gauge group and one or two different representations for each generation of matter. Of course our SM gauge group should then be a subgroup of the grand unified gauge group, and the SM fermions included in the GUT matter representations.

But what does this really mean? That we put for example all SM gauge fields together in a bigger adjoint representation of a simple group is clear, but we know that the gauge couplings of the three SM gauge interactions are numerically different. So in which sense they can unify? Here it is crucial the notion of running coupling constants. We know that the gauge (and other) couplings run with energy. So what we have to do, is to let them run and check if they meet all three together [4]. And if they do, the scale at which this happens will be the scale of (the spontaneous breaking of) grand unfication. Fortunately this is easy to do, all we need is to solve the (1-loop) renormalization group equations (RGEs):

$$\frac{dg_i}{d\log\mu} = -\frac{b_i}{(4\pi)^2} g_i^3 \qquad i = 1,2,3 \tag{43}$$

The 1-loop beta coefficient b_i can be straightforwardly calculated via (G,F,B stay for gauge bosons, fermions, bosons)

$$b = \frac{11}{3} C_2(G) - \frac{2}{3} T_F - \frac{1}{3} T_B \ . \tag{44}$$

The Dynkin index

$$T(R)\delta_{ab} = Tr\left(T_a(R) T_b(R)\right) \tag{45}$$

and the second Casimir

$$C_2(R)\delta^{ij} = \sum_a \left(T_a(R) T_a(R)\right)^{ij} \tag{46}$$

depend on the choice of the gauge group and on the representation involved. The indices a, b run over the generators of the group (N^2-1 in SU(N)), while i, j run from 1 to the dimension of the representation. The normalization usually chosen is $T = 1/2$ for the fundamental representation (quarks, leptons). Then one has in the SU(N) group for the fundamental representation $C_2 = (N^2-1)/(2N)$, and for the adjoint $T = C_2 = N$. The dimension of the representation is N for fundamentals and N^2-1 for adjoint. To remember also that in SU(2) the generators in the fundamental are the Pauli matrices $T_a^{ij} = \tau_a^{ij}/2$, while in the adjoint representations are the Levi-Civita antisymmetric tensor $T_a^{ij} = -i\epsilon_{aij}$.

For supersymmetric theories we know that for each fermion (boson) there is a boson (fermion) in the same group representation, so (44) can be written more compactly as

$$b = 3C_2(G) - T \ . \tag{47}$$

The beta coefficients in the SM are $b_i = (-41/10, 19/6, 7)$ (positive coefficients here mean asymptotic freedom). One knows the experimental values of g_i at M_Z and can evolve them towards larger scales μ using (43). It is now easy to check that there is no unification of couplings in the SM. Two loops will not help so the only possibility for unification is to add new particles in order to change the beta coefficients for energies above their mass. We will see in the next two sections two such examples.

2.1 Minimal non-supersymmetric SU(5)

The Georgi-Glashow SU(5) grand unified model [5] includes the SM three generations of fermions (the number of generations in GUTs are unfortunately not predicted, but put by hand, as in the SM) in the 10_F and $\overline{5}_F$ representations

$$10_F = \begin{pmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{pmatrix} \quad , \quad \overline{5}_F = \begin{pmatrix} d_1^c \\ d_2^c \\ d_3^c \\ \nu \\ e \end{pmatrix} \tag{48}$$

The Higgs sector is made of an adjoint 24_H, which gets a vacuum expectation value (vev) to spontaneously break SU(5)$\to$SU(3)$_C\times$SU(2)$_W\times$U(1)$_Y$:

$$\langle 24 \rangle_H = M_{GUT} \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix} \tag{49}$$

and of one fundamental representation, which contains also the SM Higgs doublet $H = (H^+, H^0)^T$:

$$5_H = (H_1^C, H_2^C, H_3^C, H^+, H^0)^T \tag{50}$$

This model is however ruled out, because it predicts wrong gauge couplings at the scale M_Z (another way of saying that the 3 gauge couplings of the SM do not unify). On top of that, this model suffers from the same problem as the SM: it predicts massless neutrinos. It is actually even worse than the SM: there we could at least phenomenologically write down an effective Weinberg operator (8). With a properly chosen values of y/M we could fit the experimental numbers. This is not allowed anymore in SU(5). Although we can write down an effective operator

$$\mathcal{L}_{Weinberg\,SU(5)} = y_{ij}\frac{(\overline{5}_F 5_H)(5_H^T \overline{5}_F^T)}{M} \tag{51}$$

the cutoff M cannot be lower than M_{GUT}. Since, due to proton decay constraints $M_{GUT} \gtrsim 10^{16}$ GeV, the resulting neutrino masses turn out too small ($y \lesssim 1$ because of perturbativity assumption).

I will show now how to overcome both problems, the missing unification and the practically vanishing neutrino mass. As we mentioned in the previous chapter, the idea is to include new degrees of freedom. For this purpose I will add to this model a fermionic adjoint [6], [7]. Under the SM it decomposes into

$$24_F = S(1,1,0) + T(1,3,0) + O(8,1,0) + X(3,2,-5/6) + \overline{X}(\bar{3},2,5/6) \tag{52}$$

Exercise: Derive (52). Hint: $24 \sim \bar{5} \times 5$.

The Higgs 24_H obviously decomposes in a similar way. We have thus the following possibility for light states (the gauge singlets do not contribute to the beta function, while the X_H, $\overline{X}_H$ get eaten by the longitudinal components of the SU(5) heavy gauge bosons via the Nambu-Goldstone mechanism):

$$\begin{aligned} S=0: &\qquad T_H(1,3,0)\,,\, O_H(8,1,1)\,,\, H^C(3,1,-1/3) \\ S=1/2: &\qquad T(1,3,0)\,,\, O(8,1,1)\,,\, X(3,2,-5/6)\,,\, \bar{X}(\bar{3},2,5/6) \end{aligned} \tag{53}$$

Although apparently a lot of freedom, there is not much choice for their masses, if we want unification. An important point is that in order to get lighter triplets and octets in 24_F, higher dimensional operators has to be used, and so the maximum mass for the leptoquark is $m_X \approx M_{GUT}^2/\Lambda$, where Λ is the cutoff of the SU(5) model, at least $100\, M_{GUT}$ or so, to make it perturbative. For this reason one can show that

$$m_T \approx 1\,\mathrm{TeV} \tag{54}$$

a neat prediction of the model. One can also show, that higher is the triplet mass, lower turns out to be the GUT scale, which means faster is the proton decay. So if we do not find it at the LHC, we should definitely find soon the proton decay, or discard the model.

Exercise: Derive (54) at 1-loop.

It is interesting that part of the spectrum is determined by the requirement of the SM being embedded in a GUT. And, even more exciting, the fermionic triplet lies in the range of the LHC.

We have now to solve the neutrino mass issue yet. We have two candidates for mediators of the see-saw mechanism, the fermionic singlet S (type I see-saw) and the fermionic triplet T (type III see-saw). They are coupled to the SM leptons as

$$\mathcal{L}_{Yuk} = y_T^i L_i T H + y_S^i L_i S H \tag{55}$$

to give the neutrino mass matrix ($M_{T,S}$ are the triplet and singlet masses)

$$m_\nu^{ij} = \frac{v^2}{2}\left(\frac{y_T^i y_T^j}{M_T} + \frac{y_S^i y_S^j}{M_S}\right) \tag{56}$$

with rank two, so the model predicts one massless neutrino.

The fermionic weak triplet $T = (T^+, T^0, T^-)$ decays through weak interactions mainly into a lepton and a gauge boson:

$$T^\pm \;\; \rightarrow \;\; W^\pm \nu \;\;\text{or}\;\; Z^0 e^\pm \tag{57}$$

$$T^0 \;\; \rightarrow \;\; W^\pm e^\mp \;\;\text{or}\;\; Z^0 \nu \tag{58}$$

with a decay width estimate

$$\Gamma_T \approx |y_T|^2 m_T \tag{59}$$

The decay rate depends on the same Yukawa couplings that are responsible for the neutrino mass. LHC could thus give us information on the yet unmeasured parameters of the neutrino sector.

To summarize, the minimal non-supersymmetric SU(5) model predicts

- a weak fermionic triplet with mass $m_T \approx 1$ TeV;
- one neutrino massless;
- neutrino mass matrix a mixture of type I and type III see-saw;
- triplet decays constrained by neutrino masses and mixings.

2.2 Minimal supersymmetric renormalizable SO(10)

In the MSSM the beta coefficients are $b_i = (-33/5, -1, 3)$. If we put all the superpartners at TeV, the three couplings unify in a single point at $\mu \approx 10^{16}$ GeV [8]. To appreciate this fact one should remember that this unification fails badly in the nonsupersymmetric case (compare the two runnings on Fig. 5. So, if we have supersymmetric partners at M_Z or close to 1 TeV as required by naturalness (hierarchy problem), then we have unification of gauge couplings for free! This is one of the (main) motivations for supersymmetry with low scale (TeV) superpartners (and of unification in supersymmetric theories).

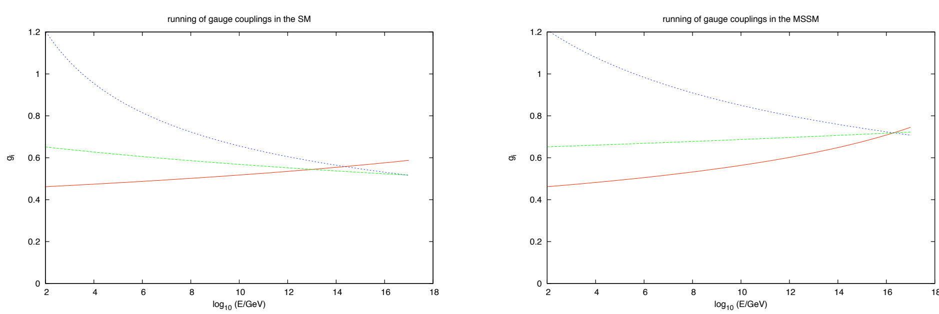

Figure 5: The running of the gauge couplings in SM (left) and low energy MSSM (right).

Let us now construct a supersymmetric GUT. Although we could use the SU(5) model, it is not the best one to constrain information on neutrinos. The previous model was an exception, only its simplicity implied constraints. In general cases the first (and probably last) non-trivial GUT model of neutrino mass is SO(10). Unfortunately we do not have much time to study it in details. I will thus just mention few most important facts about it.

First of all, the matter fields of one generation live in a single 16 dimensional (spinorial) representation of SO(10). It is great that all SM fermions are unified, and the 16^{th} element is a singlet, the right-handed neutrino.

$$16_F = (Q, u^c, d^c, L, \nu^c, e^c) \tag{60}$$

This obviously calls for the see-saw mechanism. Also, it is not strange that different Yukawas will be connected now. So one can derive in SO(10) various constraints among SM Yukawa couplings (quarks and leptons, neutrino included).

Second, only three types of Yukawas are possible, i.e. only 10, 120 and 126 dimensional Higgses of SO(10) can couple to spinorial bilinears

$$16 \times 16 = 10 + 120 + 126 \tag{61}$$

The minimal model turns out to be the one with 10 and 126 only, with the SM Higgs doublets (remember that in MSSM there must be two Higgs doublets) living in both 10 and 126 (i.e. linear combinations of doublets there). Schematically

$$\mathcal{L}_{Yukawa\,SO(10)} = 16_F^T (Y_{10} 10_H + Y_{126} 126_H) 16_F \tag{62}$$

SO(10) constraints the Yukawa matrices in generation space Y_{10} and Y_{126} to be symmetric (Y_{120} turns out to be antisymmetric).

Third, SO(10) has rank 5, the SM rank 4. So to break the rank one needs to give a vev to the SM singlet in 126 (another, non-minimal option is to add a new Higgs in a 16 dimensional representation). This same vev is the one that gives mass to the right-handed neutrino. Notice that this means that its mass matrix has the same Yukawa Y_{126} that is used for other fermion masses, a powerful consequence of SO(10) gauge invariance.

Here it is perhaps time to introduce the Pati-Salam (PS) subgroup of SO(10). It is a left-right symmetric model with 4 colors, i.e. the product group $SU(2)_L \times SU(2)_R \times SU(4)_C$. The matter fields under it are

$$16 = (2, 1, 4) + (1, 2, \bar{4}) \tag{63}$$

where the left and right handed doublets are in

$$(2,1,4) = \begin{pmatrix} u_1 & u_2 & u_3 & \nu \\ d_1 & d_2 & d_3 & e \end{pmatrix} \quad , \quad (1,2,\bar{4}) = \begin{pmatrix} u_1^c & u_2^c & u_3^c & \nu^c \\ d_1^c & d_2^c & d_3^c & e^c \end{pmatrix} \tag{64}$$

Notice that leptons are just the 4^{th} color.

The 10 and 126 dimensional Higgses get decomposed under the PS subgroup (not the SM anymore!) as

$$\begin{aligned} 10 &= (2,2,1) + (1,1,6) && (65) \\ 126 &= (2,2,15) + (3,1,\overline{10}) + (1,3,10) + (1,1,6) && (66) \end{aligned}$$

I derived the above in the following way. Remember that the PS theory is locally equivalent to SO(4)×SO(6), since locally SO(4)∼SU$(2)_L$×SU$(2)_R$ and SO(6)∼SU$(4)_C$.

10_i has one index of SO(10), i, which runs from 1 to 10. The elements in 10 with index i from 1 to 4 represent a 4 of SO(4), i.e. a (2,2,1) under Pati-Salam. The remaining elements 10_i with $i = 5, \ldots 10$ are a 6 of SO(6), thus a (1,1,6) of PS.

On the other side 126 is a 5-index completely antisymmetric matrix with a self-dual relation that modes out half of the degrees of freedom, i.e.

$$126_{ijklm} = \frac{i}{5!}\epsilon_{ijklmnopqr}126_{nopqr} \tag{67}$$

where $\epsilon_{i\ldots r}$ is the 10-dimensional completely antisymmetric Levi-Civita tensor. We can just repeat the previous case of 10, but now with 5 indeces. For example, taking all 5 indices running from 5 to 10 and antisymmetrizing them we get just a 6 of SO(6) (in 6 dimensions a 1-form is dual to a 5-form, i.e. in d-dimensions an object with p completely antisymmetric indices has the same number of components as an object with d-p completely antisymmetric indices), i.e. a (1,1,6) of PS. We continue then with 4 indices of SO(6) and one index of SO(4) to get a (2,2,15) of PS, etc.

From (62) and the above decomposition it is easy to get the SM Yukawas. For example

$$\begin{aligned} 16_F 10_H 16_F &\to (2,1,4)_F(2,2,1)_H(1,2,\overline{4})_F \\ 16_F 126_H 16_F &\to (2,1,4)_F(2,2,15)_H(1,2,\overline{4})_F + (1,2,\overline{4})_F(1,3,10)_H(1,2,\overline{4})_F \\ &+ (2,1,4)_F(3,1,\overline{10})_H(2,1,4)_F + \ldots && (68) \end{aligned}$$

The SM doublets live in $(2,2,1)_H$ and $(2,2,15)_H$, the SM singlet that break the rank of SO(10) is in $(1,3,10)_H$, while the $SU(2)_W$ triplet Higgs that gives rise to a type II see-saw is in $(3,1,\overline{10})$. Remember again that now the decomposition is under Pati-Salam, not the SM!

It is now relatively simple to guess the SM fermion masses for down quarks (D), up quarks (U), charged leptons (E) and neutrinos (N), valid for any number of generations:

$$
\begin{aligned}
M_D &= v_{10}^d Y_{10} + v_{126}^d Y_{126} && (69)\\
M_U &= v_{10}^u Y_{10} + v_{126}^u Y_{126} && (70)\\
M_E &= v_{10}^d Y_{10} - 3v_{126}^d Y_{126} && (71)\\
M_N &= -M_{\nu_D}^T M_{\nu_R}^{-1} M_{\nu_D} + M_{\nu_L} && (72)
\end{aligned}
$$

where we defined the Dirac (ν_D), left Majorana (ν_L) and right Majorana (ν_R) neutrino masses as

$$
\begin{aligned}
M_{\nu_D} &= v_{10}^u Y_{10} - 3v_{126}^u Y_{126} && (73)\\
M_{\nu_L} &= v_L Y_{126} && (74)\\
M_{\nu_R} &= v_R Y_{126} && (75)
\end{aligned}
$$

and the vevs are

$$
\begin{aligned}
v_{10}^{u,d} = \langle (2,2,1)_H^{u,d} \rangle \quad , \quad v_{126}^{u,d} = \langle (2,2,15)_H^{u,d} \rangle && (76)\\
v_R = \langle (1,3,10)_H \rangle \quad , \quad v_L = \langle (3,1,\overline{10})_H \rangle && (77)
\end{aligned}
$$

The only thing that has to be still understood is the factor of -3 in front of Y_{126} in M_E and M_{ν_D}. It is due to the vev of the (traceless) adjoint 15 of $SU(4)_C$ in $(2,2,15)_H$:

$$\langle 15_C \rangle \propto diag(1,1,1,-3) \qquad (78)$$

and thus give an extra factor -3 to leptons with respect to quarks.

Remember also that every left-right bidoublet (2,2) is (as any chiral superfield spin 0 component) complex in supersymmetry, so there are two possible vevs, which we denoted with indices u and d.

Finally, we have still to specify how SO(10) gets broken to the SM, i.e. the Higgs sector. It turns out that on top of the fields I have mentioned so far (the matter 16_F and the Higgses 10_H and 126_H) we need two other representations, the 5 indices antisymmetric and anti-self-adjoint $\overline{126}_H$ and the 4 indices antisymmetric 210.

Just to taste the predictiveness of this model, consider the case of 2 generations (let us talk about the heaviest two, the second and the third generation of the SM) and limit ourselves to the real case. We can always go into the basis in which Y_{10} for example is diagonal:

$$v_{10}^d Y_{10} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad , \quad v_{126}^d Y_{126} = \begin{pmatrix} c & d \\ d & e \end{pmatrix} \tag{79}$$

Then the number of free parameters in the charged fermion sector is 7:

$$a \, , \, b \, , \, c \, , \, d \, , \, e \, , \, v_{10}^u / v_{10}^d \, , \, v_{126}^u / v_{126}^d \tag{80}$$

They can be determined by fitting 7 experimental data:

$$m_s \, , \, m_b \, , \, m_c \, , \, m_t \, , \, m_\mu \, , \, m_\tau \, , \, \theta_{bc} \tag{81}$$

With only two new parameters,

$$v_R / v_{126}^d \, , \, v_L / v_{126}^d \tag{82}$$

we can now describe three measurable quantities from the neutrino sector

$$m_2 \, , \, m_3 \, , \, \theta_{23} \tag{83}$$

so we have one neat prediction.

Exercise: Show that the predictiveness becomes even better with increasing the number of generations, assuming all parameters real.

The realistic case of three generations and complex parameters is of course much more involved. It is however possible to fit all the data in the minimal model, providing the gaugini and higgsini of MSSM lie at about 10-100 TeV, while the sfermions and the second Higgs are much heavier (10^{13} GeV or so), which does not spoil one-step unification (one version of the so-called split supersymmetry scenario). Such a model determines all the parameters, among others predicts all proton decay rates and a large value of the yet unmeasured neutrino mixing angle θ_{13} (see [9] and references therein).

2.3 Conclusion

Here I did not have time to enter into details of GUTs. Let me just mention at the end some general properties.

The first and probably the most important of all is proton decay. As we found out, baryon and lepton numbers are not conserved anymore in GUTs, so there is nothing that prevents the proton from decaying. Since in the limit of the GUT scale to infinity proton must become stable, it is clear that the decay lifetime must be proportional to some positive power of M_{GUT}. To get it a bit more precisely, remember that the heavy GUT gauge bosons have mass M_{GUT}, and that their interaction violates B and L. So a B and L violating amplitude between four fermions gets a contribution from the exchange of such a gauge boson. The amplitude is (similar as the W exchange in muon decay, where the amplitude goes as $1/M_W^2$)

$$A(qq \to \bar{q}\bar{l}) \approx \frac{1}{M_{GUT}^2} \tag{84}$$

and thus the decay rate

$$\Gamma(p = qqq \to q\bar{q}\bar{l} = \pi^0 e^+) \approx \frac{m_p^5}{M_{GUT}^4} \tag{85}$$

One can thus estimate that the experimental lifetime $\tau_p = 1/\Gamma_p$ of 10^{34} yrs or so constrains

$$M_{GUT} \lesssim 10^{16}\,\mathrm{GeV} \tag{86}$$

which we used above.

The second important point is the electric charge quantization. This is considered a mystery of the SM. Why are all electric charges integer multiples of the down quark charge? Dirac had such an explanation. If there is only one (Dirac) magnetic monopole in the universe, all charges has to be quantized, a consequence of quantum mechanics. Another explanation is the embedding of the SM into a bigger simple group. Then all charges are quantized because originally they were just integer eigenvalues of a non-abelian simple group generator. The two explanations are however related. In fact it can be shown that each GUT that gets broken into the SM has as classical solution topological magnetic monopoles. But we are now a bit too far from neutrinos, so I will stop here.

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