BASICS OF FIELD THEORY AND SUPERSYMMETRY

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Abstract

We will review some basic notions about perturbative field theory and gauge theories, which are needed for the description of the Standard Model. We will then try to motivate and introduce supersymmetry and describe some of its peculiar ingredients. These notes are written for the (4 hour theory plus 1 hour exercises) CERN - SEENET-MTP - ICTP SCHOOL Gravitation, Cosmology and Astroparticle Physics (BS2022) 4 -10 September 2022, Belgrade, Serbia.

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1 Introduction

There are many good introductory books on field theory, for example Ryder [1], which, skipping few sections, is of the right length and deepness for a one semester course, Peskin and Schroeder [2], Weinberg [3, 4], or those free on the net, for example the notes of Siegel [5].

There are many very good reviews and books also on supersymmetry, for example, among many others: [7], [8], [9] and especially [10], [11] is a very useful introduction with all computational details, [12] is a clear overview of the main features, [13] and [14] are for those who like more formal approach, [15] and [16] are reviews on susy breaking, [17] is part of the Weinberg's famous course on quantum field theory, [18] is for fans of superspace.

During the lectures we will almost always keep the convention $c = 1$ and $\hbar = 1$. This is nothing else than the choice of particular units. In this convention mass and energy have for example the same unit, which is usually chosen to be GeV (gigaelectronvolt=10^9 eV), while time and distance have the unit GeV^{-1}. Changing these units into the usual ones is very simple: we multiply the quantity written in the $c = 1$ and $\hbar = 1$ units by proper powers of $\hbar$ and c.

2 Canonical quantisation

We very briefly repeat the case of the harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 \tag{2.1}$$

We quantise the coordinate i.e. the momentum via the commutation relation (as usual $[A, B] \equiv AB - BA$)

$$[x, p] = i \tag{2.2}$$

Let's define the creation ($a^\dagger$) and annihilation (a) operator via

$$x = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad , \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \quad , \quad [a, a^\dagger] = 1 \tag{2.3}$$

(2.1) turns into

$$H = \frac{\omega}{2}\left(a^\dagger a + aa^\dagger\right) \tag{2.4}$$

The vacuum $|0\rangle$ i defined as the state, annihilated by the annihilation operator:

$$a|0\rangle = 0 \tag{2.5}$$

The n-particle state is defined via the creation operator

$$|n\rangle = \frac{1}{\sqrt{n!}} \left(a^\dagger\right)^n |0\rangle \tag{2.6}$$

We will now try to repeat the procedure for the case of a field theory.

2.1 The real scalar field

Similarly as we quantised the coordinate and momentum in quantum mechanics via the commutation relation (2.2), we quantise in the case of field theory the field and the field momentum

$$[\phi(t,\vec{x}), \pi(t,\vec{y})] = i\delta^3(\vec{x}-\vec{y}) \tag{2.7}$$

We had to generalise the quantum mechanical commutation relation, since we have now one operator for each point in space. The relation is valid for fields defined at the *same* time t. For different times the commutator vanishes.

(2.2) becomes

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx}\right) && (2.8)\\
\pi(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (-i\omega_k) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx}\right) && (2.9)\\
[a_k, a_{k'}^\dagger] &= (2\pi)^3 2\omega_k \delta^3(\vec{k}-\vec{k}') \quad , \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 && (2.10)
\end{aligned}$$

where

$$k^\mu = (\omega_k, \vec{k}) \tag{2.11}$$

and the energy of the single particle state is

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \tag{2.12}$$

The Hamiltonian, i.e. the volume integral of the Hamiltonian density is now

$$
\begin{aligned}
H &= \int d^3x \mathcal{H} = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}\left(\nabla\phi\right)^2 + \frac{m^2}{2}\phi^2 \right) \\
&= \int d^3x \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \\
&\quad \Big[(-i\omega_k)(-i\omega_{k'}) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} - a_{k'}^\dagger e^{ik'x} \right) \\
&\quad + (i\vec{k})(i\vec{k'}) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} - a_{k'}^\dagger e^{ik'x} \right) \\
&\quad + m^2 \left(a_k e^{-ikx} + a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} + a_{k'}^\dagger e^{ik'x} \right) \Big] \\
&= \frac{1}{2} \int \frac{d^3k}{(2\pi)^2 2\omega_k} \frac{1}{2\omega_k} \\
&\quad \Big[-\omega_k^2 \left(a_k a_{-k} e^{-2\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \\
&\quad - \vec{k}^2 \left(-a_k a_{-k} e^{-2\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k - a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \\
&\quad + m^2 \left(a_k a_{-k} e^{-2\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \Big] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left(a_k^\dagger a_k + a_k a_k^\dagger \right)
\end{aligned}
\tag{2.13}
$$

where we used essentially (2.12) and

$$
\int d^3x e^{i(\vec{k}-\vec{k'})\vec{x}} = (2\pi)^3 \delta^3(k-k') \tag{2.14}
$$

The vacuum state $|0\rangle$ is defined as that one, which gets annihilated by all annihilation operators:

$$
a_k|0\rangle = 0 \tag{2.15}
$$

The one particle state with momentum k is defined as

$$
|k\rangle = a_k^\dagger|0\rangle \tag{2.16}
$$

and the Hermitian conjugated state

$$
\langle k| = \langle 0|a_k \tag{2.17}
$$

The one particle state are orthonormal:

$$
\langle k|k'\rangle = \langle 0|a_k a_{k'}^\dagger|0\rangle = \langle 0| \left[a_k, a_{k'}^\dagger \right] |0\rangle = (2\pi)^3 2\omega_k \delta^3(k-k') \tag{2.18}
$$

where we used (2.15), (2.10) and the vacuum normalisation $\langle 0|0\rangle = 1$.

Analogously we define a multiparticle state (free particles!) as

$$|n_1, k_1; \ldots; n_N, k_N\rangle = \frac{1}{\sqrt{n_1!}} \left(a^\dagger_{k_1}\right)^{n_1} \ldots \frac{1}{\sqrt{n_N!}} \left(a^\dagger_{k_N}\right)^{n_N} |0\rangle \tag{2.19}$$

According to this notation the previous one particle state $|k\rangle$ should have been written as $|1, k\rangle$.

The energy of the vacuum is, because of the second term in (2.23), strictly speaking infinite. In fact, if we use the definition (2.10), we get

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left(a^\dagger_k a_k + (2\pi)^3 \omega_k \delta^3(0)\right) \tag{2.20}$$

However, in the absence of gravity, we can put the zero of the energy anywhere, since only energy difference are measurable. Let's then take as the natural value for the energy zero. This can be formally obtained by the operation of normal ordering $\hat{N}$[2], which organises an arbitrary product of creation and annihilation operators, so that all creation operators stay on the left of all annihilation operators. In this way for example

$$\hat{N}\left(a_{k_1} a^\dagger_{k_2}\right) = a^\dagger_{k_2} a_{k_1} \tag{2.21}$$

The Hamiltonian gets defined then as

$$H = \int d^3x \hat{N}\left(\mathcal{H}\right) \tag{2.22}$$

and is in our case

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a^\dagger_k a_k \tag{2.23}$$

The energy of the multiparticle state (2.19) equals

$$E = n_1\omega_1 + \ldots + n_N\omega_N \tag{2.24}$$

Similarly as (2.23) we can define also the momentum operator:

$$\vec{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \vec{k} a^\dagger_k a_k \tag{2.25}$$

so that for the multiparticle state (2.19) it equals

$$\vec{P} = n_1\vec{k}_1 + \ldots + n_N\vec{k}_N \tag{2.26}$$

[2]in literature the normal ordered operator $\hat{\mathcal{O}}$, i.e. $\hat{N}(\hat{O})$, is often written as $:\hat{\mathcal{O}}:$.

2.2 The complex scalar field

The generalisation is easy, we have to take two types of operators, $a_k, a_k^\dagger$ and $b_k, b_k^\dagger$:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a_k e^{-ikx} + b_k^\dagger e^{ikx} \right) \tag{2.27}$$

The operator $a_k^\dagger$ (a_k) creates (annihilates) a particle, the operator $b_k^\dagger$ (b_k) creates (annihilates) and antiparticle. In the previous (real) case we had $b_k = a_k$, the particle was it own antiparticle.

Their commutation relations are now

$$\left[a_k, a_{k'}^\dagger \right] = \left[b_k, b_{k'}^\dagger \right] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{2.28}$$

$$[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger \right] = [b_k, b_{k'}] = \left[b_k^\dagger, b_{k'}^\dagger \right] = 0 \tag{2.29}$$

$$[a_k, b_{k'}] = \left[a_k^\dagger, b_{k'}^\dagger \right] = \left[a_k, b_{k'}^\dagger \right] = \left[b_k, a_{k'}^\dagger \right] = 0 \tag{2.30}$$

Completely analogously with the previous case we can now derive

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left(a_k^\dagger a_k + b_k^\dagger b_k \right) \tag{2.31}$$

which is the sum of particles' energy and antiparticles' energy.

The vacuum state $|0\rangle$ isdefined a that state, which is annihilated by all annihilation operators, those for particles (a_k) and those for antiparticles (b_k):

$$a_k|0\rangle = b_k|0\rangle = 0 \tag{2.32}$$

It is interesting to see, what happens with the interpretation for the conserved current

$$j^\mu = i \left(\phi \partial^\mu \phi^* - \phi^* \partial \phi \right) \tag{2.33}$$

Written with operators the charge is

$$Q = \int d^3x j^0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a_k^\dagger a_k - b_k^\dagger b_k \right) \tag{2.34}$$

Here we explicitly see that antiparticle contribute to the total charge oppositely as particles, from here their opposite sign. They contribute the same to the Hamiltonian, though. The current j^μ is thus the charge current of the complex scalar field.

2.3 The Dirac field

Let's consider now also fermions, i.e. particles with half spin, in our case only spin 1/2. The situation is somehow analogous to the complex scalar field, but now the free fields

should satisfy the Dirac equation instead of the Klein-Gordon equation. It is thus natural the following expansion:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_{s=1}^{2} \left(a_k^s u^s(k) e^{-ikx} + b_k^{s\dagger} v^s(k) e^{ikx} \right) \tag{2.35}$$

If we now assumed, as in the case of a complex scalar field, the commutation relations among the operators a_k, b_k, etc, we would end up with problems: the Hamiltonian would be unbounded from below. For this reason we have to assume the anti-commutation relation among them (in general this means $\{A, B\} \equiv AB + BA$):

$$\left\{ a_k^s, a_{k'}^{s'\dagger} \right\} = \left\{ b_k^s, b_{k'}^{s'\dagger} \right\} = (2\pi)^3 2\omega_k \delta^{ss'} \delta^3(\vec{k} - \vec{k}') \tag{2.36}$$

$$\left\{ a_k^s, a_{k'}^{s'} \right\} = \left\{ a_k^{s\dagger}, a_{k'}^{s'\dagger} \right\} = \left\{ b_k^s, b_{k'}^{s'} \right\} = \left\{ b_k^{s\dagger}, b_{k'}^{s'\dagger} \right\} = 0 \tag{2.37}$$

$$\left\{ a_k^s, b_{k'}^{s'} \right\} = \left\{ a_k^{s\dagger}, b_{k'}^{s'\dagger} \right\} = \left\{ a_k^s, b_{k'}^{s'\dagger} \right\} = \left\{ b_k^s, a_{k'}^{s'\dagger} \right\} = 0 \tag{2.38}$$

Stated differently, if we change the order of two arbitrary operators, we always get a minus sign, while in the case (2.36) we get also an addition.

The energy of an arbitrary system is now bounded from below. Th e Hamiltonian is in fact

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \sum_s \left(a_k^{s\dagger} a_k^s + b_k^{s\dagger} b_k^s \right) \tag{2.39}$$

and is explicitly positive definite.

The vacuum is defined via

$$a_k^s |0\rangle = b_k^s |0\rangle = 0 \tag{2.40}$$

2.4 The Maxwell's field

Very shortly we give jut the result for the massless case. The expansion used is

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \left(a_\lambda(k) \epsilon_\lambda^\mu(k) e^{-ikx} + a_\lambda^\dagger(k) \epsilon_\lambda^{\mu *}(k) e^{ikx} \right) \tag{2.41}$$

where we sum up over all polarisations λ and where the polarisation vector satisfies the constraint

$$k_\mu \epsilon_\lambda^\mu(k) = 0 \quad , \quad \sum_\lambda \epsilon_\lambda^\mu(k) \epsilon_\lambda^{\nu *}(k) = -g^{\mu\nu} \tag{2.42}$$

2.5 Exercise 1

- Derive (2.7) and (2.24).
- Calculate the vacuum expectation value of the time ordered product of two free scalar fields

$$\begin{aligned}\langle 0|\hat{T}\phi(x)\phi(y)|0\rangle &\equiv \langle 0\,|\phi(x)\phi(y)|\,0\rangle\theta(x^0-y^0)\\ &+ \langle 0|\phi(y)\phi(x)|0\rangle\theta(y^0-x^0)\end{aligned} \tag{2.43}$$

and show that it satisfies the equation (c =?)

$$\left(\partial_x^2+m^2\right)\langle 0|\hat{T}\phi(x)\phi(y)|0\rangle = c\delta^4(x-y) \tag{2.44}$$

2.6 Exercise 2

- Instead of the anti-commutation relations use for the Dirac field the (wrong!) commutation relations and show that the Hamiltonian is unbounded from below.
- In the case of a real scalar field show that causality is satisfied: check that $[\phi(x),\phi(y)] = 0$ if $(x-y)^2 < 0$.

3 Peturbation

3.1 Fields in different pictures, the operator of time evolution

So far we considered operators for free fields. The expansion of the field under the creation and annihilation operators went through (2.8). This field is in the interaction picture, i.e. it gets evolved in time via the Hamiltonian for a free field. We can see this better if we define a time independent field, i.e. the field in the Schrödinger picture via (t_0 is arbitrary)

$$\phi_S(x) \equiv \phi(t_0,\vec{x}) = e^{-iH_0(t-t_0)}\phi(x)e^{iH_0(t-t_0)} \tag{3.1}$$

where we have for example for a free real field

$$H_0 = \int d^3x \frac{1}{2}\left(\Pi^2(x) + (\nabla\phi)^2(x) + m^2\phi^2(x)\right) \tag{3.2}$$

That the right-hand side 3.1) is really time independent can be seen from the simple fact that, using (2.23),

$$e^{-iH_0(t-t_0)}a_k e^{iH_0(t-t_0)} = e^{i\omega_k(t-t_0)}a_k \quad , \quad e^{-iH_0(t-t_0)}a_k^\dagger e^{iH_0(t-t_0)} = e^{-i\omega_k(t-t_0)}a_k^\dagger \tag{3.3}$$

From here it follows that

$$H_0(\phi) = H_0(\phi_S) \tag{3.4}$$

and so

$$\phi(x) = e^{iH_0(\phi_S)(t-t_0)}\phi_S(x)e^{-iH_0(\phi_S)(t-t_0)} \tag{3.5}$$

Similarly we define the field in the Heisenberg picture as the one which transforms in time through the full (not free) but time independent Hamiltonian. In this way

$$\phi_H(x) = e^{iH(\phi_S)(t-t_0)}\phi_S(x)e^{-iH(\phi_S)(t-t_0)} \tag{3.6}$$

If we collect everything

$$\phi_H(x) = U^\dagger(t,t_0)\phi(x)U(t,t_0) \tag{3.7}$$

where we defined the time evolution operator in the interaction picture as

$$U(t,t_0) = e^{iH_0(\phi_S)(t-t_0)}e^{-iH(\phi_S)(t-t_0)} \tag{3.8}$$

We want to write this operator with $\phi(x)$. For this purpose we take the time derivative:

$$\begin{aligned} i\frac{\partial}{\partial t}U(t,t_0) &= e^{iH_0(\phi_S)(t-t_0)}\left(-H_0(\phi_S) + H(\phi_S)\right)e^{-iH(\phi_S)(t-t_0)} \\ &= H_{int}(\phi)U(t,t_0) \end{aligned} \tag{3.9}$$

where we took into account that the interaction part of the Hamiltonian

$$H_{int} \equiv H - H_0 \tag{3.10}$$

is typically a polynomial of the field.

If H_{int} were a function, the solution (3.9) would be simply $U = exp(-i\int dt H_{int})$. But now we have to do with operators, not with functions, so the equation (3.9) with the initial condition $U(t_0,t_0) = 1$ is solved at most perturbatively, for small H_{int}:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_{int}(t_1) + (-i)^2\int_{t_0}^{t} dt_1 H_{int}(t_1)\int_{t_0}^{t_1} dt_2 H_{int}(t_2) + \dots \tag{3.11}$$

We notice that products of the Hamiltonians are alway time ordered: first (on the left) are the Hamiltonians at late times, then those at initial times, i.e. $H_{int}(t_1)H_{int}(t_2)$ if $t_1 \geq t_2$. A we already mentioned, we have in general to do with non-commutative operators, so we need to take care of this orderings. To get a more symmetric notation, we introduce the operator of time ordered product $\hat{T}$:

$$\hat{T}\left(\hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)\right) \equiv \hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)\theta(t_1-t_2) + \hat{\mathcal{O}}_2(t_2)\hat{\mathcal{O}}_1(t_1)\theta(t_2-t_1) \tag{3.12}$$

Te quadratic part can be rewritten for $t > t_0$ as

$$\begin{aligned}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{int}(t_1)H_{int}(t_2) &= \frac{1}{2!}\left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{int}(t_1)H_{int}(t_2) + (t_1 \leftrightarrow t_2)\right) \\ &= \frac{1}{2!}\hat{T}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 H_{int}(t_1)H_{int}(t_2) \end{aligned} \tag{3.13}$$

Similarly we can treat an arbitrary power, which brings us to the final result

$$U(t,t_0) \equiv \hat{T}exp\left(-i\int_{t_0}^{t} dt H_{int}(\phi(x))\right) \quad , \quad t > t_0 \tag{3.14}$$

What if $t < t_0$? Then, if we repeat the above exercise,

$$U(t,t_0) = [U(t_0,t)]^{\dagger} \tag{3.15}$$

The operator (3.8) can be slightly generalised, defining

$$U(t_1,t_2) = e^{iH_0(\phi_S)(t_1-t_0)}e^{-iH(\phi_S)(t_1-t_2)}e^{-iH_0(\phi_S)(t_2-t_0)} \tag{3.16}$$

which is explicitly unitary and the same as the previous definition (3.8) when t_2 equals the reference time t_0. Now it is easy to confirm that such a unitary operator U satisfies $(t_1 > t_2 > t_3)$

$$U(t_1,t_2)U(t_2,t_3) = U(t_1,t_3) \tag{3.17}$$

$$U(t_1,t_2)U(t_2,t_1) = = 1 \tag{3.18}$$

For this reason we have for example

$$U(t_1,t_3)U(t_3,t_2) = U(t_1,t_3)\left[U(t_2,t_3)\right]^{\dagger} = U(t_1,t_2)U(t_2,t_3)\left[U(t_2,t_3)\right]^{-1} = U(t_1,t_2) \tag{3.19}$$

3.2 Correlation (Green) functions

n-point correlation functions or Green's functions are defined

$$G^{(n)}(x_1,\ldots,x_n) = \langle 0_H|\hat{T}\phi_H(x_1)\ldots\phi_H(x_n)|0_H\rangle \tag{3.20}$$

We would like to write it with the fields $\phi(x)$, which we know how to expand in creation and annihilation operators. For the field ϕ_H we use (3.7), what we still miss is relate the vacuum state $|0_H\rangle$ with the vacuum state $|0\rangle$ defined via (2.15). As is $|0\rangle$ the vacuum state of the operator H_0, so i $|0_H\rangle$ the vacuum state of the operator H. This is, the smallest eigenvalue of the energy operator H. Generically

$$H|n_H\rangle = E_n|n_H\rangle \tag{3.21}$$

Let's calculate now

$$e^{-iHT}|0\rangle = e^{-iE_0T}|0_H\rangle\langle 0_H|0\rangle + \sum_{n>0} e^{-iE_nT}|n_H\rangle\langle n_H|0\rangle \tag{3.22}$$

We can think that all energies have a small negative imaginary part. Then all higher states exponentially decay faster than the ground state for a big enough time T (so that we can safely take $T \to T + t_0$):

$$\begin{aligned}
|0_H\rangle &= \lim_{T\to\infty} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} = \lim_{T\to\infty} \frac{e^{-iH(T+t_0)}e^{iH_0(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} \\
&= \lim_{T\to\infty} \frac{\left(e^{iH_0((-T)-t_0)}e^{-iH((-T)-t_0)}\right)^\dagger|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} = \lim_{T\to\infty} \frac{[U(-T,t_0)]^\dagger|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} \\
&= \lim_{T\to\infty} \frac{U(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle}
\end{aligned} \tag{3.23}$$

where we took into account that

$$H_0|0\rangle = 0 \tag{3.24}$$

Similarly, from

$$\langle 0|e^{-iHT} = e^{-iE_0T}\langle 0|0_H\rangle\langle 0_H| + \dots \tag{3.25}$$

we can derive (homework!)

$$\langle 0_H| = \lim_{T\to\infty} \frac{\langle 0|U(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|0_H\rangle} \tag{3.26}$$

From the normalisation constraint we get

$$1 = \langle 0_H|0_H\rangle = \lim_{T\to\infty} \frac{\langle 0|U(T,-T)|0\rangle}{e^{-2iE_0T}|\langle 0|0_H\rangle|^2} \tag{3.27}$$

i.e.

$$\lim_{T\to\infty} e^{-2iE_0T}|\langle 0|0_H\rangle|^2 = \lim_{T\to\infty} \langle 0|U(T,-T)|0\rangle \tag{3.28}$$

The correlation (n-point Green) function is thus

$$\begin{aligned}
\langle 0_H|\hat{T}\phi_H(x_1)\dots\phi_H(x_n)|0_H\rangle &= \lim_{T\to\infty} \frac{1}{e^{-iE_0(T-t_0)}\langle 0|0_H\rangle}\langle 0|U(T,t_0) \\
&\quad \times\hat{T}\left[U^\dagger(x_1^0,t_0)\phi(x_1)U(x_1^0,t_0)\dots U^\dagger(x_n^0,t_0)\phi(x_n)U(x_n^0,t_0)\right] \\
&\quad \times\frac{1}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle}U(t_0,-T)|0\rangle \\
&= \lim_{T\to\infty} \frac{\langle 0|\hat{T}\phi(x_1)\dots\phi(x_n)U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}
\end{aligned} \tag{3.29}$$

and due to $\mathcal{H}_{int} = -\mathcal{L}_{int}$ we can rewritten with the free field (in the interaction picture):

$$G^{(n)}(x_1, \ldots, x_n) = \frac{\langle 0|\hat{T}\phi(x_1)\ldots\phi(x_n)\exp\left(i\int d^4x\mathcal{L}_{int}(\phi(x))\right)|0\rangle}{\langle 0|\hat{T}\exp\left(i\int d^4x\mathcal{L}_{int}(\phi(x))\right)|0\rangle} \tag{3.30}$$

where the operator of time ordering $\hat{T}$ is defined as

$$\hat{T}A(x)B(y) = A(x)B(y)\theta(x^0 - y^0) + B(y)A(x)\theta(y^0 - x^0) \tag{3.31}$$

i.e. it puts the operators in decreasing order of time.

The role of the denominator is here just to enforce the contribution of connected diagrams (those which cannot be divided in two without cutting at least one line) only. Thus, if we consider only connected diagrams, we do not need to keep track of what happens in the denominator.

This is the definition we will constantly use to calculate the correlation functions. As we will see, these correlation functions are directly related to physical quantities as for example the cross section or decay width.

3.3 The Wick theorem

Now we essentially have all we need to compute the correlation functions. In the right-hand side of eq. (3.30) we expand the exponent. The interaction term of the Lagrangian density is typically a polynomial in fields. In the so-called ϕ^4 theory we have for example

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{3.32}$$

and so

$$\mathcal{L}_{int}(\phi(x)) = -\frac{\lambda}{4!}\phi^4(x) \tag{3.33}$$

What we need are thus the correlation functions for free fields, i.e.

$$\langle 0|\hat{T}\phi(y_1)\ldots\phi(y_m)|0\rangle \tag{3.34}$$

In principle we know how to get them: Taking into account the time ordered product we expand the free fields in the interaction picture in terms of creation and annihilation operators. For a real scalar field it is the expansion (2.8). This way, although clear, is a bit long, especially if we have to do with a product of many fields. It turns out that things get much simplified if we take the vacuum expectation value of such a product. This simplification and such a rule goes under the name of Wick's theorem, which we will present now.

We split the field $\phi(x)$ into a part with positive energy $\phi^+(x)$ and a part with negative energy $\phi^-(x)$:

$$\phi(x) = \phi^+(x) + \phi^-(x) \tag{3.35}$$

$$\phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k e^{-ikx} \quad ; \quad \phi^-(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k^\dagger e^{ikx} \tag{3.36}$$

Notice that $\phi^{(+)}$ has annihilation operators a_k and a minus in the exponent $\exp(-ikx)$, while $\phi^{(-)}$ has creation operators $a_k^\dagger$ and a plus in the exponent $\exp(ikx)$, and not the other way! Then

$$\hat{N}\left[\phi(x_1)\phi(x_2)\right] = \phi^+(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \tag{3.37}$$

For two fields we have

$$\begin{aligned} \hat{T}\phi(x_1)\phi(x_2) &= \left[\phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\right]\Theta(x_1^0 - x_2^0) \\ &\quad + \left[x_1 \leftrightarrow x_2\right]\Theta(x_2^0 - x_1^0) \\ &= \left(\hat{N}\left[\phi(x_1)\phi(x_2)\right] + \left[\phi^+(x_1), \phi^-(x_2)\right]\right)\Theta(x_1^0 - x_2^0) \\ &\quad + \left(x_1 \leftrightarrow x_2\right)\Theta(x_2^0 - x_1^0) \end{aligned} \tag{3.38}$$

This notation is particularly useful since the vacuum expectation value of a normal ordered product is always zero

$$\langle 0|\hat{N}[\hat{\mathcal{O}}]|0\rangle = 0 \tag{3.39}$$

while the rest is easily computed:

$$\begin{aligned} \left[\phi^+(x_1), \phi^-(x_2)\right] &= \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} \left[a_{k_1}, a_{k_2}^\dagger\right] e^{-ik_1x_1 + ik_2x_2} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x_1 - x_2)} \end{aligned} \tag{3.40}$$

This is a c-number, not an operator anymore, so

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \left[\phi^+(x_1), \phi^-(x_2)\right]\Theta(x_1^0 - x_2^0) + \left[\phi^+(x_2), \phi^-(x_1)\right]\Theta(x_2^0 - x_1^0) \tag{3.41}$$

The integral representation for the Heaviside step function is

$$\Theta(y) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega y}}{\omega - i\epsilon} \tag{3.42}$$

since for $y > 0$ one can close the integral with the upper half circle at infinity with zero contribution ($\exp(-Im(\omega)y) \to 0$) so that the closed upper half-plane contributes exactly

1, while for $y < 0$ one needs to add the contribution of the lower half circle at infinity (now $\exp(+Im(\omega)y) \to 0$) but there is no pole in the lower half-plane giving zero.

Then

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3}\int \frac{d\omega}{2\pi}\frac{-i}{2\omega_k\left(\omega - i\epsilon\right)}e^{i(\omega-\omega_k)(x_1^0-x_2^0)+i\vec{k}(\vec{x}_1-\vec{x}_2)} + (x_1 \leftrightarrow x_2) \tag{3.43}$$

defining a integration variables as

$$k^0 = -\omega + \omega_k \tag{3.44}$$

we get

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-i}{2\omega_k\left(\omega_k - k^0 - i\epsilon\right)}\left(e^{-ik(x_1-x_2)} + e^{ik(x_1-x_2)}\right) \tag{3.45}$$

Then we change $k \to -k$ in the second integral and get

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{-ie^{-ik(x_1-x_2)}}{2\omega_k}\left(\frac{1}{\omega_k - k^0 - i\epsilon} + \frac{1}{\omega_k + k^0 - i\epsilon}\right) \tag{3.46}$$

and using

$$\omega_k^2 = \vec{k}^2 + m^2 \tag{3.47}$$

we finally arrive at the result

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}e^{-ik(x_1-x_2)} \tag{3.48}$$

Notice that here $k^\mu = (k^0, \vec{k})$ and k^0 is arbitrary (not equal to ω_k).

What would happen if we skipped the time order product? Then

$$\langle 0|\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3 2\omega_k}e^{-ik(x_1-x_2)} \tag{3.49}$$

is simply not the Green's function of the Klein Gordon operator and is not what we need. In fact we know that

$$(\partial_x^2 + m^2)\phi(x) = 0 \tag{3.50}$$

and so obviously

$$(\partial_x^2 + m^2)\langle 0|\phi(x)\phi(y)|0\rangle = 0 \tag{3.51}$$

while for the Green function we need

$$(\partial_x^2 + m^2)G^{(2)}(x,y) = -i\delta^4(x-y) \tag{3.52}$$

Such Green function is found in

$$G^{(2)}(x,y) = G^{(2)}(x-y) = \langle 0|\hat{T}\phi(x)\phi(y)|0\rangle \tag{3.53}$$

i.e. with the operator of time ordering.

The correlation function of an odd number of free fields is always zero. Repeating the procedure for four fields we get (exercise)

$$\begin{aligned}\langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = & \quad \langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle\langle 0|\hat{T}\phi(x_3)\phi(x_4)|0\rangle \\ & + \langle 0|\hat{T}\phi(x_1)\phi(x_3)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(x_4)|0\rangle \\ & + \langle 0|\hat{T}\phi(x_1)\phi(x_4)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(x_3)|0\rangle \end{aligned} \tag{3.54}$$

The correlation function for free fields is thus just the sum of propagators with all possible permutations. This can be easily generalised for an arbitrary number of free fields. This is the Wick theorem.

We can now derive the Feynman rules. Take the ϕ^4 theory:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \tag{3.55}$$

$$\mathcal{L}_0 = \frac{1}{2}\partial\phi^2 - \frac{m^2}{2}\phi^2 \quad , \quad \mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4 \tag{3.56}$$

The Feynman rule for the propagator in x-space is given by (3.48) and so in p-space

$$\frac{i}{k^2 - m^2 + i\epsilon} \tag{3.57}$$

while we derive for the vertex

$$\begin{aligned} & \langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\exp\left(i\int d^4z\mathcal{L}_{int}(\phi(z))\right)|0\rangle \\ \to \quad & -i\frac{\lambda}{4!}\int d^4z\langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(z)|0\rangle \\ = \quad & -i\lambda\int d^4z\langle 0|\hat{T}\phi(x_1)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_3)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_4)\phi(z)|0\rangle \\ = \quad & -i\lambda\int d^4z\prod_{i=1}^{4}\int\frac{d^4p_i}{(2\pi)^4}\frac{ie^{-ip_i(x_i-z)}}{p_i^2 - m^2 + i\epsilon} \\ = \quad & \prod_{i=1}^{4}\int\frac{d^4p_i}{(2\pi)^4}\frac{ie^{-ip_ix_i}}{p_i^2 - m^2 + i\epsilon}(2\pi)^4\delta^4(p_1 + \ldots + p_4)\,(-i\lambda) \end{aligned} \tag{3.58}$$

The Green function in p-space is defined as

$$G^{(n)}(x_1,\ldots,x_n)=\prod_{i=1}^{n}\int\frac{d^4p_i}{(2\pi)^4}e^{-ip_ix_i}(2\pi)^4\delta^4(p_1+\ldots+p_n)G^{(n)}(p_1,\ldots,p_n) \tag{3.59}$$

and the n-point amputated (i.e. without external legs) Green function $G^{(n)}_{amp}(p_1,\ldots,p_n)$ in p-space as

$$G^{(n)}(p_1,\ldots,p_n)=\prod_{i=1}^{n}\frac{i}{p_i^2-m^2+i\epsilon}G^{(n)}_{amp}(p_1,\ldots,p_n) \tag{3.60}$$

The Feynman rules we are interested in are written for the amputated Green function in p-space. So in our case of the ϕ^4 theory the Feynman rule for the vertex is

$$-i\lambda \tag{3.61}$$

3.4 Exercise 3

(A)

- In the ϕ^4 theory compute the 6-point Green's function for connected diagrams up to order $\mathcal{O}(\lambda^2)$. How does it look in p-space?
- Draw all connected Feynman diagrams for the 8-point G.f. up to order $\mathcal{O}(\lambda^3)$. Use the Feynman rules and write down the same G.f. in p space.
- Compute the 2-point G.f. in p space up to $\mathcal{O}(\lambda)$. What goes wrong?

(B)

Consider a model with two real scalar fieleds ϕ and χ with the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial\phi)^2+\frac{1}{2}(\partial\chi)^2-\frac{1}{2}m_\phi^2\phi^2-\frac{1}{2}m_\chi^2\chi^2-\frac{\lambda}{4}\phi^2\chi^2 \tag{3.62}$$

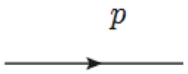
1. Derive the Feynman rules in p space.
2. Draw all connected Feynman diagrams for all possible 2-point Green functions to λ and 4 and 6 point G.f. to order λ^2.
3. Use the Feynman rules in p space for all the above cases.

3.5 The Feynman rules

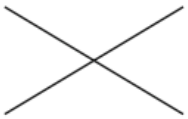
The Feynman rules for the ϕ^4 theory are simpler in p space:

1. draw all connected Feynman diagrams with given external particles $1, \ldots, n$
2. for every vertex write $-i\lambda$;
3. for each propagator take $i/(p^2 - m^2)$;
4. in every vertex the total 4-momentum is conserved;
5. the conservation of 4-momenta in all vertices does not determine all internal four-momenta; we have to integrate over all undetermined ones, for each of them we have then $\int d^4q/(2\pi)^4$;
6. the symmetry factor: if all 4! do not get canceled, one needs to take this into account.

Regarding the propagator and vertices, we denote them as



$$\frac{i}{p^2-m^2}$$



$$-i\lambda$$

3.6 Perturbation and loops

In the above case we computed the 4-point Green function to first order in powers of λ. What we get on top with this procedure are the quantum corrections, although we have not derived them. These are proportional to higher powers of λ on a given amplitude (Green function). This can be seen easily in the path integral formulation. There it turns out that in a given number of external particles the quantities λ and $\hbar$ enter only through the combination $\hbar\lambda$, which proves what we said.

What type of diagrams give higher powers of λ in the ϕ^4 theory? For a diagram with E external legs, I internal propagators and L closed loops the following relation is always satisfied:

$$4 - E = -2I + 4L \tag{3.63}$$

which follows simply from dimensional analysis. In fact

$$[\phi(x)] = M \tag{3.64}$$

and so (3.20)

$$[G^{(E)}(x)] = M^E \tag{3.65}$$

and so the Green function in p-space (3.59)

$$[G^{(E)}(p)] = M^{4-3E} \tag{3.66}$$

Each loop gives an integration over the internal four-momenta (together $4L$ integrations), each propagator contributes two inverse mass dimensions (together $-2(I+E)$):

$$4 - 3E = 4L - 2(I + E) \tag{3.67}$$

from which (3.63) follows.

Due to the fact that 4 propagators exits from each vertex and that each internal propagator is shared by two vertices, we can similarly derive that

$$E + 2I = 4V_4 \tag{3.68}$$

where V_4 is the number of vertices. From here we get the relation

$$L = 1 + V_4 - E/2 \tag{3.69}$$

which tells us that at the fixed number of external legs the number of loops and the power of the coupling constant increase the same way.

So, let's repeat. To compute physical quantities (cross sections, decay widths), with n external (initial incoming and final outgoing) particles, we need the n-point Green function. This is illustrated with connected Feynman diagrams, while the Feynman rules relate to each diagram an analytic expression as function of the quantum numbers (momenta, spin, etc.) of the external particles.

4 Physical quantities

Once we have a n-point Green function, the next step towards a physical quantity is the so called scattering or S matrix. The Green function is already almost the right thing, but surely not completely. The external particles obey the on-shell condition between energy and three-momentum, $p^2 - m^2 = 0$. If we impose this constraint, the external propagators of the Green function would diverge, since these are their poles. The correct prescription is however simple, one needs just to keep the residuum of this pole. The number of poles tells us how many external particles are involved in the process, while the residuum is the amplitude: imagine we are interested in the scattering process of particles described the above Lagrangian ϕ^4, let's say a particle with four momentum p_1 hits another particle with four-momentum p_2, while the outgoing particles have four-momenta q_j, $j = 1, ..., n$. The amplitude for such a process is simply

$$\begin{aligned} iT(p_1, p_2; q_1, ...q_n) &= \lim_{p_i^2, q_j^2 \to m^2} \prod_{i=1}^{2} \left[-i\left(p_i^2 - m^2\right)\right] \prod_{j=1}^{n} \left[-i\left(q_j^2 - m^2\right)\right] \\ &\times \; G(p_1, p_2, q_1, ...q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - \ldots - q_n) \\ &= \; G_{amp}(p_1, p_2, q_1, ...q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - ... - q_n)\big|_{p_i^2, q_j^2 \to m^2} \\ &= \; iA(p_1, p_2, q_1, ..., q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - \ldots - q_n) \end{aligned} \tag{4.1}$$

It is thus enough that for the computation of the Green functions we do not include the external propagators, those connected to external particles. We call such a Green function, G_{amp}, the amputated Green function. We see also that the total 4-momentum must be conserve. The difference between amputated Green function G_{amp} and the amplitude[3] A is in the fact that the four momenta of the external legs in the amputated Green function are arbitrary (subject only to the total conservation of four/momentum), while in the amplitude they satisfy the relation $p^2 \equiv p_0^2 - \vec{p}^{\,2} = m^2$. When this relation is satisfied, we

[3] In a sloppy way we call with the same name amplitude both the quantity T as A from the definition (4.1).

say that the external particle is on-shell: very far from the interaction point such particles behave in a good approximation as free and thus have in a good approximation a well defined four momentum (they are waves $\exp(-ip_ix_i)$ or $\exp(iq_jy_j)$).

Keep in mind that only external particles satisfy the on mass shell relation ($p^2 = m^2$). The internal particle of a Feynman diagram are virtual, their energy k^0 and three momentum $\vec{k}$ are not related in any way (they are independent quantities), so the internal propagators do not diverge (except in special cases, but this happens only when we integrate over the whole four-momentum of the virtual particle).

4.1 The cross section

As we know already from quantum mechanics, the main information on the scattering process is in the cross section. When we have the amplitude, we can get the cross section similarly as in quantum mechanics. Let us remind this derivation. To simplify the problem, let us consider the scattering of two scalar particles into two scalar particles (the generalisation to a final state with more particles is direct, while we will comment later the cases of fermions or gauge bosons).

The amplitude for such a transition is

$$iT(k_1, k_2, p_3, p_4) = (2\pi)^4\delta^4(k_1 + k_2 - p_3 - p_4)iA(k_1, k_2, p_3, p_4) \tag{4.2}$$

Strictly speaking, the initial particles are wave packets, i.e. distributions for different momenta with a sharp peak around p_1 or p_2 (such is the usual experimental situation). The initial state is described more precisely by

$$\int d\tilde{k}_1 \int d\tilde{k}_2\, f_1(k_1)f_2(k_2)|k_1k_2\rangle \tag{4.3}$$

where we used a short notation for the Lorentz invariant integration measure ($\omega^2 - \overrightarrow{k}^2 = m^2$)

$$d\tilde{k} \equiv \frac{d^3k}{(2\pi)^3 2\omega} \tag{4.4}$$

The square of the amplitude is then

$$\begin{aligned} & \int d\tilde{k}_1 \int d\tilde{k}_2 \int d\tilde{q}_1 \int d\tilde{q}_2 f_1(k_1)f_1^*(q_1)f_2(k_2)f_2^*(q_2) \\ \times\ & (2\pi)^4\delta^4(k_1 + k_2 - p_3 - p_4)(2\pi)^4\delta^4(q_1 + q_2 - p_3 - p_4) \\ \times\ & A(k_1, k_2, p_3, p_4)A^*(q_1, q_2, p_3, p_4) \end{aligned}$$

We rewrite the second δ-function in (4.5) as (using the first δ-function)

$$
\begin{aligned}
(2\pi)^4\delta^4(q_1+q_2-p_3-p_4) &= (2\pi)^4\delta^4(q_1+q_2-k_1-k_2) \\
&= \int d^4x e^{i(q_1+q_2-k_1-k_2)x}
\end{aligned}
\tag{4.5}
$$

keeping in mind that the wave packets have a sharp peak at $p_1$and. p_2, so that we can write

$$
\begin{aligned}
&(2\pi)^4\delta^4(k_1+k_2-p_3-p_4)A(k_1,k_2,p_3,p_4)A^*(q_1,q_2,p_3,p_4) \\
&\approx (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\left|A(p_1,p_2,p_3,p_4)\right|^2
\end{aligned}
\tag{4.6}
$$

and define the Fourier transform of the distribution $f_j(k_j)$

$$
\tilde{f}_j(x) = \int d\tilde{k}_j f_j(k_j) e^{-ik_j x} \tag{4.7}
$$

so that the square of the amplitude becomes now (in the remaining δ-function we have already transformed k_i with p_i before)

$$
\int d^4x \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\left|A(p_1,p_2,p_3,p_4)\right|^2 \tag{4.8}
$$

In the final state we are not looking after states with infinitely sharply defined 4-momenta $\overrightarrow{p}_{3,4}$, but all states with momenta in the interval between $\overrightarrow{p}_{3,4}$ and $\overrightarrow{p}_{3,4}+d\overrightarrow{p}_{3,4}$, so we have to multiply the above expression with the Lorentz invariant number of such states

$$
d\tilde{p}_3 d\tilde{p}_4 \tag{4.9}
$$

All together the probability for transition for a unit volume and time is

$$
\frac{dW}{VT} = \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 \left|A(p_1,p_2,p_3,p_4)\right|^2 d\,Lips_2(p_1+p_2;p_3,p_4) \tag{4.10}
$$

where the Lorentz invariant phase space for n particles with total 4-momentum P is defined generally as

$$
d\,Lips_n(P;p_1,\ldots,p_n) = (2\pi)^4\delta^4(P-\sum_{i=1}^{n} p_i)\prod_{j=1}^{n}\frac{d^3p_j}{(2\pi)^3 2\omega_j} \tag{4.11}
$$

The cross section $d\sigma$ is defined from

$$
\frac{dW}{VT} = d\sigma j\rho \tag{4.12}
$$

where the target density (let's be in the laboratory system, where the particles 1 have zero velocity) is

$$\rho = \left|\tilde{f}_1(x)\right|^2 2m \tag{4.13}$$

(we took the standard state normalisation $2p_0$), while the flux is made of particles 2:

$$j = \left|\tilde{f}_2(x)\right|^2 2|\overrightarrow{p}_2| \tag{4.14}$$

In general the seemingly non-invariant product $m|\overrightarrow{p}_2|$ can be written down in a Lorentz invariant way as

$$m|\overrightarrow{p}_2| = \left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2} \tag{4.15}$$

Finally the differential cross section is

$$d\sigma(p_1p_2 \to p_3p_4) = \frac{|A(p_1,p_2,p_3,p_4)|^2}{4\left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2}} d\,Lips_2(p_1+p_2;p_3,p_4) \tag{4.16}$$

This quantity is a Lorentz scalar, invariant of the choice of the coordinate system in which we measure.

The generalisation to scattering of two particles into $n+2$ particles is now logically unproblematic:

$$d\sigma(p_1p_2 \to p_3\dots p_{n+2}) = \frac{|A(p_1,p_2,p_3,\dots,p_{n+2})|^2}{4\left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2}} d\,Lips_n(p_1+p_2;p_3,\dots,p_{n+2}) \tag{4.17}$$

4.2 Decay width

Usually we are not interested into the possibility that three or more initial particles scatter, since the probability for three particles to meet each other is usually negligibly small. There is however another possibility, i.e. a single initial particles decaying into final state. In this case we do not speak about cross section but the decay width. As we know, this changes in different inertial systems (particles which fly by live longer according to a static observer). What we usually mention is the decay width for a particle in its own inertial system. For it we can use almost the same equation as for the cross section (we skip the flux) and get

$$d\Gamma(P \to p_1\dots p_n) = \frac{|A(P,p_1,\dots,p_n)|^2}{2m} d\,Lips_n(P;p_1,\dots,p_n) \tag{4.18}$$

where in the system of the decaying particles we have of course $P = (m, \overrightarrow{0})$.

4.3 Exercise 4

- Calculate the differential cross section $d\sigma/d\Omega$ and the total cross section σ for unpolarised fermions in the process of the previous exercise.

5 ∞

We have already encountered infinite integrals. They did not appear at the leading order (in tree level Feynman diagrams), but only in corrections (loops).

5.1 Regularisation

The first step towards taming infinities is its regularisation, i.e. a redefinition of the theory, so that everything is finite, and the original theory and infinities are recovered back only after some limit. The most used regularisation is the dimensional regularisation, which conserves gauge invariance. Essentially the infinite integral in 4-dimensions gets generalised into an integral in d dimensions, where d is not necessarily an integer. The integrals can be evaluated for arbitrary d, and the results have usually poles for integer dimensions. From here the problems when we take the limit $d \to 4$ i.e. $\epsilon \equiv 4 - d \to 0$.

Let's see it more precisely for the case of ϕ^4. First of all, let's see the dimensions of various quantities if we change the dimension of spacetime from 4 to $d = 4-\epsilon$. The action is dimensionless, so the kinetic term, gives us (remember that $[x] = M^{-1}$)

$$\left[\int d^{4-\epsilon}x \, (\partial\phi)^2\right] = M^0 \to [\phi] = M^{1-\epsilon/2} \tag{5.1}$$

So the interaction term tell us that

$$\left[\lambda \int d^{4-\epsilon}x\phi^4\right] = M^0 \to [\lambda] = M^{\epsilon} \tag{5.2}$$

The propagator is at 1-loop order

$$G^{(2)}(p) = \frac{i}{p^2 - m^2} + \frac{i}{p^2 - m^2}\frac{1}{2}(-i\lambda)\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\frac{i}{k^2 - m^2}\frac{i}{p^2 - m^2} \tag{5.3}$$

Its inverse is, again up to 1-loop order $\mathcal{O}(\lambda)$,

$$i/G^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + \frac{1}{i}\frac{1}{2}(-i\lambda)\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}}\frac{i}{k^2 - m^2} \tag{5.4}$$

The integral is infinite for $\epsilon = 0$, since for large k it is essentially $\int_{-\infty}^{+\infty} kdk$. But this is no more true for an arbitrary non-integer ϵ. Let's have a look at [2]:

$$\int \frac{d^dk}{(2\pi)^d}\frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma(n - d/2)}{\Gamma(n)}\Delta^{-n+d/2} \tag{5.5}$$

Let's use it for our case, use the relations

$$\Gamma(x+1) = x\Gamma(x) \tag{5.6}$$

$$\lim_{\epsilon\to 0}\Gamma(\epsilon/2) = 2/\epsilon - \gamma + \mathcal{O}(\epsilon) \tag{5.7}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and get up to order $\mathcal{O}(\epsilon^0)$

$$i/G^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln m^2\right) \tag{5.8}$$

The expression is clearly divergent in the physically sensible limit of 4 dimensions.
Similarly we get for the 4-point amputated G.f. up to 1-loop

$$-iG^{(4)}_{amp}(p_i, m^2, \lambda, \epsilon) = -\lambda + I(s, m^2, \lambda, \epsilon) + I(t, m^2, \lambda, \epsilon) + I(u, m^2, \lambda, \epsilon) \tag{5.9}$$

where

$$I(p^2, m^2, \lambda, \epsilon) = \frac{1}{i}\frac{1}{2}(-i\lambda)^2 \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \tag{5.10}$$

and where we introduced the Mandelstam variables (remember that $p_1 + p_2 + p_3 + p_4 = 0$)

$$s \equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 \tag{5.11}$$
$$t \equiv (p_1 + p_3)^2 = (p_2 + p_4)^2 \tag{5.12}$$
$$u \equiv (p_1 + p_4)^2 = (p_2 + p_3)^2 \tag{5.13}$$

for which

$$s + t + u = 4m^2 \tag{5.14}$$

(in general the right-handside is equal to $m_1^2 + m_2^2 + m_3^2 + m_4^2$).

We again make use of [2]. First we use

$$\frac{1}{A_1 \dots A_n} = \int_0^1 dx_1 \dots \int_0^1 dx_n \, \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n} \tag{5.15}$$

to get

$$I(p^2, m^2, \lambda, \epsilon) = \frac{\lambda^2}{2i} \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \int_0^1 dx \frac{1}{[k^2 - m^2 + (p^2 + 2pk)x]^2} \tag{5.16}$$

With a change of variables

$$k' = k + xp \tag{5.17}$$

we get rid of the linear term in k, so that we can use again eq. (5.5). The result is

$$I(p^2, m^2, \lambda, \epsilon) = \frac{\lambda^2}{2(4\pi)^2}\left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \int_0^1 dx \ln\left(m^2 - p^2 x(1-x)\right)\right] \tag{5.18}$$

5.2 Renormalisation

We regularised the two- and four-point G.f., but they still remain infinite in the limit $\epsilon \to 0$. Before performing such a limit we redefine the parameters of our model: m^2 and λ. We are allowed to do it, after all the physically measurable quantities are amplitudes (strictly speaking not directly them, but cross sections and decay widths, which are integrated squares of the amplitudes), and not necessarily the parameters of the Lagrangian. So we write

$$m^2 = m_R^2 - \delta m^2 \tag{5.19}$$
$$\lambda = \lambda_R \mu^\epsilon Z_\lambda \tag{5.20}$$

With the index R we denote renormalised, i.e. finite, quantities. The μ^ϵ term is added to have λ_R dimensionless. Of course we hide all the dangerous terms $1/\epsilon$ in δm^2, i.e. $\delta Z_\lambda = Z_\lambda - 1$, but in principle also something finite can be added. There are infinite possible choices, and the physical quantities again should not depend on that.

As we said, there is still a lot of choice to be done in defining (5.19) and (5.20). m_R and λ_R should of course be finite. We get rid of infinities if we choose for example the following renormalisation constraints:

$$i/G^{(2)}(m_R^2, m^2, \lambda, \epsilon) = 0 \tag{5.21}$$
$$-iG^{(4)}_{amp}(0, m^2, \lambda, \epsilon) = -\lambda_R \mu^\epsilon \tag{5.22}$$

Formally δm^2 and δZ_λ are of higher order in coupling constant λ_R than m_R^2 or 1. This means that

$$\delta m^2 = -\frac{\lambda_R m_R^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln\frac{m_R^2}{\mu^2}\right) \tag{5.23}$$
$$\delta Z_\lambda = \frac{3\lambda_R}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln\frac{m_R^2}{\mu^2}\right) \tag{5.24}$$

Let's define now the renormalisation Green's functions $G_R^{(n)}$, which are functions of the renormalised parameters as (as we will see later, this definition can be slightly changed in more general settings, but in our ϕ^4 example at 1-loop it is ok)

$$G_R^{(n)}(p_i, m_R^2, \lambda_R) = G^{(n)}(p_i, m^2, \lambda, \epsilon) \tag{5.25}$$

So the renormalised G.f. (we came back to four dimensions in the meantime, i.e. we took the limit $\epsilon \to 0$)

$$i/G_R^{(2)}(p^2, m_R^2, \lambda_R) = p^2 - m_R^2 \tag{5.26}$$

i.e.

$$
\begin{aligned}
-iG_R^{(4)}(p_i, m_R^2, \lambda_R) = -\lambda_R &+ I_R(s, m_R^2, \lambda_R, m_R^2) \\
&+ I_R(t, m_R^2, \lambda_R, m_R^2) + I_R(u, m_R^2, \lambda_R, m_R^2)
\end{aligned} \tag{5.27}
$$

where we used

$$
I_R(p^2, m_1^2, \lambda, m_2^2) = -\frac{\lambda^2}{2(4\pi)^2} \int_0^1 dx \ln \frac{m_1^2 - p^2 x(1-x)}{m_2^2} \tag{5.28}
$$

5.3 A further complication: field renormalisation

We saw that the corrections to the 2- and 4-point Green functions are of the form

$$
i/G^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + I^{(2)}(p^2, m^2, \lambda, \epsilon) \tag{5.29}
$$

$$
-iG^{(4)}(p_i, m^2, \lambda, \epsilon) = -\lambda + I^{(4)}(p_i, m^2, \lambda, \epsilon) \tag{5.30}
$$

In the ϕ^4 case to 1-loop it was enough to renormalise the mass and the coupling constant. At two loops or in slightly more complicated theories it turns out that also $I^{(2)}$ is a function of p^2 and in fact proportional to $1/\epsilon$. In this case we have to renormalise also the field

$$
\phi = Z_\phi^{1/2} \phi_R \tag{5.31}
$$

This must be done even in the case with finite $I^{(2)}(p^2)$ (but infinite in the $\epsilon \to 0$ limit). This can be seen in the following way. Let's define m_{POL}^2 as

$$
i/G^{(2)}(m_{POL}^2, m^2, \lambda, \epsilon) = 0 \tag{5.32}
$$

In our above examples we had $m_R^2 = m_{POL}^2$, but this is not necessarily always true. The residuum of the pole of the propagator is not 1 in general, but

$$
\frac{d}{dp^2} i/G^{(2)}(p^2, m^2, \lambda, \epsilon)\Big|_{p^2 = m_{POL}^2} = Z_\phi^{-1} \neq 1 \tag{5.33}
$$

For such cases the equation for the amplitude (4.1) is invalid. In fact we derived it for the correctly normalised ($Z_\phi = 1$) fields, for which the KG operator transforms the external legs into a δ function. Let's call such a field ϕ_R (with the canonical normalisation). Then due to (5.31)

$$
\begin{aligned}
A^{(n)} &\sim \left[i \left(\partial^2 + m^2 \right) \right]^n G_R \\
&\to Z_\phi^{-n/2} \left[i \left(\partial^2 + m^2 \right) \right]^n G
\end{aligned} \tag{5.34}
$$

Taking into account

$$G^{(n)} \sim \left[G^{(2)}\right]^n G^{(n)}_{amp} \tag{5.35}$$

and

$$-i(\partial^2 + m^2) \left\{ \begin{array}{c} G^{(2)} \\ G^{(2)}_R \end{array} \right\} \sim \left\{ \begin{array}{c} Z_\phi \\ 1 \end{array} \right\} \tag{5.36}$$

eq. (5.34) becomes

$$A^{(n)} \sim Z_\phi^{n/2} G^{(n)}_{amp} \tag{5.37}$$

On another side

$$G^{(n)} \sim \langle \phi^n \rangle = Z_\phi^{n/2} G^{(n)}_R \tag{5.38}$$

from which it follows that (5.34) can be written

$$A^{(n)} \sim G^{(n)}_{amp,R} \tag{5.39}$$

as we expected.

The renormalised G.f. is defined after comparison (5.34)-(5.39) as

$$G^{(n)}_R(p_i, m^2_R, \lambda_R) = Z_\phi^{n/2}(m^2, \lambda, \epsilon) G^{(n)}(p_i, m^2, \lambda, \epsilon) \tag{5.40}$$

The renormalisation constraints are

$$i/G^{(2)}_R(m^2_R, m^2_R, \lambda_R) = 0 \tag{5.41}$$

$$\frac{d}{dp^2} i/G^{(2)}_R(m^2_R, m^2_R, \lambda_R) = 1 \tag{5.42}$$

$$-iG^{(4)}_R(0, m^2_R, \lambda_R) = = -\lambda_R \tag{5.43}$$

from which we can compute δm^2, δZ_λ and $\delta Z_\phi = Z_\phi - 1$.

5.4 Counter-terms

We can exchange the bare quantities m^2, λ, ϕ with the renormalised m^2_R, λ_R, ϕ_R can be done from the start. The bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{5.44}$$

can be replaced by a sum of the renormalised Lagrangian

$$\mathcal{L}_R = \frac{1}{2}(\partial\phi_R)^2 - \frac{1}{2}m^2_R\phi^2_R - \frac{\lambda_R\mu^\epsilon}{4!}\phi^4_R \tag{5.45}$$

and counter-terms (CT)

$$\mathcal{L}_{CT} = \frac{1}{2}\delta Z_\phi (\partial\phi_R)^2 - \frac{1}{2}\left(\delta Z_\phi m_R^2 - Z_\phi \delta m^2\right)\phi_R^2 - \frac{\lambda_R \mu^\epsilon}{4!}\left(Z_\lambda Z_\phi^2 - 1\right)\phi_R^4 \tag{5.46}$$

We consider these new terms as real Lagrangian terms, derive from them the Feynman rules and take them into account in the Feynman diagrams. Clearly they are formally of higher power of the coupling constant λ_R. We can expand them in powers of the coupling constant (the coefficients can of course be singular in $1/\epsilon$)

$$\begin{aligned}
\delta m^2 &= \delta m_1^2 \lambda_R + \delta m_2^2 \lambda_R^2 + \ldots & (5.47)\\
\delta Z_\lambda &= \delta Z_{\lambda 1}\lambda_R + \delta Z_{\lambda 2}\lambda_R^2 + \ldots & (5.48)\\
\delta Z_\phi &= \delta Z_{\phi 1}\lambda_R + \delta Z_{\phi 2}\lambda_R^2 + \ldots & (5.49)
\end{aligned}$$

and should thus be formally taken into account at the right power of λ_R.

5.5 Different renormalisation conditions (schemes)

The renormalisation conditions above are not unique. We can choose them differently, and indeed this is what we often do. Let's now shortly describe the so called $\overline{MS}$ scheme, which is used especially in QCD.

We do not have here any constraints in special kinematic points as before, but we determine the counter-terms simply from the requirement of getting rid of all terms $2/\epsilon - \gamma + \ln 4\pi$. The name $\overline{MS}$ means minimal subtraction, the bar added is because on top of the divergent $(2/\epsilon)$ terms we included also the above finite $(-\gamma + \ln 4\pi)$ terms.

In the previous ϕ^4 case this would mean

$$\begin{aligned}
\delta m^2 &= -\frac{\lambda_R m_R^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) & (5.50)\\
\delta Z_\lambda &= \frac{3\lambda_R}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) & (5.51)\\
\delta Z_\phi &= 0 & (5.52)
\end{aligned}$$

Of course the renormalised G.f. look different than before

$$\begin{aligned}
i/G_R^{(2)}(p^2, m_R^2, \lambda_R, \mu) &= p^2 - m_R^2 + \frac{\lambda_R m_R^2}{2(4\pi)^2}\left(1 - \ln\frac{m_R^2}{\mu^2}\right) & (5.53)\\
-iG_R^{(4)}(p_i, m_R^2, \lambda_R, \mu) &= -\lambda_R + I_R(s, m_R^2, \lambda_R, \mu^2) & (5.54)\\
&+ I_R(t, m_R^2, \lambda_R, \mu^2) + I_R(u, m_R^2, \lambda_R, \mu^2)
\end{aligned}$$

However we have to remember that now m_R and λ_R are something completely different from before (different numbers).

5.6 Exercise 5

- To the ϕ^4 Lagrangian add a fermion and the terms

$$\bar{\psi}\left(i\partial\!\!\!/ - m_f\right)\psi - y\phi\bar{\psi}\psi \tag{5.55}$$

- Compute a correction to the vertex $\bar{\psi}\psi\phi$ at one loop and renormalise y in the $\overline{MS}$ scheme.

- Calculate the vertex ϕ^3 at one loop and show that it diverges.

- From here it follows that we should have added already from the beginning a term $M\phi^3$, and the renormalisation of M would cancel the divergence of the vertex ϕ^3, which appears at one loop, similarly as the renormalisation of y cancels the divergence of the vertex $\bar{\psi}\psi\phi$.

- What happens in the $m_f \to 0$ limit? Why the vertex ϕ^3 in this case does not get generated at one loop for $M = 0$? Which is the symmetry that forbids it?

5.7 Renormalisation group equations

The above expressions are a bit surprising at first sight, since it looks like the physical quantities depends on an extra arbitrary parameter μ, which entered into the result via dimensional regularisation and the requirement of dimensionless renormalised coupling constant. This would have usually[4] happened even in theories without any mass parameter: renormalisation itself generates a dimensionful factor, which we call dimensional transmutation. This way or another, the dependence over the unknown μ is only apparent. We will show that the renormalised parameters (m_R^2, λ_R in the $\overline{MS}$ scheme) depend also on μ, so that the total effect gets neutralised and the physical quantities are independent of μ.

To check this is of course important, but we will get another bonus. We will in fact show that we can, using this trick, resum up an infinite series of dangerous large logarithms, which show up in some kinematical limits, and so save the perturbative expansion.

Let's start recognising that all bare parameters are independent on μ.

$$0 = \mu\frac{d\lambda}{d\mu} = \mu\frac{d}{d\mu}\left(\lambda_R\mu^\epsilon Z_\lambda(\lambda_R,\epsilon)\right) \tag{5.56}$$

From here we get the equation

$$\epsilon\lambda_R + \mu\frac{d\lambda_R}{d\mu}\left(1 + \lambda_R\frac{\partial \ln Z_\lambda(\lambda_R,\epsilon)}{\partial\lambda_R}\right) = 0 \tag{5.57}$$

The ansatz for the solution is an expansion in positive powers of the renormalised coupling constant (higher orders become important only at the level of two loops or later)

[4] except in conformal theories

$$\mu\frac{d\lambda_R}{d\mu} = A\lambda_R + B\lambda_R^2 + \dots \tag{5.58}$$

Taking into account (5.51) we get

$$\mu\frac{d\lambda_R}{d\mu} = -\epsilon\lambda_R + \frac{3\lambda_R^2}{(4\pi)^2} \tag{5.59}$$

Now we can safely take the limit $\epsilon \to 0$ and get an equation, which describes the change of the renormalised coupling constant with the scale μ:

$$\mu\frac{d\lambda_R}{d\mu} = \frac{3\lambda_R^2}{(4\pi)^2} \quad (\equiv \beta(\lambda_R)) \tag{5.60}$$

The expansion in powers of λ_R on the righthand-side is called the β function. The equation can be integrated

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - \frac{3\lambda_R(\mu_0)}{(4\pi)^2}\ln\frac{\mu}{\mu_0}} \tag{5.61}$$

Increasing the scale we arrive at a singularity, since the beta function is positive. This means that the coupling constant gets increased with the scale at which we measure it. Of course the approximation breaks down before we reach the singularity, called the Landau pole, since λ_R becomes large enough so that we cannot stop the expansion at one loop level.

The same can be done with the mass:

$$0 = \mu\frac{dm^2}{d\mu} = \mu\frac{d}{d\mu}\left(m_R^2 - \delta m^2(m_R^2, \lambda_R, \epsilon)\right) \tag{5.62}$$

from where, using (5.50),

$$\mu\frac{dm_R^2}{d\mu} + \left(m_R^2\mu\frac{d\lambda_R}{d\mu} + \lambda_R\mu\frac{dm_R^2}{d\mu}\right)\frac{1}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) = 0 \tag{5.63}$$

Similarly as before we expand

$$\mu\frac{dm_R^2}{d\mu} = C\lambda_R + \dots \tag{5.64}$$

and using (5.59) we get at first order

$$\mu\frac{dm_R^2}{d\mu} = \frac{\lambda_R}{(4\pi)^2}m_R^2 \tag{5.65}$$

the solution of which is

$$m_R^2(\mu) = \frac{m_R^2(\mu_0)}{\left(1 - \frac{3\lambda_R(\mu_0)}{(4\pi)^2}\ln\frac{\mu}{\mu_0}\right)^{1/3}} \tag{5.66}$$

5.8 An explicit example

Imagine we have an accelerator of particles ϕ, which we postulate can be described by our old friend, the ϕ^4 Lagrangian. We are interested in elastic scattering of two particles (this means that the number and type of particles at the beginning and at the end are the same).

The incoming particles have momenta in the centre-of-mass system (let it be equal to the laboratory system)

$$p_1^\mu = (E, 0, 0, p) \tag{5.67}$$
$$p_2^\mu = (E, 0, 0, -p) \tag{5.68}$$

while the final particles have

$$p_3^\mu = -(E, p\sin\theta, 0, p\cos\theta) \tag{5.69}$$
$$p_4^\mu = -(E, -p\sin\theta, 0, -p\cos\theta) \tag{5.70}$$

where

$$E^2 - p^2 = m^2 \tag{5.71}$$

and m^2 is the pole of the propagator. At tree order this is just the mass parameter of the Lagrangian, while at the one loop level this is m_R^2 defined via (5.26) in our case, or via (5.32) in general. This is also the mass felt by gravity in the Newton's law.

Imagine we can somehow measure the scattering cross section σ_{exp} in the limit where the initial particles have a very small momentum (this is usually hard since it essentially mean that the initial particles do not move, so there is no scattering; we will imagine it as a kind of limit). We choose this definition only to get simpler equations, in the real case we measure in a different kinematical point. In this convenient limiting ($p \to 0$) kinematical phase space point the Mandelstam variables are

$$s \to 4m_R^2 \tag{5.72}$$
$$t \to 0 \tag{5.73}$$
$$u \to 0 \tag{5.74}$$

At tree order the amplitude

$$-iG^{(4)}(p_i, m, \lambda) = -\lambda \tag{5.75}$$

is independent of momenta. Since we measure the cross section, we have to evaluate the phase space etc. from formula (4.16).

Before integrating over final momenta we need to assume general form of the final momenta:

$$p_3^\mu = -(E_3, \vec{p}_3) \quad , \quad p_4^\mu = -(E_4, -\vec{p}_4) \tag{5.76}$$

where $p_{3,4} = \sqrt{m^2 + \vec{p}_{3,4}^2}$. Then the calculation of the cross section proceeds as

$$\begin{aligned} d\sigma &= \frac{|\lambda|^2}{4\sqrt{(E^2+p^2)^2 - m^4}} (2\pi)^4 \delta(2E - E_3 - E_4) \delta^3(\vec{p}_3 - \vec{p}_4) \frac{d^3\vec{p}_3}{(2\pi)^3 2\omega_{p_3}} \frac{d^3\vec{p}_4}{(2\pi)^3 2\omega_{p_4}} \\ &= \frac{|\lambda|^2}{4\sqrt{4E^4 - 4E^2m^2}} \delta(2E - 2E_3) \frac{d\Omega p_3^2 dp_3}{(2\pi)^2 4E_3^2} \\ &= \frac{|\lambda|^2}{8Ep} \frac{\delta(E - E_3)}{2} \frac{d\Omega p_3 E_3 dE_3}{(2\pi)^2 4E_3^2} \end{aligned} \tag{5.77}$$

We used $\delta(ax) = \delta(x)/|a|$ and integrating over half solid angle because the two final particles are indistinguishable, the result is

$$\sigma = \frac{\lambda^2}{128\pi E^2} \tag{5.78}$$

The coupling constant λ is determined from the measured σ_{exp} in the limit (5.72)-(5.74):

$$\sigma_{exp} = \frac{\lambda^2}{128\pi m^2} \tag{5.79}$$

So in general

$$\sigma = \sigma_{exp} \left(\frac{m}{E}\right)^2 \tag{5.80}$$

Imagine now that we are not happy with the precision of comparison between theory and experiment. In this case we can use the results of the next approximation, i.e. in the one loop approximation. For the renormalisation constraints we choose (5.41)-(5.43), so that the expression for the 4-point Green function is given in (5.27). This is in the kinematical point we measure (5.72)-(5.74) equal to (we have to evaluate the integrals of type (5.28))

$$-iG_{exp}^{(4)} = -\lambda_R + \frac{\lambda_R^2}{(4\pi)^2} \tag{5.81}$$

When squaring $\Gamma^{(4)}$we should not take into account the largest power λ_R^4, since this is already of higher order (if we calculated up to two loops, we would have also the order λ^3 in (5.81), which would, multiplied with the tree order, give also the same power), so that

$$\sigma_{exp} = \frac{1}{128\pi m_R^2} \left(\lambda_R^2 - 2\frac{\lambda_R^3}{(4\pi)^2}\right) \tag{5.82}$$

Through this equation we calculate the value of λ_R. If it is small enough, i.e. if

$$\frac{\lambda_R}{(4\pi)^2} \ll 1 \tag{5.83}$$

the perturbative expansion of the Greens' functions converge and the computation is consistent.

We calculate the scattering cross section in an arbitrary kinematical point using (4.16) and (5.27), where however now λ_R is numerically fit through (5.82).

What if, instead of (5.43), we had used a different constraint, for example in the limit (5.72)-(5.74)

$$-iG_R^{(4)} \to -\lambda'_R \tag{5.84}$$

Then

$$\sigma_{exp} = \frac{\lambda'^2_R}{128\pi m_R^2} \tag{5.85}$$

As we see, the numerical value of λ'_R is different from λ_R, the final result however is, the same, up to corrections formally of higher order in powers of the coupling constant. Numerically the physical result (the scattering cross section as a function of momenta) can slightly depend on the choice of the renormalisation scheme, but the difference is, if the convergence is good (5.83), small enough.

What if we use instead the $\overline{MS}$ scheme? To avoid extra complications, let's use this scheme only for the coupling constant, while let the mass be still the propagator pole:

$$i/G_R^{(2)} = p^2 - m_R^2 \tag{5.86}$$

$$\begin{aligned} -iG_R^{(4)} &= -\lambda_R(\mu) + I_R(s, m_R^2, \lambda_R(\mu), \mu^2) \\ &+ I_R(t, m_R^2, \lambda_R(\mu), \mu^2) + I_R(u, m_R^2, \lambda_R(\mu), \mu^2) \end{aligned} \tag{5.87}$$

The coupling constant $\lambda_R(\mu)$ is now, differently from previous one, the running one, dependent on the parameter μ through the renormalisation group equation (5.60). We should not worry about it. Using the same procedure as before we get first in the limit (5.72)-(5.74)

$$-iG_{exp}^{(4)} \to -\lambda_R(\mu) + \frac{\lambda_R^2(\mu)}{(4\pi)^2}\left(1 + 3\ln\frac{\mu}{m_R}\right) \tag{5.88}$$

At $\mu = m_R$ we have the same expression as before, see (5.81)

$$-iG_{exp}^{(4)} = -\lambda_R(m_R) + \frac{\lambda_R^2(m_R)}{(4\pi)^2} \tag{5.89}$$

so $\lambda_R(m_R)$ is numerically equal to λ_R, which is got through eq. (5.82). When we integrate (5.60), we take into account the following boundary condition:

$$\lambda_R(\mu) = \frac{\lambda_R(m_R)}{1 - \frac{3\lambda_R(m_R)}{(4\pi)^2} \ln \frac{\mu}{m_R}} \tag{5.90}$$

The careful reader can now ask, if the last expression can be maintained in this form, or if it should be expanded in powers of $\lambda_R(m_R)$, as usual. But here there is a difference, since we have here also $\ln(\mu/m_R)$, which could be in principle large. If it is small enough

$$\frac{\lambda_R(m_R)}{(4\pi)^2} \ln \frac{\mu}{m_R} \ll 1 \tag{5.91}$$

one can safely expand to quadratic order, insert it into (5.87), and get

$$\begin{aligned} \Gamma^{(4)}(p_i, m_R^2, \lambda_R) = -\lambda_R &+ I_R(s, m_R^2, \lambda_R, m_R^2) \\ &+ I_R(t, m_R^2, \lambda_R, m_R^2) + I_R(u, m_R^2, \lambda_R, m_R^2) \end{aligned} \tag{5.92}$$

totally the same as (5.27). In this case there is no difference between the original scheme and the $\overline{MS}$ scheme.

Once get a difference however, when (5.91) is not satisfied, but (5.83) is. This is obviously possible only at very (exponentially) large ratios μ/m_R. In this case the solution of the RG equation helped to sum up all powers

$$\left(\frac{\lambda_R(m_R)}{(4\pi)^2} \ln \frac{\mu}{m_R} \right)^n \tag{5.93}$$

which appear at the level of n-th loop. Of course these n-th loops for $n > 1$ has not been calculated, but the solution of the RG equation allows us to resum these dominant terms in this limit.

The careful reader will obviously be again confused. Why should we use at all these large (or small) ratios μ/m_R, after all we said we are allowed to (at least in principle) use any such ratio. Such a reason could be for example the behaviour of the cross section at very large energies, for example when $s/m_R^2 \to \infty$. In the case (5.27) the correction to tree order$-\lambda_R$ would be of the form

$$\frac{\lambda_R^2}{(4\pi)^2} \ln \frac{s}{m_R^2} \tag{5.94}$$

and this number is not necessarily less than one. Higher terms would contribute higher powers

$$\left(\frac{\lambda_R^2}{(4\pi)^2} \ln \frac{s}{m_R^2} \right)^n \tag{5.95}$$

and the expansion would not converge.

This problem is solved with the running coupling constant (5.90) and the use of (5.88) for $\mu \approx E$. Then the ratios

$$\left(\frac{\lambda_R(m_R)}{(4\pi)^2}\ln\frac{s}{\mu^2}\right)^n \tag{5.96}$$

which appear in (5.88) and eventually at higher orders, are small enough not to spoil the convergence. Of course we included all these contributions of higher orders into the running coupling constant

$$\lambda_R(E) = \frac{\lambda_R(m_R)}{1 - \frac{3\lambda_R(m_R)}{(4\pi)^2}\ln\frac{E}{m_R}} \tag{5.97}$$

The whole procedure converges under condition

$$\frac{\lambda_R(E)}{(4\pi)^2} \ll 1 \tag{5.98}$$

which is not true if we are too close to the Landau pole.

We see that resummation saved the perturbative expansion in the case of very large (or small) energies. This is why the scheme with running coupling constant is useful.

5.9 Exercise 6

- Compute the differential unpolarised cross section for $e^-\mu^- \to e^-\mu^-$. Make use of the results for $e^+e^- \to \mu^+\mu^-$. In a similar way use the known results for Compton scattering $e\gamma \to e\gamma$ to calculate the cross section for annihilation $e^+e^- \to \gamma\gamma$. Comment the "crossing" symmetry.

5.10 Exercise 7

- Consider the system with interaction $\lambda\sigma\phi_1\phi_2$. If the corresponding masses astisfy $m_\sigma > m_1 + m_2$ calculate the decay width Γ_σ at tree level. Show that at the first nonzero order in λ the optical theorem is valid

$$m_\sigma\Gamma_\sigma = Im\left(i/G^{(2)}_{\sigma\sigma}(m_\sigma^2)\right)$$

where $G^{(2)}_{\sigma\sigma}(p^2)$ is the 2-point G.f. for σ.

6 Quantum electrodynamics (QED)

6.1 The Lagrangian

This is the field theory which describes the electron and the photon., i.e. the electromagnetic interaction of a charged fermion. The Lagrangian, which describes it, has been already written down

$$\mathcal{L} = \bar{\psi}\, i\gamma^\mu \left(\partial_\mu - ieA_\mu\right) \psi - m\bar{\psi}\psi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \tag{6.1}$$

where the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{6.2}$$

is found from the requirement of gauge invariance under local (spacetime dependent) phase U(1) transformation:

$$\psi \to e^{i\alpha(x)}\psi \ , \ \ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x) \tag{6.3}$$

6.2 Feynman rules

We can derive the Feynman rules similarly as in the case of a scalar field.

6.2.1 Propagators

Let's first consider the propagators. We get them in p space as the inverse of the quadratic Lagrangian term. In the case of a real scalar field the i times the action in the exponent of the path integral was

$$\begin{aligned} iS[\phi] &= i\int d^4z \mathcal{L}(\phi(z), \partial\phi(z)) = i\int d^4z \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] + \dots \\ &= -\frac{1}{2}\int d^4z\, \phi(z)\, i(\partial^2 + m^2)\phi(z) \\ &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \phi(-p)\ \left[(-i)(p^2 - m^2)\right] \phi(p) \end{aligned} \tag{6.4}$$

where we arrived tot he second line after integration by part, and to the third one by the Fourier transform

$$\phi(z) = \int \frac{d^4p}{(2\pi)^4} \phi(p) e^{-ipz} \tag{6.5}$$

The propagator of the real scalar field in p space is defined as the inverse of the square bracket in (6.4), i.e.

$$G_S(p) = \frac{i}{p^2 - m^2} \tag{6.6}$$

In the case of a complex scalar field, there would have been no 1/2 at the beginning, but the result would have been the same as (6.6).

We can proceed in the same way for the case of fermions

$$G_F(p) = \frac{i}{\not{p} - m} \tag{6.7}$$

where

$$\not{p} \equiv \gamma^\mu p_\mu \tag{6.8}$$

A problem arises if we use the same procedure for the case of a vector field. In this case we get after the integration by parts and the transition to p space

$$i \int dz \left(-\frac{1}{4}\right) F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A^\mu(-p) \left[i \left(p^2 g_{\mu\nu} - p_\mu p_\nu \right) \right] A^\nu(p) \tag{6.9}$$

But the inverse of the square bracket does not exist, since the matrix $p^2 g_{\mu\nu} - p_\mu p_\nu$ has an eigenvector p^ν with zero eignevalue!

Although this may seem strange, we should have expected some problem since we never choose a gauge. So we now correct the mistake and without breaking Lorentz invariance add a term:

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} (\partial A)^2 \tag{6.10}$$

We repeat the above calculation

$$\begin{aligned} & i \int dz \left[\left(-\frac{1}{4}\right) F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial A)^2 \right] \\ = \; & -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A^\mu(-p) \left[i \left(p^2 g_{\mu\nu} - (1 - 1/\xi) p_\mu p_\nu \right) \right] A^\nu(p) \end{aligned} \tag{6.11}$$

We guess the inverse as

$$G_V^{\nu\sigma}(p) = A(p^2) g^{\nu\sigma} + B(p^2) p^\nu p^\sigma \tag{6.12}$$

and of course require

$$\left[i \left(p^2 g_{\mu\nu} - (1 - 1/\xi)\, p_\mu p_\nu \right) \right] G_V^{\nu\sigma}(p) = g_\mu{}^\sigma \tag{6.13}$$

The final solution is

$$G_V^{\nu\sigma}(p) = \frac{i}{p^2} \left(-g^{\nu\sigma} + (1 - \xi) \frac{p^\nu p^\sigma}{p^2} \right) \tag{6.14}$$

A particularly useful and easy choice of the gauge is the so-called 't Hooft-Feynmanova gauge $\xi = 1$, which gives asa very simple photon propagator

$$G_V^{\nu\sigma}(p) = \frac{-i g^{\nu\sigma}}{p^2} \tag{6.15}$$

This is very often used, although the general choice (6.14) can sometimes be useful, since the parameter ξ must disappear at the end (a physical quantity cannot depend on the choice of gauge), and so the general calculation represents a test.

6.2.2 Vertices

the next thing we have to determine are the Feynman rules for the vertices. We have to do only with one such vertex, since we have only one interaction term in the Lagrangian

$$\mathcal{L}_{int} = e\bar{\psi}\not{A}\psi \tag{6.16}$$

We know how to determine the Feynman rule for this vertex. We are interested into a diagram, where the component ξ of a Dirac fermion with momentum p_1 emits the component η of a Dirac fermion with a momentum p_2 and the component μ of the photon with momentum q. We thus calculate the Green function in p space

$$\int dxdydz\, G\left[\psi_\xi(x)\bar{\psi}_\eta(y)A_\mu(z)\right] e^{-ip_1x+ip_2y+iqz} \tag{6.17}$$

and then leave out all the three external propagators and the delta function. The result:

$$ie(\gamma_\mu)_{\xi\eta} \tag{6.18}$$

6.2.3 Fermion loops

When using the Wick theorem we have often to interchange two fermion fields. Since they anti-commute (different from bosons which commute), we get during such operation an extra factor (-1). Let's see more precisely when this happens. It is enough to have a look to the only two possible cases to which all diagrams reduce to. In the first one the fermion fields are external, in the second they are internal. Because of the conservation of the fermion line (these are always continuous, which can be seen from the fact that fermion fields appear, due to Lorentz invariance, only quadratically in the Lagrangian) we cannot have a third case.

In the first case the part of the Green function we are interested in looks like

$$\langle 0|\hat{T}\psi_\xi(x)\bar{\psi}_\eta(y)\prod_{i=1}^{N} i\int dz_i e\bar{\psi}\not{A}\psi(z_i)|0\rangle \tag{6.19}$$

where all the vertices in between are connected by the fermion line. It is not difficult to convince oneself that there is no minus sign here.

The second case is the one with two internal vertices. We have thus a close loop

$$\langle 0|\hat{T}\prod_{i=1}^{N} i\int dz_i e\bar{\psi}\not{A}\psi(z_i)|0\rangle \tag{6.20}$$

Here it is clear that we get exactly a factor (-1).

Let's repeat: for every closed fermion loop we get an extra factor (-1). Of course the whole loop must be fermionic, from the beginning to the end.

6.2.4 External legs

The last thing we need to specify is what to add for an external particles. The difference between the real scalar field, for which there was nothing to be added for the external line, and the vector field with spin 1 A_μ (the photon) or the fermion particle of spin 1/2 (the electron) can be traced to different forms of the quantised assimptotically free fields.

In the case of the real scalar field we expanded the external assimptotically free field (which satisfies the KG equation) with creation and annihilation operators ($p_0 = \sqrt{m^2 + \vec{p}^2}$ and m the mass of the scalar field)

$$\hat{\phi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right) \tag{6.21}$$

Similarly we do with the vector field, which satisfies the Maxwell equations in empty space ($p_0 = |\vec{p}|$, since the photon is massless)

$$\hat{A}^\mu(x) = \sum_{\lambda=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(a_\lambda(p)\epsilon_\lambda^\mu(p)e^{-ipx} + a_\lambda^\dagger(p)\epsilon_\lambda^{*\mu}e^{ipx}\right) \tag{6.22}$$

where λ denotes one of the two possible transverse photon polarisations, $\epsilon_\lambda(p)$ is the polarisation vector which satisfies

$$p_\mu \epsilon_\lambda^\mu(p) = 0 \tag{6.23}$$

we get for the free fermion field (the same relation between the energy p_0 and the three momentum $\vec{p}$ is valid as in the previous case of a free scalar field), which satisfies the Dirac equation

$$\hat{\psi}(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(b_s(p)u_s(p)e^{-ipx} + d_s^\dagger(p)v_s(p)e^{ipx}\right) \tag{6.24}$$

where s denotes one of the two possible spin choices for the fermion, $b_s(p)$ and $d_s^\dagger(p)$ are the annihilation operator for the fermion (electron) and the creation operator for the anti-fermion (positron). Similarly

$$\bar{\hat{\psi}}(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(d_s(p)\bar{v}_s(p)e^{-ipx} + b_s^\dagger(p)\bar{u}_s(p)e^{ipx}\right) \tag{6.25}$$

First let's see the case of an external photon with momentum q and polarisation λ. If it is in the initial state, we multiply the amplitude with

$$\epsilon_\lambda^\mu(q) \tag{6.26}$$

while if we find ti as a final state, we multiply with its complex conjugated value

$$\epsilon_\lambda^{\mu *}(q) \tag{6.27}$$

For a fermion with momentum p and spin s we use the following rules:

incoming particle:

$$u_s(p) \tag{6.28}$$

incoming antiparticle:

$$\bar{v}_s(p) \tag{6.29}$$

outgoing particle:

$$\bar{u}_s(p) \tag{6.30}$$

outgoing antiparticle:

$$v_s(p) \tag{6.31}$$

Each fermion line in the Feynman diagram has an arrow, which points in the case of a particle in the direction of the momentum (in the direction from initial state to the final state), while in the case of antiparticles the arrow points in the opposite direction than the momentum. We follow the rule to go in the Feynman diagram always in the opposite direction of the arrow (which shows the fermion number current): we start first with the outgoing particle, ($\bar{u}_s(p)$) or incoming antiparticle ($\bar{v}_s(p)$), and finish with the incoming particle ($u_s(p)$) or outgoing antiparticle ($v_s(p)$). So, as required by Lorentz invariance, we always have on the left spinors with a bar, and on right spinors without it.

6.2.5 Summary of Feynman rules for QED

Let's shortly summarise the rules:

- fermion propagator

$$\left(\frac{i}{\not{p} - m}\right)_{\xi\eta}$$

- photon propagator (covariant gauge)

$$\frac{i}{p^2}\left(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

- vertex

$$ie(\gamma_\mu)_{\xi\eta}$$

- factor (-1) for each closed fermion loop
- incoming (outgoing) photon:

$$\epsilon^{\mu}_{\lambda}(q) \quad (\epsilon^{\mu *}_{\lambda}(q))$$

- incoming (outgoing) electron:

$$u_s(p) \quad (\bar{u}_s(p))$$

- incoming (outgoing) positron:

$$\bar{v}_s(p) \quad (v_s(p))$$

6.3 Exercise 8

- Compute the amplitude for the process $e^+e^- \to \mu^+\mu^-$ in QED at tree order.
- Write down all possible amplitudes for polarised fermions. In the calculation take the approximation $m_e = m_\mu = 0$.

6.4 Exercise 9

- Compute the amplitude for the Compton scattering $e\gamma \to e\gamma$.
- Check that the amplitude vanishes if we replace $\epsilon_\mu(k) \to k_\mu$.
- Calculate the scattering cross section, average over the initial spins and polarisations, and sum up over the final ones. Use the above behaviour for $\epsilon_\mu(k) \to k_\mu$ to prove that the exchange

$$\sum_{\lambda} \epsilon^{\mu}_{\lambda} \epsilon^{\nu *}_{\lambda} \to -g^{\mu\nu}$$

  gives the correct result.
- Check that in the limit $\omega \to 0$ (initial photon energy) one gets the Thompson cross section ($\alpha \equiv e^2/(4\pi)$)

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

6.5 Quantum electrodynamics up to one loop

In this section we will use the method of counter-terms, all quantities will be renormalised, but for simplicity we will write them without the sign R.

The tree level renormalised Lagrangian for QED is then (we choose the Feynman gauge $\xi = 1$)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - m\bar{\psi}\psi + e\mu^{\epsilon/2}\bar{\psi} A\!\!\!/ \psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial A)^2 \tag{6.32}$$

while the counter-terms are of the form

$$\mathcal{L}_{CT} = \delta Z_2 \bar{\psi} i \partial\!\!\!/ \psi - \delta m \bar{\psi}\psi + \delta Z_1 e\mu^{\epsilon/2}\bar{\psi} A\!\!\!/ \psi - \frac{\delta Z_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{6.33}$$

We consider these new terms as small perturbations. They are formally of higher order in powers of the coupling constant (δZ_i and δm are order e), so we consider them as interaction. We write for them the following Feynman rules

- counter-term for the fermion propagator

$$i\left(\delta Z_2 p\!\!\!/ - \delta m\right) \tag{6.34}$$

- counter-term for the photon propagator

$$-i\delta Z_3\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \tag{6.35}$$

- vertex counter-term

$$i\delta Z_1 e\mu^{\epsilon/2}\gamma_\mu \tag{6.36}$$

We check the first one by considering the whole quadratic part of the fermion

$$(1+\delta Z_2)\bar{\psi} i\partial\!\!\!/\psi - (m+\delta m)\bar{\psi}\psi$$

Clearly the propagator is in this case

$$\frac{i}{(1+\delta Z_2)p\!\!\!/ - (m+\delta m)}$$

If we now expand, we get (take into account that $(M_1 M_2)^{-1} = M_2^{-1} M_1^{-1}$)

$$\begin{aligned}\frac{i}{(p\!\!\!/ - m) + (\delta Z_2 p\!\!\!/ - \delta m)} &= \frac{i}{(p\!\!\!/ - m)[1 + (p\!\!\!/ - m)^{-1}(\delta Z_2 p\!\!\!/ - \delta m)]} \\ &= \frac{i}{p\!\!\!/ - m} + \frac{i}{p\!\!\!/ - m}\left[i\left(\delta Z_2 p\!\!\!/ - \delta m\right)\right]\frac{i}{p\!\!\!/ - m} + \ldots\end{aligned}$$

which coincides with the above rule.

It si not difficult to check also the complete quadratic photon term

$$-\frac{1+\delta Z_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial A)^2$$

gives the following propagator

$$\frac{-i}{(1+\delta Z_3)p^2}\left(g_{\mu\nu} + \delta Z_3 \frac{p_\mu p_\nu}{p^2}\right)$$

which can be expanded

$$\frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\alpha}}{p^2}\left[-i\delta Z_3\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right)\right]\frac{-ig_{\beta\nu}}{p^2} + \dots$$

as with the above rules.

The vertex correction is obvious.

6.5.1 Electron propagator

The one loop contribution to it is usually denoted as

$$\frac{i}{\not p - m}\left(-i\Sigma(p)\right)\frac{i}{\not p - m}$$

which gives for the invderse of the 2-point G.f. for the electron

$$i/G^{(2)}_\psi(p) = \not p - m + \delta Z_2 \not p - \delta m - \Sigma(p) \tag{6.37}$$

Now let's calculate:

$$-i\Sigma(p) = \left(ie\mu^{\epsilon/2}\gamma^\alpha\right)\int \frac{d^dk}{(2\pi)^d}\frac{i}{\not k - m}\frac{-ig_{\alpha\beta}}{(k-p)^2}\left(ie\mu^{\epsilon/2}\gamma^\beta\right) \tag{6.38}$$

As common we use the identity

$$\frac{1}{\not k - m} = \frac{\not k + m}{k^2 - m^2} \tag{6.39}$$

The definition for the γ matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{6.40}$$

is valid in arbitrary (also non-integer) dimensions, so ($g^\alpha{}_\alpha = d$)

$$\gamma^\alpha\gamma_\alpha = d \tag{6.41}$$
$$\gamma^\alpha\gamma^\mu\gamma_\alpha = -(d-2)\gamma^\mu \tag{6.42}$$

We first get

$$\Sigma(p) = -ie^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)\not{k} + dm}{(k^2-m^2)(k-p)^2} \tag{6.43}$$

and withe the usual trick

$$\Sigma(p) = -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)\not{k} + dm}{\left[k^2 - m^2(1-x) + (p^2-2kp)x\right]^2} \tag{6.44}$$

We introduce a new variable $k' = k - xp$ (and skip the prime)

$$\begin{aligned}
\Sigma(p) &= -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)x\not{p} + dm}{\left[k^2 - m^2(1-x) + p^2x(1-x)\right]^2} \\
&= \frac{e^2\mu^\epsilon}{(4\pi)^d}\Gamma(2-d/2) \int_0^1 dx \frac{-(d-2)x\not{p} + dm}{\left[m^2(1-x) - p^2x(1-x)\right]^{2-d/2}} \\
&= -\frac{e^2}{(4\pi)^2}(\not{p} - 4m)\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) + \mathcal{O}(1)
\end{aligned} \tag{6.45}$$

Compare with (6.37) and get in the $\overline{MS}$ scheme

$$\delta Z_2 = -\frac{e^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{6.46}$$
$$\delta m = -4m\frac{e^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{6.47}$$

6.5.2 Photon propagator

All contributions at one loop order (with counterterms included) can be written as

$$\frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\alpha}}{p^2}\left[i\Pi^{\alpha\beta}(p) - i\delta Z_3\left(p^2g^{\alpha\beta} - p^\alpha p^\beta\right)\right]\frac{-ig_{\beta\nu}}{p^2} + \ldots \tag{6.48}$$

where (remember the (-1) factor for the fermion loop!)

$$\begin{aligned}
i\Pi^{\alpha\beta}(p) &= (-1)\int \frac{d^dk}{(2\pi)^d} Tr\left(\frac{i}{\not{k}+\not{p}-m} ie\mu^{\epsilon/2}\gamma^\alpha \frac{i}{\not{k}-m} ie\mu^{\epsilon/2}\gamma^\beta\right) \\
&= -e^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{Tr\left[\left(\not{k}+\not{p}+m\right)\gamma^\alpha\left(\not{k}+m\right)\gamma^\beta\right]}{\left[(k+p)^2 - m^2\right]\left[k^2-m^2\right]}
\end{aligned} \tag{6.49}$$

We need to generalise the dimensionality of the γ matrices for d dimensions. We can define completely consistently

$$\begin{aligned} Tr\,(1) &= f(d) && (6.50)\\ Tr\left(\gamma^\alpha\gamma^\beta\right) &= f(d)g^{\alpha\beta} && (6.51)\\ Tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) &= f(d)\left(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\mu}g^{\beta\nu} + g^{\alpha\nu}g^{\beta\mu}\right) && (6.52)\end{aligned}$$

with the only constraint that $f(4) = 4$.

We get

$$\begin{aligned} i\Pi^{\alpha\beta}(p) &= -e^2\mu^\epsilon f(d)\int\frac{d^dk}{(2\pi)^d}\frac{2k^\alpha k^\beta + k^\alpha p^\beta + p^\alpha k^\beta + (m^2 - k(k+p))\,g^{\alpha\beta}}{\left[(k+p)^2 - m^2\right]\left[k^2 - m^2\right]}\\ &= -e^2\mu^\epsilon f(d)\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{2k^\alpha k^\beta + k^\alpha p^\beta + p^\alpha k^\beta + (m^2 - k(k+p))\,g^{\alpha\beta}}{\left[k^2 - m^2 + (p^2 + 2kp)x\right]^2}\end{aligned}$$

We again introduce a new variable $k' = k + xp$ (and again we skip the prime)

$$\begin{aligned} i\Pi^{\alpha\beta}(p) &= -e^2\mu^\epsilon f(d)\int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2 - m^2 + p^2x(1-x)\right]^2} && (6.53)\\ &\times \left[-2x(1-x)p^\alpha p^\beta + \left((2/d-1)k^2 + m^2 + p^2x(1-x)\right)g^{\alpha\beta}\right]\end{aligned}$$

where we used the relation

$$k^\alpha k^\beta \to \frac{k^2}{d}g^{\alpha\beta} \qquad (6.54)$$

which is of course valid under the integration.

Let's now check if the photon gets a mass due to one loop corrections. This would be catastrophic, since it would mean the breaking of gauge invariance. As we will see, nothing of this happens, which confirms our belief that dimensional regularisation explicitly conserves gauge invariance.

We would potentially get a mass term in the $p \to 0$ limit of the above expression. It follows

$$\begin{aligned} \Pi^{\alpha\beta}(0) = -e^2\mu^\epsilon f(d)g^{\alpha\beta} \quad & \int_0^1 dx\int\frac{d^dk}{(2\pi)^d}\frac{m^2 + (2/d-1)k^2}{\left(k^2 - m^2\right)^2}\\ = -e^2\mu^\epsilon f(d)g^{\alpha\beta} \quad & \left[\, m^2\frac{i}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}\left(\frac{1}{m^2}\right)^{2-d/2}\right. && (6.55)\\ & \left. + (2/d-1)\frac{-i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(2-d/2-1)}{\Gamma(2)}\left(\frac{1}{m^2}\right)^{2-d/2-1}\right]\end{aligned}$$

Using (5.6) the result is exactly

$$i\Pi^{\alpha\beta}(0) = 0 \tag{6.56}$$

as we expected (and hoped).

Now let's see also the divergent pieces. In the proof of a massles photon we showed that

$$\int \frac{d^dk}{(2\pi)^d} \frac{m^2 + (2/d-1)k^2}{(k^2-m^2)^2} = 0 \tag{6.57}$$

From here it follows

$$\int \frac{d^dk}{(2\pi)^d} \frac{m^2 + (2/d-1)k^2}{\left[k^2 - m^2 + p^2x(1-x)\right]^2} = \int \frac{d^dk}{(2\pi)^d} \frac{p^2x(1-x)}{\left[k^2 - m^2 + p^2x(1-x)\right]^2} \tag{6.58}$$

and then

$$i\Pi^{\alpha\beta}(p) = -\frac{2e^2\mu^\epsilon f(d) i}{(4\pi)^{2-\epsilon/2}} \left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \frac{x(1-x)}{\left[m^2 - p^2x(1-x)\right]^{\epsilon/2}} \tag{6.59}$$

The limit $\epsilon \to 0$ gives

$$i\Pi^{\alpha\beta}(p) = -i\frac{4}{3}\frac{e^2}{(4\pi)^2} \left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) + \mathcal{O}(1) \tag{6.60}$$

We require that δZ_3 in (6.48) exactly cancels the divergent pieces:

$$\delta Z_3 = -\frac{4}{3}\frac{e^2}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{6.61}$$

6.6 Exercise 10

- In the $\overline{MS}$ scheme show that in QED $Z_1 = Z_2$ at the one loop level, which follows from the requirement of gauge invariance

$$D_\alpha = \partial_\alpha - ieA_\alpha = \partial_\alpha - ie_R A_{R\alpha}$$

6.7 Exercise 11

- Calculate the anomalous magnetic moment of the electron in QED at one loop. Compare the result with the latest measurements and theoretical predictions in PDG (to be found in [6]).

7 Supersymmetry

7.1 Introduction

In field theory there are Lorentz (Poincare) symmetries and internal (gauge or global) symmetries. Any other possibility? Sohnius et al. showed that it is possible the so called supersymmetry, a symmetry that connects bosons with fermions. For this to happen the susy generator must have fermionic (Grassman) character. It predicts (super)multiplets in which different elements have different spins. All these elements should have the same mass. In the early days it was hoped that one could use known particles to form these multiplets (neutrino as the fermionic partner of the photon), but it was soon realised (Fayet) that this is impossible, and that one needs to really double all the known particles (a new boson for each known fermion and a new fermion for each known boson). Not only, two Higgses are needed in the supersymmetric version of the standard model, not just one.

Supersymmetric theories thus contain the same number of complex bosons and chiral fermions. This means that for each particle we need a partner with all the internal quantum number the same, but with a different spin:

$$\begin{aligned} FERMIONS\ (\psi_i) &\leftrightarrow SFERMIONS\ (\phi_i)\,, \\ GAUGE\ BOSONS\ (W^a) &\leftrightarrow GAUGINOS\ (\lambda^a)\,, \\ HIGGS\ (H) &\leftrightarrow HIGGSINO\ (\tilde{H})\,. \end{aligned}$$

The particles on the left are our standard model particles, the ones on the right are their superpartners.

7.2 Some basics on supersymmetry

Let us shortly summarise some basic points in supersymmetry.

There are chiral multiplets, which members are a spin 1/2 Weyl fermion ψ_i (2 degrees of freedom on-shell) and a complex spin 0 boson ϕ_i (2 d.o.f.o.s.), both with the same quantum numbers.

The kinetic terms for the **chiral multiplets** are the same as in ordinary, non supersymmetric case, what changes is the potential, as well as the mass and Yukawa terms of the fermions. They are all described by a single holomorphic function, called superpotential, $W(\phi)$, such that the (F-term) potential is

$$V_F = \sum_i \left| \frac{\partial W}{\partial \phi_i} \right|^2 \,, \tag{7.1}$$

while the terms with fermion bilinears are given by

$$-\frac{1}{2}\frac{\partial^2 W}{\partial \phi_i \partial \phi_j}\psi_i \psi_j + h.c. \tag{7.2}$$

If the superpotential is invariant under some internal symmetries, so is the potential. If one considers renormalisable theories, then the superpotential is a polynomial up to the third power in fields.

There are also **vector** massless **multiplets**, which members are a real vector field A_μ (2 d.o.f.o.s.) and a Weyl fermion λ (2 d.o.f.o.s.). Such multiplets take care of the gauge interactions.

On top of changing all derivatives into covariant ones as usually, the introduction of gauge interactions adds two terms in the Lagrangian: the (D-term) potential

$$V_D = \frac{g^2}{2} \sum_a \left(\sum_{i,j} \phi_i^* T_{ij}^a \phi_j \right)^2 , \tag{7.3}$$

where T^a is the a^{th} generator of the gauge group in the representation of the ϕ chiral multiplets.

The other additions are new Yukawa interactions:

$$-g\sqrt{2}\phi_i^* T_{ij}^a \psi_j \lambda^a + h.c. , \tag{7.4}$$

7.3 The standard model supersymmetrised

Now we are well equipped to try to write down the superymmetric version of the standard model Lagrangian. From above it is clear, that the only unknown part is the superpotential W. Instead of the notation ψ_i let's use the usual Q for the quark weak doublet, u^c for the (charge conjugated) up quark weak singlet ($(u^c)_L \equiv C\bar{u}_R^T$), d^c for the down singlet, L for the leptonic weak doublet and e^c for the leptonic singlet, while the same notation with a tilde will denote their scalar (complex) partners.

The known Yukawas from the standard model make some terms a must:

$$W_Y = Q_i H_u \left(Y_u\right)_{ij} u_j^c + Q_i H_d \left(Y_d\right)_{ij} d_j^c + L_i H_d \left(Y_e\right)_{ij} e_j^c \tag{7.5}$$

where i, j run over generations. Notice that the standard model Higgs would be H_u (or $\tau_2 H_d^*$). This makes it clear, why we need two Higgses, since the superpotential is a holomorphic function of the fields.

The gauge quantum numbers of the fields are as in the standard model, with the hypercharge of the two Higgses determined by the invariance of the superpotential. This is the second reason for the necessity to double the Higgs sector: the chiral anomaly cancellations.

The higgsinos would be massless here, so one adds the term

$$W_\mu = \mu H_u H_d . \tag{7.6}$$

In the SM gauge interactions automatically conserve baryon and lepton numbers at the renormalisable level. Here this is not true anymore, and one could write down in principle

$$\Delta W = \epsilon_i L_i H_u + \lambda_{ijk} L_i L_j e^c_k + \lambda'_{ijk} L_i Q_j d^c_k + \lambda''_{ijk} u^c_i d^c_j d^c_k \tag{7.7}$$

It is easy to see that such terms would be catastrophic for the stability of the proton. Since proton longevity is extremely well measured, the above terms are bound to be very small. It is thus quite usual to assume that they are zero. In this limit, there is a new Z_2 discrete symmetry of the potential, under which the known SM particles and both Higgses are even, while the new partners are odd. We will assume it from now on, although from the SM point of view there is absolutely no reason for it. In some SO(10) grand unified theories this low-energy extra symmetry can be derived from reasonable assumptions, and are thus motivated.

We will call the Lagrangian described by the usual (gauge invariant) kinetic terms plus terms (7.1)-(7.6) the supersymmetric limit of the standard model. As we will see next time, this model is unrealistic. Before noticing it, let us first mention its virtues.

7.4 Hierarchy problem

The first one is the stabilisation of the gauge hierarchy problem. This problem is connected with the fact, that usually in field theory scalar masses are not protected by quantum effects. Even if the tree order scalar mass is small, it can get arbitrary high corrections by loops with heavy fields running inside. This is not true for spin 1 and spin 1/2 fields, since their corrections are protected by gauge symmetry and chiral symmetry. Supersymmetry solves this problem by linking a scalar (spin 0) mass to its superpartner fermion mass.

Although these thoughts sound rather philosophical, there is actually also a prediction coming from them. In fact, once you break supersymmetry and give masses to the sfermions (we will see in the next lecture that this is unavoidable), supersymmetry still preserves some correlation among different masses. It can be shown, that the stop and Higgs mass are related, a fact that we will hopefully be able to check at LHC or at some next collider. In the limit of a large stop mass this reduces to the approximate relation between the lightest neutral Higgs mass m_{h^0} and the stop mass $m_{\tilde{t}}$ (see for example [19] and references therein)

$$m^2_{h^0} \approx m^2_Z \cos^2 2\beta + 6 \left(\frac{\alpha_2}{4\pi}\right) \frac{m^4_t}{m^2_W} \log\left(\frac{m^2_{\tilde{t}}}{m^2_t}\right) , \tag{7.8}$$

where $\tan\beta \equiv \langle H_u \rangle / \langle H_d \rangle$.

7.5 Radiative electroweak symmetry breaking

In the SM the potential is

$$V = m^2_H |H|^2 + \lambda |H|^4 \ . \tag{7.9}$$

To have the electroweak symmetry breaking, one needs $m^2_H < 0$. There is no explanation why this is so. Also, it is known that this mass cannot change sign with running in the SM.

In supersymmetric extensions of the SM there are also other bosons with masses $m_{\tilde{t}}^2$, $m_{\tilde{t}^c}^2$, etc. Clearly they must be positive (only H_u and H_d need negative square masses), otherwise we would for example break too much gauge symmetry, as for example the colour. So here one would like to understand why is the Higgs special with respect to the other bosons, for example the stop. In other words, one would like to see why the Higgs mass square is negative, while the masses square for the other bosons are positive. It is a remarkable fact that supersymmetry combined with gauge symmetry can give a possible explanation of this behaviour.

The differences among different fields is coming from the different renormalisation group equations. Consider the system of the Higgs and the two stop masses (similar equations can be written for H_d, $\tilde{b}$ and $\tilde{b}^c$, changing $y_t \to y_b$, which is relevant if $\tan\beta$ is large) :

$$\mu\frac{dm_{H_u}^2}{d\mu} = 3\frac{y_t^2}{16\pi^2}\left(m_{H_u}^2 + m_{\tilde{t}^c}^2 + m_{\tilde{t}}^2\right) + ... \,, \tag{7.10}$$

$$\mu\frac{dm_{\tilde{t}^c}^2}{d\mu} = 2\frac{y_t^2}{16\pi^2}\left(m_{H_u}^2 + m_{\tilde{t}^c}^2 + m_{\tilde{t}}^2\right) + ... \,, \tag{7.11}$$

$$\mu\frac{dm_{\tilde{t}}^2}{d\mu} = 1\frac{y_t^2}{16\pi^2}\left(m_{H_u}^2 + m_{\tilde{t}^c}^2 + m_{\tilde{t}}^2\right) + ... \,, \tag{7.12}$$

where the dots denote other contributions not necessary for the understanding of the phenomenon. We see that the righthandsides are the same except for the numerical factor in front of them. These numerical factors are just consequences of gauge symmetry, i.e. of the representation for the Higgs and stops under the gauge groups, which is not difficult to read from the relevant 1-loop Feynman diagrams (different number of fields running in the loop).

So we see that all the three masses increase with the scale (the righthandsides are positive) and that the Higgs mass increase faster than the other two. So one can think that the three masses are for example equal at a large scale, for example at the Planck scale, $\mu = M_{Planck}$. Then one evolves the three masses down using the RGE (7.10)-(7.12). Since $m_{H_u}^2$ decreases faster than the other two masses squared with decreasing scale μ, one could finish with a negative $m_{H_u}^2 < 0$ at small scales, but positive $m_{\tilde{t}^c,\tilde{t}}^2 > 0$. This is indeed what happens when one chooses realistic values for the parameters, giving a very appealing explanation for the electroweak symmetry breaking in the MSSM.

One can solve the system (7.10)-(7.12) with the result

$$m_{H_u}^2(m_W) = m_0^2\left[\frac{3}{2}\left(\frac{m_W}{\Lambda}\right)^{3/(8\pi^2)} - \frac{1}{2}\right] \,, \tag{7.13}$$

$$m_{\tilde{t}^c}^2(m_W) = m_0^2\left[\left(\frac{m_W}{\Lambda}\right)^{3/(8\pi^2)}\right] \,, \tag{7.14}$$

$$m_{\tilde{t}}^2(m_W) = m_0^2\left[\frac{1}{2}\left(\frac{m_W}{\Lambda}\right)^{3/(8\pi^2)} + \frac{1}{2}\right] \,, \tag{7.15}$$

where we assumed a universal boundary condition at the large scale Λ:

$$m_{H_u}^2(\Lambda) = m_{\tilde{t}^c}^2(\Lambda) = m_{\tilde{t}}^2(\Lambda) = m_0^2 \; . \tag{7.16}$$

A reasonable value for Λ is for example the Planck scale M_{Pl}, which gives the mass term for $m_{H_u}^2$ negative, while the other two turn out to be safely positive.

7.6 Unification of couplings

The 1-loop RGE for the gauge couplings g_i are

$$\mu \frac{dg_i}{d\mu} = \frac{1}{16\pi^2} b_i g_i^3 \; , \tag{7.17}$$

where $i = 1, 2, 3$ denotes the gauge group and b is the so called β function coefficient. It can be straightforwardly calculated in any theory via (G,F,B stay for gauge bosons, fermions, bosons)

$$b = -\frac{11}{3} C_2(G) + \frac{2}{3} T_F + \frac{1}{3} T_B \; . \tag{7.18}$$

The Dynkin index

$$T(R)\delta_{ab} = Tr\left(T_a(R) T_b(R)\right) \tag{7.19}$$

and the second Casimir

$$C_2(R)\delta^{ij} = \sum_a \left(T_a(R) T_a(R)\right)^{ij} \tag{7.20}$$

depend on the choice of the gauge group and on the representation involved. The indices a, b run over the generators of the group ($N^2 - 1$ in SU(N)), while i, j run from 1 to the dimension of the representation. The normalisation usually chosen is $T = 1/2$ for the fundamental representation (quarks, leptons). Then one has in the SU(N) group for the fundamental representation $C_2 = (N^2 - 1)/(2N)$, and for the adjoint $T = C_2 = N$. The dimension of the representation is N for fundamentals and $N^2 - 1$ for adjoint. To remember also that in SU(2) the generators in the fundamental are the Pauli matrices $T_a^{ij} = \tau_a^{ij}/2$, while in the adjoint representations are the Levi-Civita antisymmetric tensor $T_a^{ij} = -i\epsilon_{aij}$.

For supersymmetric theories we know that for each fermion (boson) there is a boson (fermion) in the same group representation, so (7.18) can be written more compactly as

$$b = -3C_2(G) + T \; . \tag{7.21}$$

A simple exercise shows that

$$\text{in MSSM} \;\; : \;\; b_i = (33/5, 1, -3) \tag{7.22}$$

$$\text{in SM} \;\; : \;\; b_i = (41/10, -19/6, -7) \tag{7.23}$$

Negative coefficients here mean asymptotic freedom. One knows the experimental values of g_i at M_Z and can evolve them towards larger scales μ using (7.17). It is then surprising that in the supersymmetric case the three couplings unify in a single point at $\mu \approx 10^{16}$ GeV. To appreciate this fact one should notice that this unification fails badly in the non-supersymmetric case. So, if we have supersymmetric partners at M_Z or close to 1 TeV as required by naturalness (hierarchy problem), then we have unification of gauge couplings for free!

7.7 Candidate for dark matter

Once R-parity is preserved, one has at his disposal a stable massive particle (the lightest supersymmetric partner - LSP), i.e. the lightest particle with negative R-parity. If this particle is electrically and colour neutral, but feels weak interactions (the so-called neutralino), like for example the Higgsino, bino or neutral wino, then its mass of the order few TeV is automatically of the right order of magnitude to be a possible dark matter candidate. This is in contrast with the standard model, where no such candidate exists.

8 Supersymmetry breaking

8.1 Soft terms

So far we described only the good qualities of the supersymmetric version of the standard model. Obviously there are many drawbacks, the main is that the theory described so far is completely unrealistic. In fact, it predicts the doubling of all the particles we know, i.e. for any known fermion there is a new scalar with the same quantum numbers. This is clearly impossible, there is for example no scalar particle with the quantum numbers and the mass of the electron. In other words, supersymmetry must be broken. The most pragmatic way to obtain it is to add by hand a new set of terms, which break supersymmetry softly. In this way one maintain some of the good features encountered before, but still make the theory possible. In general this amounts to adding the following dimension two (scalars square) and three (scalars cube or fermions square) terms to the Lagrangian:

$$\mathcal{L}_{soft} = m_i^2 \phi_i^* \phi_i + \left(B_{ij} \phi_i \phi_j + A_{ijk} \phi_i \phi_j \phi_k + m_A \lambda^A \lambda^A + h.c. \right) \ , \tag{8.1}$$

where i, j, k run over all different representations and A over all different gauge groups. The new parameters are undetermined, but can be sometimes restricted by experimental data. For example, from the fact that we have never found the sfermions, we conclude that their masses must be larger than around few TeV (however these limits typically assume something). Other restrictions come from the rare flavour changing neutral currents: in the standard model these processes are suppressed by the one-loop factor ($\approx 1/(4\pi)^2$), the GIM mechanism ($\approx m_i/m_W$) and by the small off-diagonal elements of the CKM mixing matrix. The new terms in (8.1) bring new contributions to the FCNC (in $K - \bar{K}$,

$B-\bar{B}$ mixing, $b \to s\gamma$, etc). These are automatically compatible with the experimentally measured or bounded values only in some cases, for example among others:

- if the soft masses in (8.1) are much higher than naively expected form naturalness, i.e. on the order of 10^{3-4} TeV or more;

- if the coefficients A in (8.1) are very small, generation independent or proportional to the relevant Yukawa matrices in the superpotential.

As you can see, the bad side of such a model (which we call the minimal supersymmetric standard model - MSSM) is a big proliferation of unknown parameters, something like 100 or so. The desire to simplify things adding a new symmetry (supersymmetry) clashed with the facts of nature, forcing us to add the new terms (8.1). Theoretically it would be thus much better to have a model of calculating or deriving the unknown coefficients. This will be studied in detail in the remaining part of this lecture. We will present two different scenarios.

8.2 Spontaneous susy breaking is not easy

The best option would be to break supersymmetry spontaneously, as we do for example in gauge theories. We will see that this is not easy to do in two examples.

In supersymmetric theories the hamiltonian is proportional to the sum of the squares of the generators of supersymmetry algebra. So the energy is zero iff susy is preserved, and positive if susy is spontaneously broken. We saw in the previous lecture that the potential (energy) is made from two terms, the F-term (7.1) and the D-term (7.3). We will consider here only the so called F-term breaking:

$$\frac{\partial W}{\partial \phi_i} \neq 0 \quad \text{for at least some } i \ . \tag{8.2}$$

Notice that such a requirement automatically implies a massless fermion. In fact, from the minimisation of (7.1), the second derivative of the superpotential in in the field space direction defined by (8.2) must vanish. This is expected from the Goldstone theorem. In the case of spontaneously broken internal symmetry a massless Goldstone boson appears. Here the spontaneously broken supersymmetry generator is a Grassmanian object, thus a fermionic zero mode follows. This object is called the goldstino.

Although W is a gauge singlet, in the SM ϕ_i are not. Thus the only field that could break supersymmetry without breaking some unwanted extra gauge symmetry is one of the two Higgses H_u or H_d. Imagine we do it with H_d. Then suppose that we are able to properly change the superpotential so that

$$F_{H_d} \equiv \left\langle \frac{\partial W}{\partial H_d} \right\rangle \neq 0 \ . \tag{8.3}$$

To see why this cannot work, let us concentrate on the mass of the selectron:

$$W = y_e e h_d^0 e^c + ... \tag{8.4}$$

From (7.1) and (8.3) the mass terms for the selectron in the potential are

$$V = \begin{pmatrix} \tilde{e}^* & \tilde{e}^c \end{pmatrix} \begin{pmatrix} y_e^2 v_d^2 & y_e F^*_{H_d} \\ y_e F_{H_d} & y_e^2 v_d^2 \end{pmatrix} \begin{pmatrix} \tilde{e} \\ \tilde{e}^{c*} \end{pmatrix} , \tag{8.5}$$

where v_d is the vev of H_d. The eigenvalues satisfy the mass sum rule

$$m^2_{\tilde{e}} + m^2_{\tilde{e}^c} - 2m^2_e = 0 . \tag{8.6}$$

This is in contradiction with what we know from low energy physics: there is no scalar with quantum numbers of the electron and a smaller (or equal) mass.

It is possible to show that any spontaneously broken global supersymmetric model has similar unacceptable mass sum rules at tree order. There are two possible ways out: either one goes local, i.e. to supergravity, or one transmits the information of susy breaking not at tree order, but at one loop. In both cases the mass sum rules change and unwanted constraints get relaxed.

Either way, the mechanism should roughly look as follows: a sector with no interaction with the SM fields (and thus called the hidden sector) is responsible for the spontaneous breaking of supersymmetry. The information that susy is broken in this hidden sector gets thus transmitted to our SM sector either by $1/M_{Pl}$ suppressed higher dimensional terms (as in supergravity) or through an intermediate (messenger) field, that couples to both SM and hidden sector fields. In this scenario loops with external SM fields and internal messenger fields (with susy breaking couplings and/or masses) transmit the information on susy breaking to our sector.

Let us see now in more details the above mechanisms of susy breaking mediation.

8.3 Gravity mediation

In the case of local supersymmetry (supergravity), one needs to introduce the **gravity multiplet**, which essentially means the spin 2 graviton (2 d.o.f.o.s.) plus the spin 3/2 gravitino (also 2 d.o.f.o.s.). Analogous to the case of spontaneously breaking of a local symmetry, where the would-be goldstone boson gets eaten by the longitudinal component of the vector boson, here the gravitino eats the goldstino, acquiring a nonzero mass, $m_{3/2}$. This mass turns out to be the typical scale, and soft masses will be proportional to it.

The potential in the supergravity case becomes (φ_i denote both SM fields ϕ_i and hidden sector fields X)

$$V_F = e^{K/M_*^2} \left[\left(\frac{\partial W}{\partial \varphi^i} + \frac{\partial K}{\partial \varphi^i} \frac{W}{M_*^2} \right) \left(K^{-1}\right)^i_j \left(\frac{\partial W^*}{\partial \varphi^*_j} + \frac{\partial K}{\partial \varphi^*_j} \frac{W^*}{M_*^2} \right) - 3 \frac{|W|^2}{M_*^2} \right] , \tag{8.7}$$

where $M_* = M_{Pl}/\sqrt{8\pi} \approx 2 \times 10^{18}$ GeV is the so called reduced Planck mass and $\left(K^{-1}\right)^i_j$ is the inverse matrix of $\partial^2 K / \partial \varphi^i \partial \varphi^*_j$. The Kähler function $K(\varphi, \varphi^*)$ is a real function of the fields, which we call canonical, iff $K_{can} = \sum_i \varphi^*_i \varphi_i$, but in a general nonrenormalizable model can be actually modified with higher dimensional polinomials.

Assuming that it is a singlet field X that breaks susy, the order parameter in supergravity is defined as

$$F_X = \frac{\partial W}{\partial X} + \frac{\partial K}{\partial X}\frac{W}{M_*^2} \ . \tag{8.8}$$

It must be nonzero, so to break supersymmetry. In flat spacetime one can parametrize this breaking by the gravitino mass:

$$F_X = \sqrt{3} m_{3/2} M_* \ . \tag{8.9}$$

As we said before, this field X should have small enough couplings with the SM fields ϕ_i. This is most easily obtained, assuming

$$W(X, \phi) = W(X) + W_{SM}(\phi) \ . \tag{8.10}$$

Then all the couplings between the two sectors are through $1/M_{Pl}$ suppressed operators.

One then needs the following requirements at $X = \langle X \rangle$:

$$W = m_{3/2} M_*^2 \qquad \text{(zero cosmological constant)} \ , \tag{8.11}$$

$$\frac{\partial V}{\partial X} = 0 \qquad \text{(extremum of the potential)} \ , \tag{8.12}$$

$$\left| \frac{\partial^2 V}{\partial X^2} \right| \leq \frac{\partial^2 V}{\partial X \partial X^*} \qquad \text{(no tachyons)} \ . \tag{8.13}$$

As an example take the so-called Polonyi superpotential:

$$W(X) = aX + b \tag{8.14}$$

and a canonical Kähler. With properly chosen constants a and b, so to satisfy (8.8), (8.11)-(8.13), it is possible to determine all the soft parameters in (8.1) in terms of few parameters at the Planck scale, which are at the moment still compatible with any experimental constraint.

We were thus able to reduce the 100 and so parameters to very few of them. It has to be kept in mind however, that this is no more than just a reparamentrisation of the original soft terms. In fact, there is absolutely no real reason to believe that the superpotential should look like (8.14) and even less that the Kaehler potential is canonical. In fact, taking for example

$$K = X^* X + \phi_i^* \phi_i + c_{ij} \frac{X^* X \phi_i^* \phi_j}{M_{Pl}^2}, \tag{8.15}$$

gives the sfermion soft mass squares

$$\left(m_{\tilde{f}}^2 \right)_{ij} = (\delta_{ij} + c_{ij}) \frac{|F_X|^2}{M_{Pl}^2} \ , \tag{8.16}$$

which not only introduces new couplings (c's), but may very easily be in contradiction with experiment due to possible large contribution to the flavour changing neutral currents. Although there are some ways of making these c's small in the infrared via running, this requires extra physics, and is certainly not a minimalist's approach. It is safe to conclude that supergravity cannot explain the structure, less the particular values of the soft parameters.

8.4 Gauge mediation

The problem with the previous example was, that gravity is not actually flavour blind (the masses of fermions of different generations are very different!), which can be parametrised by a nontrivial matrix c above. This led people to use as mediators of susy breaking gauge interactions, which are known to be flavour blind

As before, we have to forbid that supersymmetry breaking is transmitted to the SM particles at tree order, so no SM field can satisfy (8.2). Thus imagine that we have a gauge singlet field X with a non zero F-term

$$F_X = \left\langle \frac{\partial W}{\partial X} \right\rangle \neq 0 \, . \tag{8.17}$$

Can it be coupled to any of the standard model fields? Allowing only renormalisable couplings the only possibility is to transform the dimension 2 term (7.6) to a dimension 3 term [20]

$$W_\mu = \lambda_X X H_u H_d \, . \tag{8.18}$$

The higgsino mass appears from the vev of X,

$$\mu = \lambda_X \langle X \rangle \ . \tag{8.19}$$

The sum rule (8.6) applied to the Higgs supermultiplets is not dangerous in this case, since no mass of this multiplet has been measured yet. On the other side, the sfermions do not couple directly to X, so they are not influenced by it at tree order and no mass sum rule thus applies. Unfortunately the one loop correction to the masses of the sfermions gets negative and proportional to the relevant Yukawas [21]. This would destabilise the stop, breaking SU(3).

In other words, one cannot couple only the Higgs to the X field, but must use another pair of multiplets, call them Φ and $\bar{\Phi}$:

$$W = W(X) + \lambda_X X \bar{\Phi} \Phi + W_{SM} \, . \tag{8.20}$$

These new multiplets are not gauge singlets, so they can interchange gauge bosons (and gauginos) with the SM fields. Thus at leading order in a small F_X/M_Φ^2 ratio, where $M_\Phi = \lambda_X \langle X \rangle$ is the susy preserving Φ mass, the gaugino mass gets a 1-loop contribution

$$m_\lambda = c_i \frac{\alpha_i}{4\pi} \frac{\lambda_X F_X}{M_\Phi} \, , \tag{8.21}$$

while the sfermions get a two-loop contribution

$$m_{\tilde{f}}^2 = d_i \left(\frac{\alpha_i}{4\pi}\right)^2 \frac{\left|\lambda_X F_X\right|^2}{M_\Phi^2} \,, \tag{8.22}$$

with c_i, d_i $(i = 1, 2, 3)$ depending on the representation under the SM gauge group SU(3)×SU(2)×U(1). Similar relations are possible also for the other soft parameters. Although this seems to solve the flavour problem mentioned above, one must be careful. In fact, there is no guarantee that the Φ and $\bar{\Phi}$ do not couple or mix with the SM fields. Typically these mediators have the quantum numbers of the sfermions. This mixing can in principle spoil the flavour blindness of the mediation [21].

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