QUANTUM FIELD THEORY

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Abstract

These notes are intended for a one semester course (30 hours for theory and 15 hours for exercises) at the graduate level.

Contents

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1 Introduction, relativity (45min)

These notes will get changed and improved continuously. It is a short review of what was lectured, with the exception of the sections marked with asterisk. These are written for a reader who would like to know more about quantisation via the path integral. For more detailed descriptions and derivations one should take for example the book of Ryder [1], which, skipping few sections, is of the right length and deepness for a one semester course. For those particularly interested I would recommend the book of Peskin and Schroeder [2], or for example the notes of Siegel [3], which are for free on the net (they are very long though), or Weinberg [4, 5]. Of course, for arbitrary information on particle physics it is recommended to use INSPIRE [6] and for daily news (new papers) [7].

During the lectures we will almost always keep the convention $c = 1$ and $\hbar = 1$. This is nothing else than the choice of particular units. In this convention mass and energy have for example the same unit, which is usually chosen to be GeV (gigaelectronvolt=10^9 eV), while time and distance have the unit GeV^{-1}. Changing these units into the usual ones is very simple: we multiply the quantity written in the $c = 1$ and $\hbar = 1$ units by proper powers of $\hbar$ and c.

1.1 Motivation and goals

During the undergraduate study of physics one gets used to both quantum mechanics and relativity (here and in the following we think always about special relativity), although separately. In particle physics we are fast coming close to speed of light. Quantum mechanics without relativity is thus very fast incomplete. The purpose of this course is to learn how to compute measurable physical quantities as the cross section and decay width in a relativistic invariant way.

1.2 Lorentz transformations

The essence of relativity are the Lorentz transformations. We will derive them as rotations in 4-dimensional spacetime (the time x^0 and spatial coordinates x^i, in short x^α). Indices with latin letters will denote spatial coordinates and will run from 1 to 3, while indices with greek letters will run from 0 to 3.

Let's remind first how we describe rotations in a plane (2-dimensional space). They are simply

$$\begin{pmatrix} x' \\ y\prime \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{1.1}$$

The rotation matrix can be written (check it!) in an apparently strange way as

$$O = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \exp\left(i\alpha T\right) \tag{1.2}$$

where we call the matrix

$$T = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{1.3}$$

the generator of rotations in 2-dimensional space. The matrices (1.2) are representations of the Lie group SO(2). This is a Lie group because the parameter α is continuous, while SO(d) in general means orthogonal ($OO^T = O^T O = I$) rotations in d-dimensional space with unit determinant ($detO = 1$).

This form is very useful for generalisations. In a d-dimensional space we rotate a vector $x = (x^1, ..., x^d)^T$ with a group element O ($d \times d$ matrix). All these elements make up the group SO(d), while each element is described by the $d(d-1)/2$ angles $\alpha^{ab} = -\alpha^{ba}$ via

$$O = \exp\left(\frac{i}{2}\alpha^{ab} T_{ab}\right) \tag{1.4}$$

and the same number of generators of rotation (the Kronecker δ is 1 when two indices are equal and 0 otherwise)

$$(T_{ab})^{kl} = -i\left(\delta_a^k\delta_b^l - \delta_b^k\delta_a^l\right) \tag{1.5}$$

Keep in mind that the indices a and b determine the generator (we could have determined them differently, for example with a single number from 1 to $d(d-1)/2$, $T_{12} \to T_1$, $T_{13} \to T_2$, etc.), while the indices k and l tell us, which element of the matrix we are talking about.

In the case of SO(2) we have only one element, T_{12}, which has been written so far without any index for simpilicity.

The representation of generators via (1.5) is only one of the infinite many others. These can be represented as $n \times n$ matrices, the representation (1.5) is the simplest one[2] dimensional (d) and are called the fundamental representations of the group SO(d). In general all SO(d) group generators (in any representation) satisfy the algebra defined by the commutator

$$[T_{ab}, T_{cd}] = i\left(\delta_{ac}T_{bd} + \delta_{bd}T_{ac} - \delta_{bc}T_{ad} - \delta_{ad}T_{bc}\right) \tag{1.6}$$

Keeping in mind that $T_{ab} = -T_{ba}$ and $\delta_{ab} = +\delta_{ba}$, we can check the consistency of the above form by change 1) $a \leftrightarrow b$, 2) $c \leftrightarrow d$, 3) $a \leftrightarrow c$ and $b \leftrightarrow d$ together. It is not difficult to check that the fundamental representation (1.5) really satisfy the definition (1.6).

If we allowed the orthogonal matrices to have the determinant also -1, we would get the group O(d) instead of SO(d). In three dimensions this means for example that we have also mirroring on arbitrary plane ($x^i \to -x^i$ for a single i) or inversion ($x^i \to -x^i$ for all i).

Let's go back to the Lorentz transformations. The spacetime is 4-dimensional but of special type, since time is not space. This can be seen for example in the invariants. Remember that the length in the four-dimensional flat spacetime is $c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$, and not the sum of squares as in the usual Euclidean space. This means that we get the product of two vectors in Minkowski space if we insert between them a matrix - the metric tensor, which takes care of these extra minuses. If we have for example two vectors $a^\mu = (a^0, a^i)$ and $b^\mu = (b^0, b^i)$, then the product is

$$a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 = a^\mu g_{\mu\nu} b^\nu \tag{1.7}$$

where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.8}$$

[2]usually also the smallest except for $d < 4$ for the the spinorial representations, see next section.

We will always stick to the convention that two equal Lorentz indices implicitly mean summing over them (i.e. without writing explicitly the sum), one of them must be written up, one down (never both up or both down). For this reason it is useful to lower or raise the index by the metric tensor. The above equation can thus be written in different equivalent ways

$$a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\mu g^{\mu\nu} b_\nu = a_\mu b^\mu \tag{1.9}$$

where we defined

$$a_\mu \equiv g_{\mu\nu} a^\nu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) \tag{1.10}$$

and the inverse of the metric tensor, i.e.

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1.11}$$

which satisfies

$$\left(g g^{-1}\right)_\mu{}^\nu = g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu{}^\nu \tag{1.12}$$

$$\left(g^{-1} g\right)^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu \tag{1.13}$$

From here we see that $g_\mu{}^\nu = \delta_\mu{}^\nu$.

The previous definitions for the group SO(d) can be now generalised to SO(d_+,d_-), where the metric tensor has in the diagonal d_+ value plus one and d_- values minus one (in our case we are interested in SO(1,3)). All the above definitions are still ok, providing we systematically change all δ with g and understand all matrix products mediated by g in between. So the definition of the commutator in the SO(1,3) algebra becomes

$$[T_{\alpha\beta}, T_{\mu\nu}] = i \left(g_{\alpha\mu} T_{\beta\nu} + g_{\beta\nu} T_{\alpha\mu} - g_{\beta\mu} T_{\alpha\nu} - g_{\alpha\nu} T_{\beta\mu}\right) \tag{1.14}$$

while the definition for the group element (1.4), which is a matrix in the 4-dimensional Minkowski space, must be understood in the series expansion

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \frac{\theta}{2}^{\alpha_1\beta_1} (iT_{\alpha_1\beta_1})^\mu{}_\nu + \frac{1}{2}\frac{\theta}{2}^{\alpha_1\beta_1} \frac{\theta}{2}^{\alpha_2\beta_2} (iT_{\alpha_1\beta_1})^\mu{}_\lambda (iT_{\alpha_2\beta_2})^\lambda{}_\nu + ... \tag{1.15}$$

Single matrices can be found from the definition (1.5)

$$(iT_{\alpha\beta})^\mu{}_\nu = (iT_{\alpha\beta})^{\mu\lambda} g_{\lambda\nu} = \left(\delta^\mu_\alpha \delta^\lambda_\beta - \delta^\lambda_\alpha \delta^\mu_\beta\right) g_{\lambda\nu} \tag{1.16}$$

Now it is not difficult to check that (1.15) really describe the Lorentz transformations

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \tag{1.17}$$

For example, let's check that the rotation $\theta^{01} = \alpha$ represnets the Lorentz transformation along direction x. In this case

$$\Lambda^{\mu}{}_{\nu} = \left[\delta + \alpha \left(iT_{01}.g\right) + \frac{\alpha^2}{2}\left(iT_{01}.g\right)^2 + ...\right]^{\mu}{}_{\nu} \tag{1.18}$$

From (1.16) we get

$$iT_{01}.g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{1.19}$$

and

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0\,0 \quad 0 \quad 1 \quad 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{1.20}$$

Finally we introduce a new variable, $\cosh\alpha = 1/\sqrt{1-v^2}$, and find exactly the known Lorentz transformation (v is the velocity).

Similarly an arbitrary tensor transforms under Lorentz as

$$T'^{\alpha_1 \ldots \alpha_n} = \Lambda^{\alpha_1}{}_{\beta_1} \ldots \Lambda^{\alpha_n}{}_{\beta_n} T^{\beta_1 \ldots \beta_n} \tag{1.21}$$

It is good to remember that, due to antisymmetry of the generators¡ we get from (1.15)

$$\Lambda_{\mu}{}^{\sigma} = \left(\Lambda^{-1}\right)^{\sigma}{}_{\mu} \tag{1.22}$$

and from here the relation

$$\Lambda^{\mu}{}_{\nu} \Lambda_{\mu}{}^{\sigma} = \delta^{\sigma}{}_{\nu} \tag{1.23}$$

Products of two vectors are thus Lorentz scalars:

$$a'^{\mu} b'_{\mu} = \Lambda^{\mu}{}_{\nu} a^{\nu} \Lambda_{\mu}{}^{\sigma} b_{\sigma} = a^{\nu} b_{\nu} \tag{1.24}$$

2 More on the Lorentz group (45min)

2.1 Spinors

At first sight it looks like the fundamental representation of the generators (1.5) is also the simplest one, form which we can get the transformations of all higher tensors as in

(1.21). It turns out this is not true, since it exists an even simpler represnetation fo the Lorentz group, from which we can derive even the transformation matrix $\Lambda^{\mu}{}_{\nu}$, which is needed in (1.17) or (1.21).

This can be seen from the following derivation (let's stick to four dimensions): imagine there exist four 4×4 matrices γ^{μ}, which satisfy the Dirac algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \tag{2.1}$$

Then the matrices

$$\Sigma^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{2.2}$$

satisfy the commutation relations (1.14).

The matrices we need can be written for example as (in the so called chiral represnetation) as

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \tag{2.3}$$

where $\sigma^{\mu} = (1, \sigma^{i})$, $\bar{\sigma}^{\mu} = (1, -\sigma^{i})$ and σ^{i} are the Pauli matrices. The spinorial represen-tation Ψ of the Lorentz gruop is thus the one that transforms as

$$\Psi' = \Lambda_{1/2}\Psi = \exp\left(\frac{i}{2}\alpha^{\mu\nu}\Sigma_{\mu\nu}\right)\Psi \tag{2.4}$$

In our case (4d) this is the 4-dimensional Dirac spinor. Since the spacetime dimension is even, the generators of the Lorentz transformations in the spinorial representations are block diagonal. Irreducible representations of the Lorents group are thus two-dimensional. In the basis (2.3) they are the Weyl spinors ψ_L in ψ_R:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.5}$$

Under parity we have

$$\Psi \to \gamma^0\Psi \tag{2.6}$$

i.e.

$$\psi_L \leftrightarrow \psi_R \tag{2.7}$$

The Lorentz transformation matrix in the fundamental representation $\Lambda^{\mu}{}_{\nu}$ can be written, as promised, with the matrix in the spinorial representation $\Lambda_{1/2}$:

$$\Lambda_{1/2}^{-1}\gamma^{\mu}\Lambda_{1/2} = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{2.8}$$

For later use we define the 4×4 matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.9}$$

and

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 \tag{2.10}$$

which transforms as

$$\bar{\Psi}' = \bar{\Psi}\Lambda_{1/2}^{-1} \tag{2.11}$$

All 16 possible bilinear combinations of Dirac spinors can be written as

$$\begin{aligned} S &= \bar{\Psi}\Psi && (2.12)\\ P &= \bar{\Psi}\gamma^5\Psi && (2.13)\\ V^\mu &= \bar{\Psi}\gamma^\mu\Psi && (2.14)\\ A^\mu &= \bar{\Psi}\gamma^\mu\gamma^5\Psi && (2.15)\\ T^{\mu\nu} &= \bar{\Psi}\left[\gamma^\mu, \gamma^\nu\right]\Psi && (2.16) \end{aligned}$$

The chosen letters (S =scalar, P =pseudoscalar, V =vector, A =axial vector, T =tensor) denote their behaviour under Lorentz transformations

$$\begin{aligned} (S', P') &= (S, P) && (2.17)\\ (V'^\mu, A'^\mu) &= \Lambda^\mu{}_\nu\,(V^\nu, A^\nu) && (2.18)\\ T^{\mu\nu} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} && (2.19) \end{aligned}$$

and parity

$$\begin{aligned} S &\to +S && (2.20)\\ P &\to -P && (2.21)\\ V^\mu &\to (-1)^{\delta_{\mu 0}+1}\, V^\mu && (2.22)\\ A^\mu &\to (-1)^{\delta_{\mu 0}}\, A^\mu && (2.23)\\ T^{\mu\nu} &\to (-1)^{\delta_{\mu 0}+\delta_{\nu 0}}\, T^{\mu\nu} && (2.24) \end{aligned}$$

2.2 All representations of the Lorentz group

Generic SO(1,3) generators can be written as označimo kot

$$T_{0a} = K_a \quad , \quad T_{ab} = \epsilon_{abc} J_c \tag{2.25}$$

where the Levi-Civite tensor ϵ_{abc} is antisymmetric under the exchange of arbitrary two indices and $\epsilon_{123} = 1$. The generators K represent Lorentz boosts, while generators J are rotations in 3-dimensional space (angular momentum!). The commutation rules (1.14) get written as

$$[K_a, K_b] = -i\epsilon_{abc}J_c \tag{2.26}$$
$$[J_a, K_b] = i\epsilon_{abc}K_c \tag{2.27}$$
$$[J_a, J_b] = i\epsilon_{abc}J_c \tag{2.28}$$

Although this looks less compact as before, it gives some physical insight. The last equation is nothing else than the commutation relation for the operator of angular momentum. Defining new linear combinations

$$A_a = \frac{1}{2}(J_a + iK_a) \tag{2.29}$$
$$B_a = \frac{1}{2}(J_a - ik_a) \tag{2.30}$$

whose commutation relations simplify as

$$[A_a, A_b] = i\epsilon_{abc}A_c \tag{2.31}$$
$$[A_a, B_b] = 0 \tag{2.32}$$
$$[B_a, B_b] = i\epsilon_{abc}B_c \tag{2.33}$$

These are commutation relations for rotations in 2-dimensional complex space (see the above rules for angular momentum oparators). We can talk about the group SU(2), which has 3 elements (a generic SU(n), these are rotations - no more orthogonal, but unitary - in n-dimensional complex space, has $n^2 - 1$ generators). A and B are thus generators of two SU(2).

Generators of the SO(4) group thus get divided into generators of two unrelated rotations, two SU(2) (one with generators A, teh second one with generators B). The group SO(4) lis locally equivalent to the group SU(2)×SU(2).

Fields describing elementary particles transform as irreducible representations of the Lorentz group, which can be characterised by two multiples of 1/2, i.e. with one spin number for each SU(2). In this way the simplest representation is the Lorentz scalar $(0, 0)$. Then we have two types of spinors. we denote the above SU(2) groups with the index L (left) and R (right), so that we have two types of basic (Weylovih) spinors $\psi_L \sim (1/2, 0)$ or $\psi_R \sim (0, 1/2)$, called the left-handed or right-handed spinor. Each of them can describe only massless fermions. For massive fermions we need either both of them (i.e a Dirac spinor) or a combination of the two (i.e. a Majorana spinor). All known massive fermions (electron, muon, tau, quarks) are Dirac particles, except the neutrino which may be either Dirac or Majorana.

The Dirac fermion transforms under Lorentz transformations with $\Lambda_{1/2}$, see (2.4), from here the sign 1/2, i.e. from spin 1/2 involved. The last representation we will use is for the vector boson $A^\mu \sim (1/2, 1/2)$, which is partly 0 $(= 1/2 - 1/2)$ and partly spin 1 $(= 1/2 + 1/2)$.

2.3 Exercise 1

- Check that $\Sigma_{\mu\nu} = c\,[\gamma_\mu, \gamma_\nu]$ satisfy the commutation relations for SO(d_+, d_-) and determine c.
- Show that $\Sigma_{\mu\nu}$ are block diagonal in the chiral representation of the γ matrices.
- Prove the equation (2.8).
- Check equations (2.26)-(2.28), i.e. check the proportionality factor in the definition of the angular momentum operator from T_{ab}.
- Show that the Dirac equation is Lorentz covariant.

3 Quantm mechanics, equations for different spins (45min)

3.1 Schrödinger equation

In quantum mechanics one is typically solving the Schrödinger equation of the form

$$i\frac{\partial}{\partial t}\Psi = \left(-\frac{1}{2m_k}\nabla_k^2 + V\right)\Psi \tag{3.1}$$

where we sum over all particles $k = 1, ..., N$, Ψ jis the wavefunction of the system, and V the potential. From here it is easy to derive the continuity equation

$$\frac{\partial}{\partial t}\rho = -\nabla_k . \overrightarrow{j}_k \tag{3.2}$$

where

$$\rho = \Psi^*\Psi \tag{3.3}$$

a positive definite quantity, can be interpreted as the probability density (probability, that the system is in a given state), while

$$\overrightarrow{j}_k = -\frac{i}{2m_k}\left(\Psi^*\nabla_k\Psi - \Psi\nabla_k\Psi^*\right) \tag{3.4}$$

is the probability current of the k-th particle. The equation (3.2) tells us that probability is conserved

3.2 The Klein-Gordon equation

the Schrödinger equation is explicitly non-relativistic. It follows from the non-relativistic relation for the energy

$$E = \frac{\overrightarrow{p}^2}{2m} + V \tag{3.5}$$

and the exchange of the energy and momentum with the appropriate operators

$$E \to i\frac{\partial}{\partial t} \quad , \quad \overrightarrow{p} \to -i\nabla \tag{3.6}$$

The generalisation for the relativistic case seems thus simple: from

$$E^2 = \overrightarrow{p}^2 + m^2 \tag{3.7}$$

we get the Klein-Gordon (KG) equation (for a free particle)

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\Phi = -m^2\Phi \tag{3.8}$$

Denoting

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right) \tag{3.9}$$

the KG equation can be explicitly written in a relativistic invariant form:

$$\partial^\mu\partial_\mu\Phi = -m^2\Phi \tag{3.10}$$

This equation has two main shortcomings. First, it includes also negative energies from the square root of the equation (3.7)

$$E = \pm\sqrt{\overrightarrow{p}^2 + m^2} \tag{3.11}$$

If we included an arbitrary interaction (the above form of the KG equation describes only kinematics, it is only for free particles), a particle with positive energy could tarnsform into a similar with negative energy, which obviously does not happen. Second, similarly as in the case of the Schrödinger equation we could find the relativistic current

$$j^\mu = \frac{i}{2m}\left(\Phi^*\partial^\mu\Phi - \Phi\partial^\mu\Phi^*\right) \tag{3.12}$$

which automatically satisfies teh continuity eqaution due to KG

$$\partial_\mu j^\mu = 0 \tag{3.13}$$

However now

$$\rho = j^0 = \frac{i}{2m}\left(\Phi^*\frac{\partial}{\partial t}\Phi - \Phi\frac{\partial}{\partial t}\Phi^*\right) \tag{3.14}$$

is not positive definite anymore and so cannot be interpreted as the probability density.

Said differently, the solution of the KG equation Φ is not a wavefunction anymore. We will see that all problems get solved if we interpret the field Φ as an operator, which can change the number of particles.

3.3 The Dirac equation

In the previous section we have not explicitly specified for which particles (of what spin) the KG equation applies. The answer is simple: since this is just a kinematical constraint, there is no interaction and it follows just as a relation between energy, momentum and mass. So it should be valid for every spin. And indeed this is the case, although this is the whole story only for scalars, i.e. for particles with spin 0, or, said differently, for the representation $(0,0)$ of the Lorentz group. For particles with different spin we have additional constraints.

Particles with spin 1/2 satisfy the well known Dirac equation

$$(i\partial\!\!\!/ - m)\,\Psi = 0 \tag{3.15}$$

where we used the known Feynman convention

$$a\!\!\!/ \equiv \gamma^\mu a_\mu \tag{3.16}$$

The Dirac equation is covariant under Lorentz transformations, for massless particles is reduced to two independent equations, one for ψ_L, one for ψ_R.

The relativistic invariant current

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi \tag{3.17}$$

has a positive definite zero component, which can be thus interpreted as the probability density, similarly as in the Schrödinger equation.

With the ansatz ($px \equiv p_\mu x^\mu$)

$$\Psi = u^{(\alpha)}e^{-ipx} \quad (\text{ali } \; v^{(\alpha)}e^{+ipx}) \tag{3.18}$$

the equation can be solved, with two eigenvalues ($\alpha = 1, 2$) for energy positive (u), and two negative (v). But since The Dirac equation describes partilces with spin 1/2, these negative energies are innocuous. Not only, due to these negative energy solutions, Dirac predicted the existence of antiparticles, i.e. particles with all charges opposite to the usual particles. To forbid transitions of states with positive energy to those with negative energy, Dirac postulated that all states with negative energy are filled. The vacuum (the ground state of the system) has all negative energy states filled. The state of a positive energy particle is thus stable, since due to the Pauli exclusion principle, it cannot transform into a negative energy state, since all of them are already populated. On the other side, such a vacuum allows, by addition of energy, a particle with negative energy to get excited to a state with positive energy, leaving the previous negative energy state empty. The hole

in the negative energy states means antiparticle, so such a process is nothing than the generation of a particle-antiparticle pair.

The Dirac equation behaves much better than the the KG or Schrödinger euqtaions, since it avoids the previously mentioned problems and it is relativistic. However, although it describes a fermion at low energy , it fails at higher energies. The reason is that it fails to describe processes which change the number of particles. Such processes are however possible, as we have seen in the example of creation of a pair of particle-antiparticle. This problem will be definitely solved only with a consistent quantum field theory description.

3.4 Maxwell's equations

Let's mention only briefly the Maxwell's equations. These can be written in an explicit relativistic covariant form as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.19}$$

$$\partial^\mu F_{\mu\nu} = j_\nu \tag{3.20}$$

where A_μ is the electromagnetic potential. $F_{\mu\nu}$ transforms under the Lorentz transformations as a tensor with two indices, A_μ and j_μ as vectors. The quantity j_μ is the souce (current) of the electromagnetic field and it is conserved ($\partial^\mu j_\mu = 0$). The equation (3.20) has too much freedom. In fact, if $A_\mu(x)$ is the solution, so is also $A_\mu(x) + \partial_\mu \alpha(x)$, where $\alpha(x)$ is an arbitrary function of coordinates and time. This can be cleverly used to simplify the above equations. The physical quantities - then electric field $\vec{E}$ and the magnetic field $\vec{B}$ - doe not depend on the choice of the parameter α, since

$$E_i = F_{0i} \qquad B_i = \frac{1}{2}\epsilon_{ijk}\partial_j A_k \tag{3.21}$$

As an example of simplifying the equations (3.20) take such a $\alpha(x)$, for which $\partial^\mu A_\mu = 0$. This can always be done. Suppose that $\tilde{A}_\mu$ does not satisfy it. Then we can redefine $A_\mu = \tilde{A}_\mu + \partial_\mu \alpha$ and choose α to satisfy the equation $\partial^2 \alpha = -\partial.\tilde{A}$. We say that we choose a gauge.

In such a (Lorentz) gauge it is enaugh to solve, instead of (3.20), only the simplified equation

$$\partial^2 A_\nu = j_\nu \tag{3.22}$$

with an extra constraint

$$\partial.A = 0 \tag{3.23}$$

3.5 Exercise 2

- Solve the Dirac equation for a free field, write explicitly $u^{(\alpha)}(p)$, $v^{(\alpha)}(p)$.

- Normalising

$$\bar{u}^{(\alpha)}(p)u^{(\alpha')}(p) = 2m\,\delta^{\alpha\alpha'} \tag{3.24}$$
$$\bar{v}^{(\alpha)}(p)v^{(\alpha')}(p) = -2m\,\delta^{\alpha\alpha'} \tag{3.25}$$

  compute the quantities

$$P_+ \equiv \sum_{\alpha=1}^{2} u^{(\alpha)}(p)\bar{u}^{(\alpha)}(p) \tag{3.26}$$
$$P_- \equiv -\sum_{\alpha=1}^{2} v^{(\alpha)}(p)\bar{v}^{(\alpha)}(p) \tag{3.27}$$

- Assuming that

$$u^{(\alpha)}\bar{u}^{(\alpha)} = \frac{1}{2}\left(1+\gamma^5 \not{S}^{(\alpha)}\right) P_+ \tag{3.28}$$

  compute explicitly the polarisation vector $S_\mu^{(\alpha)}$ (spin) and check that $p^\mu S_\mu = 0$. Repeat the exercise also for $v^{(\alpha)}$.

- Calculate for $n = 0, ..., 4$

$$Tr\left(\gamma_{\mu_1}...\gamma_{\mu_n}\right) \tag{3.29}$$
$$Tr\left(\gamma_5\gamma_{\mu_1}...\gamma_{\mu_n}\right) \tag{3.30}$$

4 Equations of motion, internal symmetries (1h30min)

4.1 The action

All the above equations of motion can be derived from the known principle of extreme action, which we know already from classical mechanics. Here we generalise it for the case of fields in 4-dimensional spacetime. Imagine that we have a Lagrangian density (from here I will usually call it simply Lagrangian), which is a scalar under Lorentz (and other, see later) transformations. This Lagrangian is a function of the field ϕ and its first derivative $\partial_\mu\phi$, so $\mathcal{L}(\phi, \partial_\mu\phi)$.

The action is defined as the space and time integral of the Lagrangian

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi) \tag{4.1}$$

According to the principle of extreme action the equations of motion are obtained by requiring that the action does not change for small changes of the field $\phi \to \phi' = \phi + \delta\phi$

$$\begin{aligned} S[\phi'] - S[\phi] &= \int d^4x \left[\mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)\right] \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right] \\ &= \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right)\delta\phi\right] = 0 \end{aligned} \tag{4.2}$$

In going to the last line we took into account that the difference of derivatives os equal to the derivative of the difference $\delta(\partial_\mu \phi) = \partial_\mu \delta\phi$ and we integrated by parts.

The first term is a total derivative. Its integral depends thus only on the values of the fields at the boundary (at infinity). If we limit ourselves to small enough fields and/or difference of fields $\delta\phi$, then this term does not contribute. The action is then extremal for fields that satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0 \tag{4.3}$$

What type of Lagrangians will we use? As we already said, let the Lagrangian be a Lorentz scalar. In the case of free fields we would also like to correctly reproduce the equations for different spins derived in the previous section, i.e. the Klein-Gordon equation (spin 0), the Diracovo equation (spin 1/2) and the Maxwell equation (spin 1).

It is not difficult to check that the Lagrangian for a free scalar field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 \tag{4.4}$$

where the normalisation to 1/2 of the first term (with derivatives) is similar to the defintion in quantum mechanics:

$$L = \frac{1}{2}m\left(\frac{d\overrightarrow{x}}{dt}\right)^2 \tag{4.5}$$

Time t and the particle coordinate $\overrightarrow{x}(t)$ in classical mechanics corresponds now in quantum field theory the four-vector $x^\mu = (t, \overrightarrow{x})$ and the field $\phi_i(x^\mu)$.

For a free complex field we write

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{4.6}$$

The normalisation is chosen here so that the complex field can be expressed with two real fields as $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

Similarly for fermions

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{4.7}$$

where we must consider the fields ψ and $\bar{\psi}$ in the derivation of the Euler-Lagrange equations as independent fields (similarly as ϕ and ϕ^* in (4.6)).

Finally we get the Maxwell equations from

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{gf} \tag{4.8}$$

where the last term depends on the choice of gauge (gf=gauge fixing). If the Lorentz gauge is chosen

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \tag{4.9}$$

with ξ a Lagrange multiplicator.

The factor $-1/4$ in (4.8) is determined by the requirement that the product of time derivatives of the spatial components A^i satisfy the same normalisation as in (4.4). The time components A^0 on the other side do not have this kinetic term, which indicates that not all four components of A^μ are physical, which further motivates the gauge fixing term (for example (4.9)).

4.2 The Noether theorem

Let us go back to the equation (4.2). Let's assume that the field ϕ satisfies the equation of motion. We are interested now in the internal symmetries of the Lagrangian, i.e. on such $\delta\phi$, for which the Lagrangian is invariant. In this case the first term under the integral is zero in the whole spacetime (and not only after the integration, or stated differently, at infinity, as for arbitrary $\delta\phi$)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = 0 \tag{4.10}$$

If there are more fields, we need to add the single contributions. We will be interested here in continuous transformations, which are described by a Lie group (T_a are the generators)

$$\phi' = e^{i\alpha^a T_a} \phi \tag{4.11}$$

Then $\delta\phi = i\alpha^a T_a \phi$. All together we get as many conserved currents as the number of the generators of transformations we have:

$$\partial_\mu j_a^\mu = 0 \ , \ j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i T_a \phi \tag{4.12}$$

These relations are exact only after using the equations of motion.

Let's see the known example of a Lagrangian for a free complex scalar field (4.6). The simmetry here is the U(1) phase:

$$\phi' = e^{i\alpha}\phi \ , \ \phi^{*\prime} = e^{-i\alpha}\phi^* \tag{4.13}$$

under which the Lagrangian (4.6) is invariant. Then the current is

$$j^\mu = i\left(\phi\partial^\mu\phi^* - \phi^*\partial^\mu\phi\right) \tag{4.14}$$

If we differentiate

$$\partial_\mu j^\mu = i\left(\phi\partial^2\phi^* - \phi^*\partial^2\phi\right) = i\left(-\phi m^2\phi^* + \phi^* m^2\phi\right) = 0 \tag{4.15}$$

where we took into account the equation of motion

$$\left(\partial^2 + m^2\right)\phi = 0 \tag{4.16}$$

We encountered this current (4.14) already in the previous section, where it came from the Klein-Gordon equation, while here it comes directly from the symmetry of the Lagrangian. We say that the current is conserved, since the quantity

$$Q_a = \int d^3x j_a^0 \tag{4.17}$$

which is called the charge, is time independent:

$$\frac{dQ_a}{dt} = \int d^3x \partial_0 j_a^0 = \int d^3x \partial_\mu j_a^\mu = 0 \tag{4.18}$$

As usual we took into account that the fields decrease fast enough towards infinity, and use the Gauss law (the space integral of a total derivative is equal to the integrand at the boundary). We call the current (4.12) and the charge (4.17) the Noether current and the Noether charge. The Noether theorem states that these charges are conserved.

4.3 Gauge invariance

So far we took transformations (4.11) with constant parameters. In this section we will consider the very important case, when these parameters are functions of coordinates, $\alpha_a = \alpha_a(x)$. These are the gauge transformations.

The motivation for them comes from the gauge invariance of the Maxwell equations. We can couple a fermion

$$\mathcal{L} = \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{4.19}$$

with the EM field with the known quantum mechanical receipe

$$i\partial_\mu \to p_\mu \to p_\mu + eA_\mu \to i\partial_\mu + eA_\mu \to iD_\mu \tag{4.20}$$

We defined the covariant derivative

$$D_\mu\psi = \left(\partial_\mu - ieA_\mu\right)\psi \tag{4.21}$$

The Lagrangian

$$\mathcal{L}_1 = \bar{\psi} i\gamma^\mu D_\mu \psi \tag{4.22}$$

is invariant under gauge transformations

$$\psi \to e^{i\alpha(x)}\psi \ , \ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha \tag{4.23}$$

since the definition of the covariant derivative on a field is to transform exactly as the field itself.

$$(D_\mu \psi)' = e^{i\alpha(x)} D_\mu \psi \tag{4.24}$$

Similarly the kinetic term for the EM field

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{4.25}$$

is invariant under gauge transformations, since it is invariant already the field strength itself.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{4.26}$$

$$F'_{\mu\nu} = F_{\mu\nu} \tag{4.27}$$

The sum (4.22) and (4.25) is the Lagrangian for a fermion and the EM field.

Transformations (4.23) form, as we know, the group U(1), i,e. the change of the fermion phase. This can be generalised to other groups, which has been first done by Yang and Mills in the fifties, who first wrote down the Lagrangian invariant under the transformation group SU(2).

More genrally, we would like to understand how to write Lagrangians invariant under

$$\psi' = U\psi \ , \ U = U(\alpha(x)) \ , \ U^\dagger = U^{-1} \tag{4.28}$$

where U are unitary matrices, elements of the SU(N) group. Terms of the form$\bar{\psi}\psi$ are invariant since U is unitary, $U^\dagger U = 1$. Problems appear though when we have to do with derivatives:

$$\partial_\mu \psi' = U\partial_\mu \psi + (\partial_\mu U)\,\psi \tag{4.29}$$

so that the Lagrangian is not invariant due to the second term

$$\bar{\psi}' i\gamma^\mu \partial_\mu \psi' \neq \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{4.30}$$

Similarly as before with the photon field A_μ we would like to transform also here the usual derivative intro a covariant one. Now we have the group SU(N), and so $N^2 - 1$ gauge bosons, the same number as the number of generators of transformations. We first define the matrix

$$A_\mu \equiv A_\mu^a T^a \tag{4.31}$$

where the generators T^a are in the same representation as ψ. If for example ψ is in the fundamental representation, for the case of SU(2) the generators T^a are the properly normalised Pauli matrices $\sigma^a/2$, while in the case of SU(3) the Gell-Mann matrices. In any case we stick to the normalisation convention for the genrators in the fundamental representation

$$Tr\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{4.32}$$

(a different choice would just redefine the coupling constant g). Let us now use a similar trick as before. We introduce the covariant derivative

$$D_\mu \psi \equiv \left(\partial_\mu - igA_\mu\right)\psi \tag{4.33}$$

which under gauge transformations

$$\psi' = e^{i\alpha_a(x)T^a}\psi \ \ , \ \ A'_\mu = UA_\mu U^\dagger + \frac{c}{g}\partial_\mu\left(U\right)U^\dagger \tag{4.34}$$

behaves as (4.24). We guessed this change of the vector field (of the matrix (4.31)), and the constant c is determined from the condition (4.24) for $\alpha(x) \equiv \alpha_a(x)T^a$ finding $c = -i$.

Ho do we get the invariant for the kinetic terms of the gauge boson A_μ? It makes sense to have a similar form as before, but taking a trace since we have to do with matrices:

$$\mathcal{L} = -\frac{1}{4c_2}Tr\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{4.35}$$

which does not tell us much, since we have not defined $F_{\mu\nu}$ for a general SU(N) group. The simplest ansatz (4.26) turns out wrong, so we try another guess (we demand antisymmetry under exchange of indices)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + c_1 g\left[A_\mu, A_\nu\right] \tag{4.36}$$

To have an invariant trace (4.35), we need

$$F'_{\mu\nu} = UF_{\mu\nu}U^\dagger \tag{4.37}$$

which determines $c_1 = -i$. The last constant c_2 is fixed by normalisation. Fo every gauge field A_μ^a we want to have the same normalisation of its kinetic term as for the case of electromagnetism. This shows that c_2 is just the normalisation factor for the generators (in a given representation)

$$Tr\left(T^a T^b\right) = c_2\delta^{ab} \tag{4.38}$$

and so equal to 1/2 in the case of fundamental representation.

Two more comments.

First, we can use the same general formula for SU(N) also for the case of electromagnetism, where the only generator $T^a = q$ is now a constant, which is not necessarily normalised the same way for different fields. This means that each field ψ_q can have its own q. In this way we can for example interpret e as the electron charge (it has $q = 1$), so that the up quark has $q = 2/3$, down quark $q = -1/3$, etc. The product qe is the U(1) charge of the field ψ_q. Here there is a difference between the abelian U(1) in the nonabelian SU(N) groups. While in the abelian case each field can have its own charge, in the nonabelian case the charge depends uniquely on the group representation.

Second, in the same way as we defined the covariant derivative for fermions, we could have done it for bosons.

$$D_\mu \phi \equiv (\partial_\mu - igA_\mu)\,\phi \tag{4.39}$$

while the invariant kinetic term is

$$\mathcal{L} = (D_\mu \phi)^\dagger\, D^\mu \phi \tag{4.40}$$

4.4 Exercise 3

- Show that the Noether theorem for translations and rotations mean conservation of energy, momentum and angular momentum.
- Compute c_1 from equation (4.36), by checking that the electromagnetic field strength transforms correctly as $F'_{\mu\nu} = UF_{\mu\nu}U^\dagger$. Check also if the same tensor can be writtena as $D_\mu A_\nu - D_\nu A_\mu$ or $[D_\mu, D_\nu]$.

5 Perturbation (3h)

5.1 * Quantum mechanics

The Schrödinger equation keeps the number of particles constant, its solution (the wavefunction) tells us how the initial $\psi(x_a, t_a)$ (x_a are the coordinates of all partilces in the initial moment t_a) changes into $\psi(x_b, t_b)$ (x_b are the coordinates of all partilces in the final moment t_b). This can be written via the function $G(x_b, t_b; x_a, t_a)$:

$$\psi(x_b, t_b) = \int G(x_b, t_b; x_a, t_a)\psi(x_a, t_a)dx_a \tag{5.1}$$

If we know the function G, then we know the system completely. Something similar will be true also later on in field theory, when the number of particles can change, while the interpretation (5.1) clearly will not be correct.

The equation we try to solve is in general of the form

$$\mathcal{O}_{x_b,t_b}\psi(x_b, t_b) = 0 \tag{5.2}$$

This can for example be

$$\mathcal{O}_{x_b,t_b} = -\frac{1}{2m_b}\nabla^2_{x_b} + V(x_b,t_b) - i\frac{\partial}{\partial t_b} \tag{5.3}$$

If we knew how to solve such equation exactly,this section would finish here. But since we do not know exact solutions almost never, we must make use of approximations. The most common one is a systematic perturbation expansion. We split he operator $\mathcal{O}$ into a piece which can be solved exactly (this piece is typicall without interactions) and the rest:

$$\mathcal{O} = \mathcal{O}^0 + \mathcal{O}^1 \tag{5.4}$$

We choose the operator $\mathcal{O}^0$ so that we are able to solve the homogeneous equation

$$\begin{aligned} \mathcal{O}^0_{x,t}\psi_0(x,t) &= 0 & (5.5)\\ \mathcal{O}^0_{x_2,t_2}G_0(x_2,t_2;x_1,t_1) &= -i\delta(x_2-x_1)\delta(t_2-t_1) & (5.6) \end{aligned}$$

In the case (5.3) we choose for a particle in D spatial dimensions for example (the Heaviside function $\theta(t) = 1$ for $t > 0$ and 0 otherwise)

$$\begin{aligned} \mathcal{O}^0 &= -\frac{1}{2m}\nabla^2 - i\frac{\partial}{\partial t} & (5.7)\\ \mathcal{O}^1 &= V & (5.8)\\ \psi_0(x,t) &\propto e^{-i\left(Et - \overrightarrow{p}\,\overrightarrow{x}\right)} \;,\; E = \frac{\overrightarrow{p}^2}{2m} & (5.9)\\ G_0(x_2,t_2;x_1,t_1) &= \left(\frac{m}{2\pi i t}\right)^{D/2}\exp\left[\frac{im\overrightarrow{x}^2}{2t}\right]\theta(t) & (5.10) \end{aligned}$$

where $\overrightarrow{x} \equiv \overrightarrow{x}_2 - \overrightarrow{x}_1$ and $t \equiv t_2 - t_1$.

Let derive the result (5.10) from (5.6) for $D = 1$. AS usual, the invariance under translations tells us that the Green function G_0 depends only on differences $x_x = x_2 - x_1$ and $x_t = t_2 - t_1$:

$$G_0(x_2,t_2;x_1,t_1) \to G_0(x) \tag{5.11}$$

We guess the solution (5.6) through the integral

$$G_0(x) = \int \frac{d^2k}{(2\pi)^2}\tilde{G}_0(k)e^{-ikx} \tag{5.12}$$

We get

$$\tilde{G}_0(k) = \frac{i}{k_t - k_x^2/(2m)} \tag{5.13}$$

This has been a formal calculation, for the integration under we need also a prescription how to avoid the pole on the real axis. $k_t = k_x^2/2m$. This can be done by adding in the denominator a term $+i\epsilon$, where $\epsilon > 0$ is infinitesimally small and will be made vanish at the end of the calculation. Then the new pole is

$$k_t = \frac{k_x^2}{2m} - i\epsilon \tag{5.14}$$

i.e. in the fourth quadrant of the complex plane $(Re(k_t), Im(k_t))$. The integration under (real) k_t from $-\infty$ to $+\infty$ can be transformed into a closed integral around the upper half-plane if $t < 0$ or into a closed integral around the lower half-plane if $t > 0$ (the integral on the half-circle at infinity is zero if $Re(-ik_t t) = Im(k_t)t < 0$). The only pole in the whole k_t plane is, as we said, in the fourth quadrant, so the integral vanishes automatically for $t < 0$. In the opposite ($t > 0$) case w eget

$$G_0(x) = \theta(t) \int_{-\infty}^{+\infty} \frac{dk_x}{2\pi} \exp\left[-i\left(\frac{k_x^2}{2(m+i\epsilon)}t - k_x x\right)\right] \tag{5.15}$$

(that the new ϵ is not completely the same as the previous one is not really important, what is important is only that they both have the same sign). The Heaviside takes care of the vanishing of the integral for $t < 0$.

The last integral can be evaluated with the help of

$$\int_{-\infty}^{+\infty} dx e^{-\lambda x^2} = \sqrt{\frac{\pi}{\lambda}} \tag{5.16}$$

which is true providing $Re(\lambda) > 0$, which is true in the above example thanks to the positiveϵ.With all this we arrive at (5.10), which is what we wanted to show.

A typical process we will study will be scattering when particles come from very far. At that time ($t_a \to -\infty$) these particles are essentially free and so solve homogeneous equation (5.5). Of course this cannot be completely true since the plane wave is present in the whole of space, so also close to the interaction. In reality te particles are wave-packets and at infinity they do not feel the interaction. The approximation that at infinity the wavefunction is for free fields must thus be valid:

$$\psi(x_a, t_a) = \psi_0(x_a, t_a) \tag{5.17}$$

The solution (5.2) satisfies the integral equation

$$\psi(x_b, t_b) = \psi_0(x_b, t_b) - i \int G_0(x_b, t_b; x, t) \mathcal{O}^1_{x,t} \psi(x, t) dx dt \tag{5.18}$$

which we check by acting on it with the operator $\mathcal{O}^0_{x_b,t_b}$.

And here we go: we expand ψ on the right-hand-side according to the equation itself and find the perturbative expansion (for $\mathcal{O}^1 = V$)

$$\psi(x_b,t_b) = \psi_0(x_b,t_b) + \int G_0(x_b,t_b;x,t)(-i)V(x,t)\psi_0(x,t)dxdt \tag{5.19}$$
$$+ \int G_0(x_b,t_b;x,t)(-i)V(x,t)G_0(x,t;x',t')(-i)V(x',t')\psi_0(x',t')dxdtdx'dt' + ...$$

$\psi_0(x,t)$ is the same function of space and time we had at the beginning (same momentum), i.e.

$$\psi_0(x,t) = \int G_0(x,t;x_a,t_a)\psi_0(x_a,t_a)dx_a \tag{5.20}$$

which applies for $t > t_a$, so that (5.5) is satisfied.

A similar expansion can be repeated for

$$\begin{aligned} G(x_b,t_b;x_a,t_a) &= G_0(x_b,t_b;x_a,t_a) \\ &+ \int G_0(x_b,t_b;x,t)(-i)V(x,t)G_0(x,t;x_a,t_a)dxdt + ... \end{aligned} \tag{5.21}$$

5.2 Classical field theory, Feynman rules and diagrams

The above derivation of the solution to the Schrödinger equation is pictorially clear, since we can follow the change of the propagator in the interacting theory G: this is made of the free propagator G_0, two free propagators G_0 connected with one interacting vertex V, three free propagators connected with two interacting vertices, etc. For small interaction term V we hope this expansion will converge.

There are two shortcomings of this approach: 1) everything is nonrelativistic; 2) the interaction is external, i.e. the number of particles does not change. Point 1 will be solved by considering the Klein-Gordon equation instead of the Schrödinger equation, while point 2 will be taken care of by generalising the linear equation in the wavefunction ψ by adding nonlinear terms. Let's see this in more detail.

We will consider the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - V(\phi) \tag{5.22}$$

The equation of motion is

$$\partial^2\phi = -\frac{\partial V}{\partial\phi} + J \tag{5.23}$$

where we added on the right-handside the source of particles J. Similarly as in the previous case we are unable to solve in full generality, so will will do ti perturbatively. To dom that we will make an expansion in powers of a fictitious parameter λ. At the end of the calculation we will put it to one, $\lambda \to 1$.

$$\phi \to \sum_{n=0}^{\infty} \lambda^n \phi_n \quad , \quad J \to \lambda J \tag{5.24}$$

The equation of motion (5.23) becomes an infinite system of equations:

$$\begin{aligned}
\partial^2\phi_0 + V'(\phi_0) &= 0 && (5.25)\\
\left(\partial^2 + V''(\phi_0)\right)\phi_1 &= J && (5.26)\\
\left(\partial^2 + V''(\phi_0)\right)\phi_2 &= -\frac{1}{2}V'''(\phi_0)\phi_1^2 && (5.27)\\
\left(\partial^2 + V''(\phi_0)\right)\phi_3 &= -V'''(\phi_0)\phi_1\phi_2 - \frac{1}{3!}V''''(\phi_0)\phi_1^3 && (5.28)\\
\cdots &\quad \cdots
\end{aligned}$$

The first equation determines the background around which we expand our solution. Usually this background solves the equation with a constant ϕ_0, which we call the vacuum expectation value of the field ϕ. Sometimes the solution is non-trivial, time and/or space dependent. In this case we talk about solitonic solutions in Minkowski spacetime, and vacuum transitions or instantons if the spacetime is Euclidean. The solution ϕ_0 represents the vacuum of the system, and we need to check that this extremum is a minimum and not a maximum of the action. This requirement is satisfied if the operator $\partial^2 + V''(\phi_0)$ has only nonnegative eigenvalues. We assume here that this is true.

The next equation (5.26) is solved with the ansatz

$$\phi_1(x) = \int dy\, i\Delta(x,y) iJ(y) \tag{5.29}$$

where the propagator $\Delta(x,y)$ is the Green function of the equation

$$\left(\partial_x^2 + V''(\phi_0(x))\right) i\Delta(x,y) = -i\delta^4(x-y) \tag{5.30}$$

satisfying

$$\Delta(x,y) = \Delta(y,x) \tag{5.31}$$

All these i-factors are here for historic reasons.

The next equation is (5.27), with the solution

$$\begin{aligned}
\phi_2(x) &= \frac{1}{2}\int dy_1\, i\Delta(x,y_1)(-iV'''(y_1)) && (5.32)\\
&\times \int dy_2 i\Delta(y_1,y_2) iJ(y_2) \int dy_3 i\Delta(y_1,y_3) iJ(y_3)
\end{aligned}$$

where we used the notation

$$V^{(n)}(x) \equiv \left.\frac{\partial^n V(\phi)}{\partial \phi^n}\right|_{\phi=\phi_0(x)} \tag{5.33}$$

We continue the iteration

$$\begin{aligned}
\phi_3(x) &= \int dy_1\, i\Delta(x,y_1)(-iV'''(y_1)) \int dy_2\, i\Delta(y_1,y_2) iJ(y_2) \\
&\times \frac{1}{2}\int dy_3\, i\Delta(y_1,y_3)(-iV'''(y_3)) \int dy_4\, i\Delta(y_3,y_4) iJ(y_4) \\
&\times \int dy_5\, i\Delta(y_3,y_5) iJ(y_5) \\
&+ \frac{1}{3!}\int dy_1\, i\Delta(x,y_1)(-iV''''(y_1)) \int dy_2\, i\Delta(y_1,y_2) iJ(y_2) \\
&\times \int dy_3\, i\Delta(y_1,y_3) iJ(y_3) \int dy_4\, i\Delta(y_1,y_4) iJ(y_4)
\end{aligned} \tag{5.34}$$

All this can be put in a more symmetric and useful form if we introduce the generator of connected diagrams $W[J]$, which is a functional of the source J via

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x) \tag{5.35}$$

where the functional derivative is generically defined as

$$\frac{\delta F[f(t)]}{\delta f(\tau)} = \lim_{\epsilon\to 0} \frac{F[f(t)+\epsilon\delta(t-\tau)] - F[f(t)]}{\epsilon} \tag{5.36}$$

If we now integrate and expand the solution as powers of sources, we get (we can always choose $W[0]=0$)

$$W[J] = \frac{1}{i}\sum_{n=1}^{\infty}\frac{1}{n!}\int dx_1 \ldots \int dx_n G(x_1,\ldots,x_n) iJ(x_1)\ldots iJ(x_n) \tag{5.37}$$

where we defined with

$$G(x_1,\ldots,x_n) \equiv \left.\frac{\delta^n iW[J]}{\delta iJ(x_1)\ldots\delta iJ(x_n)}\right|_{J=0} \tag{5.38}$$

the n-point connected Green function. Let's write down few lower ones:

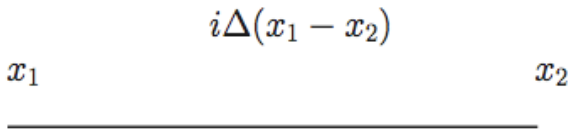


Figure 1: The Feynman rule for the 2-point Green function (propagator) in x-space.

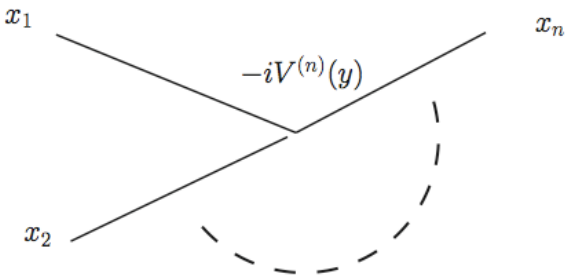


Figure 2: The Feynman rule for the n-point Green function in x-space.

$$
\begin{aligned}
G(x_1) &= \phi_0(x_1) && (5.39)\\
G(x_1,x_2) &= i\Delta(x_1,x_2) && (5.40)\\
G(x_1,x_2,x_3) &= \int dy\, i\Delta(x_1,y)i\Delta(x_2,y)i\Delta(x_3,y)(-iV'''(y)) && (5.41)\\
G(x_1,x_2,x_3,x_4) &= \int dy\, i\Delta(x_1,y)i\Delta(x_2,y)i\Delta(x_3,y)i\Delta(x_4,y)(-iV''''(y))\\
&+ \int dy_1 \int dy_2 (-iV'''(y_1))i\Delta(y_1,y_2)(-iV'''(y_2))\\
&\times (i\Delta(x_1,y_1)i\Delta(x_2,y_1)i\Delta(x_3,y_2)i\Delta(x_4,y_2)\\
&\quad +i\Delta(x_1,y_1)i\Delta(x_3,y_1)i\Delta(x_2,y_2)i\Delta(x_4,y_2)\\
&\quad +i\Delta(x_1,y_1)i\Delta(x_4,y_1)i\Delta(x_2,y_2)i\Delta(x_3,y_2)) && (5.42)
\end{aligned}
$$

The result becomes more and more complicated, but there is a natural pictorial interpretation of the solution through diagrams. The propagator $i\Delta(x,y)$ is denoted by a line between x and y, the nteraction $(-iV^{(n)}(x))$ is a n-point vertex (with n legs). This correspondence is called the Feynman rules. The diagrams drawn in this way are the Feynman diagrams.

The n-point classical Green function can be got through these diagrams . We first draw all possible tree (with this we mean all diagrams which do not enclose any point in the figure) diagrams with n external legs, and then using the Feynman rules we write down the corresponding function. In doing that we should not forget the integration over all internal points (y_i in equations (5.41)-(5.42)), which is implicit in the diagram.

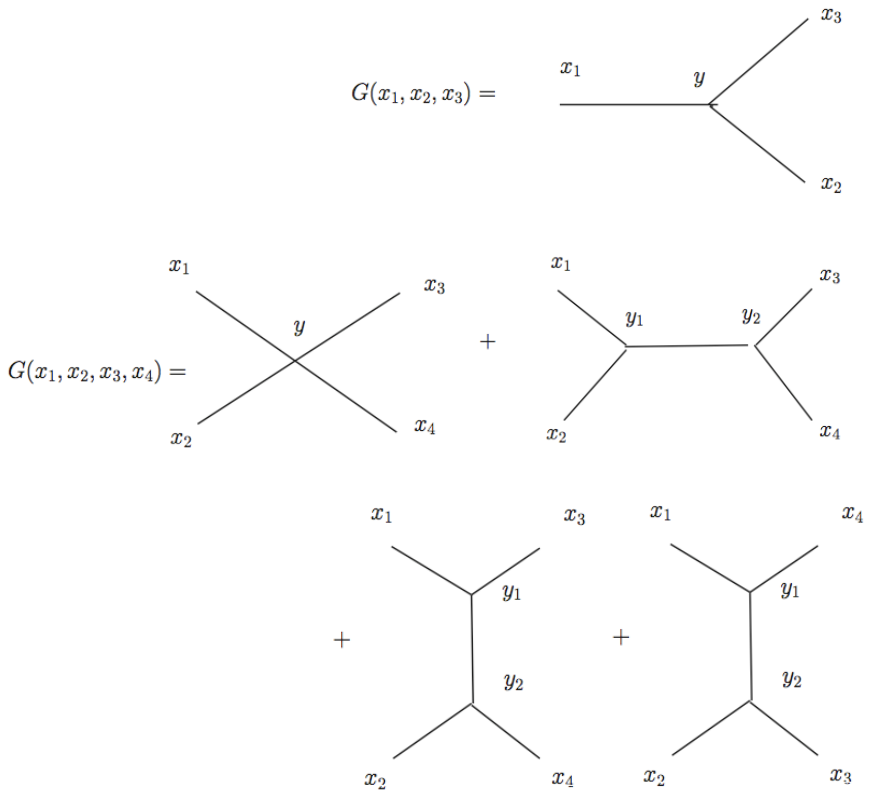


Figure 3: 3 and 4-point Green functions.

5.3 Backgrounds with Lorentz simmetry, Feynman rules and diagrams in p-space

Although everything is written very compactly, it is very difficult if not impossible to compute the Green's functions analytically for a generic background $\phi_0(x)$. Everything simplifies drastically for backgrounds which maintain the Lorentz symmetry, i.e. with a constant ϕ_0. In the following we will consider such backgrounds.

In such a case the constraint for nonnegative eigenvalues of the Klein-Gordon operator $\partial^2 + V''(\phi_0)$ becomes a constraint on the positive mass square of the excitation

$$m^2 = V''(\phi_0) \geq 0 \tag{5.43}$$

If we did not take care of this constarint, the next equation (5.26) would give an exponentially increasing solution, signalling an instability (at the maximum of the action the system is unstable).

We can take advantage of the Lorentz symmetry by transforming all quantities through a Fourier transformation into p space.

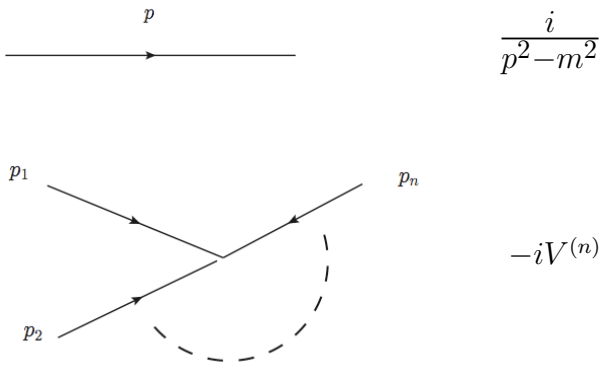


Figure 4: Feynman rules in p-space

$$\begin{aligned} G(x_1,\ldots,x_n) &= \int \frac{d^4p_1}{(2\pi)^4}\ldots\frac{d^4p_1}{(2\pi)^4}e^{ip_1x_1}\ldots e^{ip_nx_n} \\ &\times \ (2\pi)^4\delta^4(p_1+\ldots+p_n)G(p_1,\ldots,p_n) \end{aligned} \tag{5.44}$$

where the δ-function in the integrand follows from translation invariance

$$G(x_1+y,\ldots,x_n+y) = G(x_1,\ldots,x_n) \tag{5.45}$$

It is now easy to derive (for the propagator we prefer to use $G(p)$ instead of $G(p,-p)$)

$$G(p) = \frac{i}{p^2-m^2} \tag{5.46}$$

$$G(p_1,p_2,p_3) = G(p_1)G(p_2)G(p_3)(-iV''') \tag{5.47}$$

$$\begin{aligned} G(p_1,p_2,p_3,p_4) &= G(p_1)G(p_2)G(p_3)G(p_4)\left[(-iV''''\right) \\ &+ \ (-iV''')^2(G(p_1+p_2)+G(p_1+p_3)+G(p_1+p_4))\big] \\ \ldots \quad &\ldots \end{aligned} \tag{5.48}$$

Remember that $V^{(n)} = \partial^n V(\phi_0)/\partial\phi_0^n$.

We got the result in p-space, which has no addditional integration. The solution of classical equations of motion can be thus written in momentum p-space as the sum of all Feynman diagrams, where to each line corresponds a propagator (5.46) and to each n-point vertex a constant term $(-iV^{(n)})$.

As we will see the Green function $G(p_1,\ldots,p_n)$ is directly connected to a physical quantity. More precisely, it is proportional to the classical amplitude for a physical process, where there are n number of (in this case same) external particles. All we have to do is to skip in the expression for the Green function the contributions of the external propagators,

and put all external propagators on-shell (this means they need to satisfy the relations $p_i^2 = m_i^2$). If we multiply all this with i we get

$$\begin{aligned} A(p_1, p_2, p_3) &= V''' \\ A(p_1, p_2, p_3, p_4) &= V'''' - i(V''')^2 \left(G(p_1 + p_2) + G(p_1 + p_3) + G(p_1 + p_4)\right) \\ \dots &\quad \dots \end{aligned}$$

6 To quantum field theory (1h30min)

So far we derived a way how to get the solution of the equation of motion assuming relativistic invariance, allowing through interaction that the number fo particles get changed. We already mentioned that this corresponds to the classical level. This follows somehow from the definition: the solution of the equation of motion is a classical solution. For example the solution to Maxwell equation is valid in classical electrodynamics. In this section we will generalise the calculation of the Greens' functions for quantum-mechanical systems. We will first put the Schrödinger equation into such a special form, which will be easily generalised in field theory. Then we will show that such a generalised definition of the generator of connected diagrams behaves as we expect in the classical limit. Finally we will define the canonical quantisation and compare it with the path integral method.

6.1 Path integral

The idea of the path integral, which has been first introduced by Feynman in the forties of the last century, comes form a simple argument: the interference in the two slot experiment, which is valid in classical optics for light, is valid in quantum mechanics also for particles. This essentially means that the particle moves through all possible paths. Imagine that at time t_a the particle wavefunction is $\psi(x_a, t_a)$. We want to compute the wavefunction at final time $\psi(x_b, t_b)$. What we need is the propagator $G(x_b, t_b; x_a, t_a)$. All possible paths must be taken, which from the initial position x_a at time t_a go to an arbitrary final position x_b at time t_b. For a given $x(t)$ with correct initial and final constraint $x(t_{a,b}) = x_{a,b}$ we get a change of phase

$$\Delta\phi = \int_a^b (pdx - Edt) = \int_{t_a}^{t_b} dt \left(p\dot{x} - H(x,p)\right) = \int_{t_a}^{t_b} dt L(x, \dot{x}) \tag{6.1}$$

where as usual $p = \partial L(x, \dot{x})/\partial\dot{x}$. If only one path $x_1(t)$ is possible

$$G(x_b, t_b; x_a, t_a) \propto \ e^{i\Delta\phi_1} \tag{6.2}$$

if two are possible, $x_1(t)$ and $x_2(t)$, we have

$$G(x_b, t_b; x_a, t_a) \propto \ e^{i\Delta\phi_1} + e^{i\Delta\phi_2} \tag{6.3}$$

while in general we can have an arbitrary path, which means that we have to integrate $x(t_i)$ for every $t_i \in [t_a, t_b]$ from $-\infty$ to $+\infty$. Of course there is an infinite number of such t_i, i.e. there is an infinite number of integrations over $x(t_i)$, so we could define such an operation as the limit

$$\begin{aligned} G(x_b, t_b; x_a, t_a) &= \lim_{n\to\infty} C_n \int_{-\infty}^{+\infty} dx_1 ... \int_{-\infty}^{+\infty} dx_n \\ &\times \exp\left[i \sum_{i=0}^{n} \Delta t \; L\left(\frac{x_{i+1}+x_i}{2}, \frac{x_{i+1}-x_i}{\Delta t} \right) \right] \end{aligned} \tag{6.4}$$

where we divided the whole interval in equal parts $\Delta t = (t_b - t_a)/n$, and took into account the constraints $x_0 = x_a$ and $x_{n+1} = x_b$. The constanto C_n will be determined soon.

This definition can be written as

$$G(x_b, t_b; x_a, t_a) = \int \mathcal{D}x(t) \exp\left[i \int_{t_a}^{t_b} dt L(x, \dot{x}) \right] \tag{6.5}$$

although we always have in mind (6.4).

Let's now convince ourselves that the above is correct and calculate the unknown constant. To this end we check that the Green function (6.4) satisfies the Schrödinger equation, as we expect from (5.1):

$$C_n = \left(\frac{m}{2\pi i \Delta t} \right)^{\frac{n+1}{2}} \tag{6.6}$$

6.2 Quantum field theory, finally

Similarly as for the solution of the Schrödinger equation, we can define via the path integral also the generator of the connected Greens functions $W[J]$, which we call the generating functional. The following correspondence between quantum mechanics and field theory can be used:

$$t \;\to\; x^\mu = (t, x^i) \tag{6.7}$$

$$x(t) \;\to\; \phi(x^\mu) = \phi(x) \tag{6.8}$$

The generating functional is

$$\exp\left(iW[J]\right) = \frac{\int \mathcal{D}\phi(x) \exp\left[i \int d^4x \left(\mathcal{L}(\phi, \partial\phi) + J\phi \right) \right]}{\int \mathcal{D}\phi(x) \exp\left[i \int d^4x \mathcal{L}(\phi, \partial\phi) \right]} \tag{6.9}$$

Let's check if we get at first order the already known relations. At the same time we will show that corrections to them are quantum. For this purpose we will reintroduce the Planck constant $\hbar$. The Lagrangeva density has the same mass dimension as the Hamiltonian density, i.e. mass per volume:

$$[\mathcal{L}] = [\mathcal{H}] = ML^{-3} \tag{6.10}$$

Since

$$[\hbar] = ML \tag{6.11}$$

the definition (6.9) gets correctly written

$$\exp\left(\frac{i}{\hbar}W[J]\right) = \frac{\int \mathcal{D}\phi(x) \exp\left[\frac{i}{\hbar}\int d^4x \left(\mathcal{L}(\phi,\partial\phi) + J\phi\right)\right]}{\int \mathcal{D}\phi(x) \exp\left[\frac{i}{\hbar}\int d^4x \mathcal{L}(\phi,\partial\phi)\right]} \tag{6.12}$$

Instead of the full case let's have a look to the analogous simplified case

$$\exp\left(\frac{i}{\hbar}w[j]\right) = \frac{\int_{-\infty}^{+\infty} dx \exp\left[\frac{i}{\hbar}\left(s(x) + jx\right)\right]}{\int_{-\infty}^{+\infty} dx \exp\left[\frac{i}{\hbar}\left(s(x)\right)\right]} \tag{6.13}$$

We expand the function $s(x)$ in the numerator around x_{cl}, which satisfies

$$s'(x_{cl}) + j = 0 \tag{6.14}$$

and around x_{cl}^0 in the denominator

$$s'(x_{cl}^0) = 0 \tag{6.15}$$

We introduce a new variable $x = x_{cl} + \hat{x}$ in the numerator and a similar one in the denominator, getting

$$\begin{aligned} \exp\left(\frac{i}{\hbar}w[j]\right) &= \exp\left[\frac{i}{\hbar}\left(s(x_{cl}) + jx_{cl} - f(x_{cl}^0)\right)\right] \\ &\times \frac{\int_{-\infty}^{+\infty} d\hat{x} \exp\left[\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{s^{(n)}(x_{cl})}{n!}\hat{x}^n\right]}{\int_{-\infty}^{+\infty} d\hat{x} \exp\left[\frac{i}{\hbar}\sum_{n=2}^{\infty}\frac{s^{(n)}(x_{cl}^0)}{n!}\hat{x}^n\right]} \end{aligned} \tag{6.16}$$

We redefine $\hat{x} \to \hbar^{1/2}\hat{x}$, assume that $s^{(2)}(x_{cl})$ has a small positive imaginary component (in field theory this is the prescription $m^2 \to m^2 - i\epsilon$) and finally obtain

$$\begin{aligned} \exp\left(\frac{i}{\hbar}w[j]\right) &= \exp\left[\frac{i}{\hbar}\left(s(x_{cl}) + jx_{cl} - s(x_{cl}^0)\right)\right] \\ &\times \left[\frac{s^{(2)}(x_{cl})}{s^{(2)}(x_{cl}^0)}\right]^{-1/2}\left(1 + \mathcal{O}(\hbar^{1/2})\right) \end{aligned} \tag{6.17}$$

The first approximation (classical) is

$$w[j] = s(x_{cl}) + jx_{cl} - s(x_{cl}^0) + \mathcal{O}(\hbar) \tag{6.18}$$

The third term on the right-handside, $-s(x^0_{cl})$, has no special meaning, it just enforces $w[0] = 0$.

Let's now repeat the same for the case of field theory (6.12). It is not difficult to guess that the analogue of (6.18) is

$$W[J] = S[\phi_{cl}] + \int d^4x J\phi_{cl} + \mathcal{O}(\hbar) \tag{6.19}$$

If we take a functional derivative over $J(x)$ we get ϕ_{cl} as the solution of

$$\frac{\delta S[\phi_{cl}]}{\delta\phi_{cl}(x)} + J(x) = 0 \tag{6.20}$$

Since the action is

$$S[\phi] \equiv \int d^4x\, \mathcal{L}(\phi, \partial\phi) = \int d^4x \left(\frac{1}{2}(\partial\phi)^2 - V(\phi)\right) \tag{6.21}$$

we can change (6.20) using (5.36) and integration by parts into

$$-\partial^2\phi_{cl}(x) - V'(\phi_{cl}(x)) + J(x) = 0 \tag{6.22}$$

Of course this is nothing else than our old friend, the equation (5.23).

We thus have the definition (6.13) for the generating functional $W[J]$, which is valid quantum mechanically. At first order (6.19) gives the already known solution to the classical equation of motion (6.22).

we obtain the n-point connected Green function in general as

$$G(x_1, ..., x_n) = \left.\frac{\delta^n iW[J]}{\delta iJ(x_1)...\delta iJ(x_n)}\right|_{J=0} \tag{6.23}$$

Let's stress some points on the above equations.

- In field theory to each type of particle there corresponds a field. So for example a field is for the electron, one for the up quark, etc. The above definition we used is for a generic field ϕ. If we have many different fields, then we have to integrate over all of them. For example if we study the a system with particle ϕ (whatever this may be) and the photon A_μ, then we have to integrate over $\mathcal{D}\phi$ and over $\mathcal{D}A_\mu$. Strictly speaking, the above definition can be used only for particles with spin zero. For fermions and gauge bosons we will need some more explanations.

- A system with many particles of the same type (for example 3 electrons) does not introduce any new integration, just more derivatives. So for example (6.23) is related to the probability (this is not the S-matrix yet) for a transition from n_1 ϕ-particles into n_2 ϕ-particles, where $n_1 + n_2 = n$. The n functional derivatives of generating functional mean this the total number of initial and final particles in the process. Each particle has appeared or disappeared at its own time t_i at position $\overrightarrow{x}_i$.

- Now we can see the meaning of teh denominator (6.9): the transition from vacuum to vacuum without intermediate sources is normalised to 1, i.e. nothing happens.
- The formulation via the path integral is explicitly relativistic invariant (providing of course the Lagrangian is a Lorentz scalar).

The method of path integral via direct integration as we have shown in the previous section is very useful in the computation of various quantities, as for example the quantum-mechanical corrections to the equations of motion)the so-called Coleman-Weinberg potential). For the amplitude we will however use another method. This will be done in the next section.

7 Canonical quantisation

In previous sections we introduced everything we need to calculate the physical quantities in a quantum field theory. We quantised the classical fields via the path integral without even noticing it. Historically another method is more known, the canonical quantisation. Since some interesting details and computational bonuses are connected with it, we will spend this section to get an idea of this method. At the same time we will come back to our old problems: the existence of solutions with negative energy and the interpretation of the zeroth component of the conserved current. In spite of having already constructed the rules of a quantum field theory, solutions to these issues are not seen clearly yet.

7.1 The Hamiltonian

Let's first define the Hamiltonian density (or shortly the Hamiltonian) via the known method. We have a Lagrangian as a function of fields and its first derivatives:

$$\mathcal{L} = \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{7.1}$$

The field momentum is (analogouly as the coordinate momentum in quantum mechanics)

$$\pi(x) \equiv \frac{\partial \mathcal{L}(\phi(x), \partial_\mu \phi(x))}{\partial(\partial_0 \phi(x))} \tag{7.2}$$

The Hamiltonian is then

$$\mathcal{H}(\phi(x), \pi(x)) = \pi(x)\partial_0 \phi(x) - \mathcal{L}(\phi(x), \partial_\mu \phi(x)) \tag{7.3}$$

where $\partial_0 \phi(x)$ on the right-handside has to be replaced by $\phi(x)$ and $\pi(x)$. Let's see some typical examples.

7.1.1 The free real scalar field

We know already its Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \tag{7.4}$$

By using the above rules we get

$$\pi = \partial_0\phi \tag{7.5}$$

and then

$$\mathcal{H} = \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{m^2}{2}\phi^2 \tag{7.6}$$

The energy (the Hamiltonian) i explicitly positive definite, so there is no danger for states with negative energy. The problem has been thu solved by replacing a particle with its (classical) field.

7.1.2 Free complex scalar field

From the Lagrangiana

$$\mathcal{L} = \partial\phi^*\partial\phi - m^2\phi^*\phi \tag{7.7}$$

we get, similarly as as before,

$$\pi = \frac{\mathcal{L}}{\partial(\partial_0\phi)} = \partial_0\phi^* \quad , \quad \pi^* = \frac{\mathcal{L}}{\partial(\partial_0\phi^*)} = \partial_0\phi \tag{7.8}$$

and again the Hamiltonian is positive definite

$$\mathcal{H} = |\pi|^2 + |\nabla\phi|^2 + m^2|\phi|^2 \tag{7.9}$$

so again no problems with negative energy states.

7.1.3 Free Dirac field

From the Lagrangian

$$\mathcal{L} = \bar{\psi}\left(i\partial\!\!\!/ - m\right)\psi \tag{7.10}$$

we first get the field canonical momentum

$$\Pi = \frac{\partial\mathcal{L}}{\partial(\partial_0\psi)} = \bar{\psi}i\gamma^0 \tag{7.11}$$

and then the Hamiltonian

$$\mathcal{H} = \Pi\partial_0\psi - \mathcal{L} = \psi^\dagger i\partial_0\psi \tag{7.12}$$

where in the last line we used the Dirac equation. The result is not surprising (in quantum mechanics the time derivative corresponds to the Hamiltonian), but, differently form the previous cases, the operator is not explicitly positive definite. To solve this problem we have to change the classical fields with operators - we have to quantise. To tell this operation from the quantisation of coordinate and momenta in quantum mechanics (first quantisation) we call this new quantisation of fields the second quantisation.

7.2 Quantisation of fields

We now very briefly repeat the case of the harmonic oscillator with the Hamiltonian

$$H = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2 \tag{7.13}$$

We quantise the coordinate i.e. the momentum via the commutation relation (as usual $[A, B] \equiv AB - BA$)

$$[x, p] = i \tag{7.14}$$

Let's define the creation ($a^\dagger$) and annihilation (a) operator via

$$x = \frac{1}{\sqrt{2\omega}}(a + a^\dagger) \quad , \quad p = -i\sqrt{\frac{\omega}{2}}(a - a^\dagger) \quad , \quad [a, a^\dagger] = 1 \tag{7.15}$$

(7.13) turns into

$$H = \frac{\omega}{2}\left(a^\dagger a + aa^\dagger\right) \tag{7.16}$$

The vacuum $|0\rangle$ is defined as the state, annihilated by the annihilation operator:

$$a|0\rangle = 0 \tag{7.17}$$

The n-particle state is defined via the creation operator

$$|n\rangle = \frac{1}{\sqrt{n!}}\left(a^\dagger\right)^n |0\rangle \tag{7.18}$$

We will now try to repeat the procedure for the case of a field theory.

7.2.1 The real scalar field

Similarly as we quantised the coordinate and momentum in quantum mechanics via the commutation relation (7.14), we quantise in the case of field theory the field and the field momentum

$$[\phi(t, \vec{x}), \pi(t, \vec{y})] = i\delta^3(\vec{x} - \vec{y}) \tag{7.19}$$

We had to generalise the quantum mechanical commutation relation, since we have now one operator for each point in space. The relation is valid for fields defined at the *same* time t. For different times the commutator vanishes.

(7.14) becomes

$$\begin{aligned}
\phi(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a_k e^{-ikx} + a_k^\dagger e^{ikx} \right) && (7.20)\\
\pi(x) &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} (-i\omega_k) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx} \right) && (7.21)\\
\left[a_k, a_{k'}^\dagger \right] &= (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k'}) \quad , \quad [a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger \right] = 0 && (7.22)
\end{aligned}$$

where

$$k^\mu = (\omega_k, \vec{k}) \tag{7.23}$$

and the energy of the single particle state is (we will se this later)

$$\omega_k = \sqrt{\vec{k}^2 + m^2} \tag{7.24}$$

The Hamiltonian, i.e. the volume integral of the Hamiltonian density (7.6) is now

$$\begin{aligned}
H &= \int d^3x \mathcal{H} \\
&= \int d^3x \frac{1}{2} \int \frac{d^3k}{(2\pi)^3 2\omega_k} \int \frac{d^3k'}{(2\pi)^3 2\omega_{k'}} \\
&\quad \Big[(-i\omega_k)(-i\omega_{k'}) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} - a_{k'}^\dagger e^{ik'x} \right) \\
&\quad + (i\vec{k})(i\vec{k'}) \left(a_k e^{-ikx} - a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} - a_{k'}^\dagger e^{ik'x} \right) \\
&\quad + m^2 \left(a_k e^{-ikx} + a_k^\dagger e^{ikx} \right) \left(a_{k'} e^{-ik'x} + a_{k'}^\dagger e^{ik'x} \right) \Big] \\
&= \frac{1}{2} \int \frac{d^3k}{(2\pi)^2 2\omega_k} \frac{1}{2\omega_k} \\
&\quad \Big[-\omega_k^2 \left(a_k a_{-k} e^{-2\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \\
&\quad - \vec{k}^2 \left(-a_k a_{-k} e^{-2\omega_k t} - a_k a_k^\dagger - a_k^\dagger a_k - a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \\
&\quad + m^2 \left(a_k a_{-k} e^{-2\omega_k t} + a_k a_k^\dagger + a_k^\dagger a_k + a_k^\dagger a_{-k}^\dagger e^{i2\omega_k t} \right) \Big] \\
&= \int \frac{d^3k}{(2\pi)^3 2\omega_k} \frac{\omega_k}{2} \left(a_k^\dagger a_k + a_k a_k^\dagger \right) && (7.25)
\end{aligned}$$

where we used essentially (7.24) and

$$\int d^3x e^{i(\vec{k}-\vec{k'})\vec{x}} = (2\pi)^3\delta^3(k-k') \tag{7.26}$$

The vacuum state $|0\rangle$ is defined as that one, which gets annihilated by all annihilation operators:

$$a_k|0\rangle = 0 \tag{7.27}$$

The one particle state with momentum k is defined as

$$|k\rangle = a_k^\dagger|0\rangle \tag{7.28}$$

and the Hermitian conjugated state

$$\langle k| = \langle 0|a_k \tag{7.29}$$

The one particle state are orthonormal:

$$\langle k|k'\rangle = \langle 0|a_k a_{k'}^\dagger|0\rangle = \langle 0|\left[a_k, a_{k'}^\dagger\right]|0\rangle = (2\pi)^3 2\omega_k\delta^3(k-k') \tag{7.30}$$

where we used (7.27), (7.22) and the vacuum normalisation $\langle 0|0\rangle = 1$.

Analogously we define a multiparticle state (free particles!) as

$$|n_1, k_1; \ldots; n_N, k_N\rangle = \frac{1}{\sqrt{n_1!}}\left(a_{k_1}^\dagger\right)^{n_1} \ldots \frac{1}{\sqrt{n_N!}}\left(a_{k_N}^\dagger\right)^{n_N}|0\rangle \tag{7.31}$$

According to thi notation the previous one particle state $|k\rangle$ should have been written as $|1, k\rangle$.

The energy of the vacuum is, because of the second term in (7.35), strictly speaking infinite. In fact, if we use the definition (7.22), we get

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k\left(a_k^\dagger a_k + (2\pi)^3\omega_k\delta^3(0)\right) \tag{7.32}$$

However, in the absence of gravity, we can put the zero of the energy anywhere, since only energy difference are measurable. Let's then take as the natural value for the energy zero. This can be formally obtained by the operation of normal ordering $\hat{N}$[3], which organises an arbitrary product of creation and annihilation operators, so that all creation operators stay on the left of all annihilation operators. In this way for example

$$\hat{N}\left(a_{k_1}a_{k_2}^\dagger\right) = a_{k_2}^\dagger a_{k_1} \tag{7.33}$$

The Hamiltonian gets defined then as

$$H = \int d^3x \hat{N}\left(\mathcal{H}\right) \tag{7.34}$$

[3]in literature the normal ordered operator$\hat{\mathcal{O}}$, i.e. $\hat{N}(\hat{O})$, is seldom written as $:\hat{\mathcal{O}}:$.

and is in our case

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k a_k^\dagger a_k \tag{7.35}$$

The energy of the multiparticle state (7.31) equals

$$E = n_1\omega_1 + \ldots + n_N\omega_N \tag{7.36}$$

Similarly as (7.35) we can define also the momentum operator:

$$\vec{P} = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \vec{k} a_k^\dagger a_k \tag{7.37}$$

so that for the multiparticle state (7.31) it equals

$$\vec{P} = n_1\vec{k}_1 + \ldots + n_N\vec{k}_N \tag{7.38}$$

7.2.2 The complex scalar field

The generalisation is easy, we have to take two types of operators, $a_k, a_k^\dagger$ and $b_k, b_k^\dagger$:

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(a_k e^{-ikx} + b_k^\dagger e^{ikx} \right) \tag{7.39}$$

The operator $a_k^\dagger$ (a_k) creates (annihilates) a particle, the operator $b_k^\dagger$ (b_k) creates (annihilates) and antiparticle. In the previous (real) case we had $b_k = a_k$, the particle was it own antiparticle.

Their commutation relations are now

$$\left[a_k, a_{k'}^\dagger\right] = \left[b_k, b_{k'}^\dagger\right] = (2\pi)^3 2\omega_k \delta^3(\vec{k} - \vec{k}') \tag{7.40}$$

$$[a_k, a_{k'}] = \left[a_k^\dagger, a_{k'}^\dagger\right] = [b_k, b_{k'}] = \left[b_k^\dagger, b_{k'}^\dagger\right] = 0 \tag{7.41}$$

$$[a_k, b_{k'}] = \left[a_k^\dagger, b_{k'}^\dagger\right] = \left[a_k, b_{k'}^\dagger\right] = \left[b_k, a_{k'}^\dagger\right] = 0 \tag{7.42}$$

Completely analogously with the previous case we can now derive

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \omega_k \left(a_k^\dagger a_k + b_k^\dagger b_k \right) \tag{7.43}$$

which is the sum of particles' energy and antiparticles' energy.

The vacuum state $|0\rangle$ isdefined a that state, which is annihilated by all annihilation operators, those for particles (a_k) and those for antiparticles (b_k):

$$a_k|0\rangle = b_k|0\rangle = 0 \tag{7.44}$$

It is interesting to see, what happens with the interpretation for the conserved current

$$j^\mu = i\left(\phi\partial^\mu\phi^* - \phi^*\partial\phi\right) \tag{7.45}$$

Written with operators the charge is

$$Q = \int d^3x j^0 = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\left(a_k^\dagger a_k - b_k^\dagger b_k\right) \tag{7.46}$$

Here we explicitly see that antiparticle contribute to the total charge oppositely as particles, from here their opposite sign. They contribute the same to the Hamiltonian, though. The current j^μ is thus the charge current of the complex scalar field.

7.2.3 The Dirac field

Let's consider now also fermions, i.e. particles with half spin, in our case only spin 1/2. The situation is somehow analogous to the complex scalar field, but now the free fields should satisfy the Dirac equation instead of the Klein-Gordon equation. It is thus natural the following expansion:

$$\psi(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\sum_{s=1}^{2}\left(a_k^s u^s(k)e^{-ikx} + b_k^{s\dagger} v^s(k)e^{ikx}\right) \tag{7.47}$$

If we now assumed, as in the case of a complex scalar field, the commutation relations among the operators a_k, b_k, etc, we would end up with problems: the Hamiltonian would be unbounded from below. For this reason we have to assume the anticommutation relation among them (in general this means $\{A, B\} \equiv AB + BA$):

$$\left\{a_k^s, a_{k'}^{s'\dagger}\right\} = \left\{b_k^s, b_{k'}^{s'\dagger}\right\} = (2\pi)^3 2\omega_k \delta^{ss'}\delta^3(\vec{k} - \vec{k}') \tag{7.48}$$

$$\left\{a_k^s, a_{k'}^{s'}\right\} = \left\{a_k^{s\dagger}, a_{k'}^{s'\dagger}\right\} = \left\{b_k^s, b_{k'}^{s'}\right\} = \left\{b_k^{s\dagger}, b_{k'}^{s'\dagger}\right\} = 0 \tag{7.49}$$

$$\left\{a_k^s, b_{k'}^{s'}\right\} = \left\{a_k^{s\dagger}, b_{k'}^{s'\dagger}\right\} = \left\{a_k^s, b_{k'}^{s'\dagger}\right\} = \left\{b_k^s, a_{k'}^{s'\dagger}\right\} = 0 \tag{7.50}$$

Stated differently, if we change the order of two arbitrary operators, we always get a minus sign, while in the case (7.48) we get also an addition.

The energy of an arbitrary system is now bounded from below. Th e Hamiltonian is in fact

$$H = \int \frac{d^3k}{(2\pi)^3 2\omega_k}\omega_k \sum_s \left(a_k^{s\dagger} a_k^s + b_k^{s\dagger} b_k^s\right) \tag{7.51}$$

and is explicitly positive definite.

The vacuum is defined via

$$a_k^s|0\rangle = b_k^s|0\rangle = 0 \tag{7.52}$$

7.2.4 The Maxwell's field

Very shortly we give jut the result for the massless case. The expansion used is

$$A_\mu(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \sum_\lambda \left(a_\lambda(k)\epsilon_\lambda^\mu(k)e^{-ikx} + a_\lambda^\dagger(k)\epsilon_\lambda^{\mu *}(k)e^{ikx} \right) \tag{7.53}$$

where we sum up over all polarisations λ and where the polarisation vector satisfies the constraint

$$k_\mu \epsilon_\lambda^\mu(k) = 0 \quad , \quad \sum_\lambda \epsilon_\lambda^\mu(k)\epsilon_\lambda^{\nu *}(k) = -g^{\mu\nu} \tag{7.54}$$

7.3 Exercise 4

- Derive (7.19) and (7.36).
- Calculate the vacuum expectation value of the time ordered product of two free scalar fields

$$\begin{aligned} \langle 0 \left| \hat{T}\phi(x)\phi(y) \right| 0 \rangle &\equiv \langle 0 \left| \phi(x)\phi(y) \right| 0 \rangle \Theta(x^0 - y^0) \\ &+ \langle 0 \left| \phi(y)\phi(x) \right| 0 \rangle \Theta(y^0 - x^0) \end{aligned} \tag{7.55}$$

  and show that it satisfies the equation (c =?)

$$\left(\partial_x^2 + m^2\right) \langle 0 \left| \hat{T}\phi(x)\phi(y) \right| 0 \rangle = c\delta^4(x-y) \tag{7.56}$$

7.4 Exercise 5

- Instead of the anticommutation relations use for the Dirac field the (wrong!) commutation relations and show that the Hamiltonian is unbounded from below.
- In the case of a real scalar field show that causality is satisfied: check that $[\phi(x), \phi(y)] = 0$ if $(x-y)^2 < 0$.

7.5 Fields in different pictures, the operator of time evolution

So far we considered operators for free fields. The expansion of the field under the creation and annihilation operators went through (7.20). This field is in the interaction picture, i.e. it gets evolved in time via the Hamiltonian for a free field. We can see this better if we define a time independent field, i.e. the field in the Schrödinger picture via (t_0 is arbitrary)

$$\phi_S(x) \equiv \phi(t_0, \vec{x}) = e^{-iH_0(t-t_0)}\phi(x)e^{iH_0(t-t_0)} \tag{7.57}$$

where we have for example for a free real field

$$H_0 = \int d^3x \frac{1}{2} \left(\Pi^2(x) + (\nabla\phi)^2(x) + m^2\phi^2(x) \right) \tag{7.58}$$

That the right-hand side 7.57) is really time independent can be seen from the simple fact that, using (7.35),

$$e^{-iH_0(t-t_0)} a_k e^{iH_0(t-t_0)} = e^{i\omega_k(t-t_0)} a_k \quad , \quad e^{-iH_0(t-t_0)} a_k^\dagger e^{iH_0(t-t_0)} = e^{-i\omega_k(t-t_0)} a_k^\dagger \tag{7.59}$$

From here it follows that

$$H_0(\phi) = H_0(\phi_S) \tag{7.60}$$

and so

$$\phi(x) = e^{iH_0(\phi_S)(t-t_0)} \phi_S(x) e^{-iH_0(\phi_S)(t-t_0)} \tag{7.61}$$

Similarly we define the field in the Heisenberg picture as the one which transforms in time through the full (not free) but time independent Hamiltonian. In this way /

$$\phi_H(x) = e^{iH(\phi_S)(t-t_0)} \phi_S(x) e^{-iH(\phi_S)(t-t_0)} \tag{7.62}$$

If we collect everything

$$\phi_H(x) = U^\dagger(t, t_0) \phi(x) U(t, t_0) \tag{7.63}$$

where we defined the time evolution operator in the interaction picture as

$$U(t, t_0) = e^{iH_0(\phi_S)(t-t_0)} e^{-iH(\phi_S)(t-t_0)} \tag{7.64}$$

We want to write this operator with $\phi(x)$. For this purpose we take the time derivative:

$$\begin{aligned} i\frac{\partial}{\partial t} U(t, t_0) &= e^{iH_0(\phi_S)(t-t_0)} \left(-H_0(\phi_S) + H(\phi_S) \right) e^{-iH(\phi_S)(t-t_0)} \\ &= H_{int}(\phi) U(t, t_0) \end{aligned} \tag{7.65}$$

where we too into account that the interaction part of the Hamiltonian

$$H_{int} \equiv H - H_0 \tag{7.66}$$

is typically a polynomial of the field.

If H_{int} were a function, the solution (7.65) would be simply $U = exp(-i \int dt H_{int})$. But now we have to do with operators, not with functions, so the equation (7.65) with the initial condition $U(t_0, t_0) = 1$ is solved at most perturbatively, for small H_{int}:

$$U(t,t_0) = 1 + (-i)\int_{t_0}^{t} dt_1 H_{int}(t_1) + (-i)^2 \int_{t_0}^{t} dt_1 H_{int}(t_1) \int_{t_0}^{t_1} dt_2 H_{int}(t_2) + \ldots \quad (7.67)$$

We notice that products of the Hamiltonians are alway time ordered: first (on the left) are the Hamiltonians at late times, then those at initial times, i.e. $H_{int}(t_1)H_{int}(t_2)$ if $t_1 \geq t_2$. A we already mentioned, we have in general to do with noncommutative operators, so we need to take care of this orderings. To get a more symmetric notation, we introduce the operator of time ordered product $\hat{T}$:

$$\hat{T}\left(\hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)\right) \equiv \hat{\mathcal{O}}_1(t_1)\hat{\mathcal{O}}_2(t_2)\theta(t_1 - t_2) + \hat{\mathcal{O}}_2(t_2)\hat{\mathcal{O}}_1(t_1)\theta(t_2 - t_1) \quad (7.68)$$

Te quadratic part can be rewritten for $t > t_0$ as

$$\begin{aligned} \int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{int}(t_1)H_{int}(t_2) &= \frac{1}{2!}\left(\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2 H_{int}(t_1)H_{int}(t_2) + (t_1 \leftrightarrow t_2)\right) \\ &= \frac{1}{2!}\hat{T}\int_{t_0}^{t} dt_1 \int_{t_0}^{t} dt_2 H_{int}(t_1)H_{int}(t_2) \quad (7.69) \end{aligned}$$

Similarly we can treat an arbitrary power, which brings us to the final result

$$U(t,t_0) \equiv \hat{T}exp\left(-i\int_{t_0}^{t} dt H_{int}(\phi(x))\right) \quad , \quad t > t_0 \quad (7.70)$$

What if $t < t_0$? Then, if we repeat the above exercise,

$$U(t,t_0) = [U(t_0,t)]^{\dagger} \quad (7.71)$$

The operator (7.64) can be slightly generalised, defining

$$U(t_1,t_2) = e^{iH_0(\phi_S)(t_1-t_0)}e^{-iH(\phi_S)(t_1-t_2)}e^{-iH_0(\phi_S)(t_2-t_0)} \quad (7.72)$$

which is explicitly unitary and the same as the previous definition (7.64) when t_2 equals the reference time t_0. Now it is easy to confirm that such a unitary operator U satisfies $(t_1 > t_2 > t_3)$

$$\begin{aligned} U(t_1,t_2)U(t_2,t_3) &= U(t_1,t_3) \quad (7.73) \\ U(t_1,t_2)U(t_2,t_1) = &= 1 \quad (7.74) \end{aligned}$$

For this reason we have for example

$$U(t_1,t_3)U(t_3,t_2) = U(t_1,t_3)\left[U(t_2,t_3)\right]^{\dagger} = U(t_1,t_2)U(t_2,t_3)\left[U(t_2,t_3)\right]^{-1} = U(t_1,t_2) \quad (7.75)$$

7.6 Correlation (Green) functions

Correlation function are defined

$$\langle 0_H|\hat{T}\phi_H(x_1)\dots\phi_H(x_n)|0_H\rangle \tag{7.76}$$

We would like to write it with the fields $\phi(x)$, which we know how to expand in creation and annihilation operators. Fo the field ϕ_H we use (7.63), what we still miss is relate the vacuum state $|0_H\rangle$ with the vacuum state $|0\rangle$ defined via (7.27). As is $|0\rangle$ the vacuum state of the operator H_0, so i $|0_H\rangle$ the vacuum state of the operator H. This is, the smallest eigenvalue of the energy operator H. Generically

$$H|n_H\rangle = E_n|n_H\rangle \tag{7.77}$$

Let's calculate now

$$e^{-iHT}|0\rangle \;=\; e^{-iE_0T}|0_H\rangle\langle 0_H|0\rangle + \sum_{n>0} e^{-iE_nT}|n_H\rangle\langle n_H|0\rangle \tag{7.78}$$

We can think that all energies have a small negative imaginary part. Then all higher states exponentially decay faster than the ground state for a big enough time T (so that we can safely take $T \to T + t_0$):

$$\begin{aligned}
|0_H\rangle &= \lim_{T\to\infty} \frac{e^{-iH(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} = \lim_{T\to\infty} \frac{e^{-iH(T+t_0)}e^{iH_0(T+t_0)}|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} \\
&= \lim_{T\to\infty} \frac{\left(e^{iH_0((-T)-t_0)}e^{-iH((-T)-t_0)}\right)^\dagger|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} = \lim_{T\to\infty} \frac{[U(-T,t_0)]^\dagger|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle} \\
&= \lim_{T\to\infty} \frac{U(t_0,-T)|0\rangle}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle}
\end{aligned} \tag{7.79}$$

where we took into account that

$$H_0|0\rangle = 0 \tag{7.80}$$

Similarly, from

$$\langle 0|e^{-iHT} = e^{-iE_0T}\langle 0|0_H\rangle\langle 0_H| + \dots \tag{7.81}$$

we can derive (homework!)

$$\langle 0_H| = \lim_{T\to\infty} \frac{\langle 0|U(T,t_0)}{e^{-iE_0(T-t_0)}\langle 0|0_H\rangle} \tag{7.82}$$

From the normalisation constraint we get

$$1 = \langle 0_H|0_H\rangle = \lim_{T\to\infty} \frac{\langle 0|U(T,-T)|0\rangle}{e^{-2iE_0T}|\langle 0|0_H\rangle|^2} \tag{7.83}$$

i.e.

$$\lim_{T\to\infty} e^{-2iE_0T}|\langle 0|0_H\rangle|^2 = \lim_{T\to\infty}\langle 0|U(T,-T)|0\rangle \tag{7.84}$$

The correlation function is thus

$$\begin{aligned}\langle 0_H|\hat{T}\phi_H(x_1)\dots\phi_H(x_n)|0_H\rangle &= \lim_{T\to\infty}\frac{1}{e^{-iE_0(T-t_0)}\langle 0|0_H\rangle}\langle 0|U(T,t_0)\\ &\quad\times\hat{T}\left[U^\dagger(x_1^0,t_0)\phi(x_1)U(x_1^0,t_0)\dots U^\dagger(x_n^0,t_0)\phi(x_n)U(x_n^0,t_0)\right]\\ &\quad\times\frac{1}{e^{-iE_0(T+t_0)}\langle 0_H|0\rangle}U(t_0,-T)|0\rangle\\ &= \lim_{T\to\infty}\frac{\langle 0|\hat{T}\phi(x_1)\dots\phi(x_n)U(T,-T)|0\rangle}{\langle 0|U(T,-T)|0\rangle}\end{aligned} \tag{7.85}$$

and due to $\mathcal{H}_{int} = -\mathcal{L}_{int}$ we can rewritten with the free field (in the interaction picture):

$$\langle 0_H|\hat{T}\phi_H(x_1)\dots\phi_H(x_n)|0_H\rangle = \frac{\langle 0|\hat{T}\phi(x_1)\dots\phi(x_n)\exp\left(i\int d^4x\mathcal{L}_{int}(\phi(x))\right)|0\rangle}{\langle 0|\hat{T}\exp\left(i\int d^4x\mathcal{L}_{int}(\phi(x))\right)|0\rangle} \tag{7.86}$$

We will constantly use this equality and calculate the correlation functions from the right-hand side. As we will see, these correlation functions are directly related to physical quantities as for example the cross section or decay width.

7.7 The Wick theorem

Now we essentially have all we need to compute the correlation functions. In the right-hand side of eq. (7.86) we expand the exponent. The interaction term of the Lagrangian density is typically a polynomial in fields. In the so-called "ϕ^4" theory we have for example

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{7.87}$$

and so

$$\mathcal{L}_{int}(\phi(x)) = -\frac{\lambda}{4!}\phi^4(x) \tag{7.88}$$

What we need are thus the correlation functions for free fields, i.e.

$$\langle 0|\hat{T}\phi(y_1)\dots\phi(y_m)|0\rangle \tag{7.89}$$

In principle we know how to get them: Taking into account the time ordered product we expand the free fields in the interaction picture in terms of creation and annihilation operators. For a real scalar field it is the expansion (7.20). This way, although clear, is a bit long, especially if we have to do with a product of many fields. It turns out that

things get much simplified if we take the vacuum expectation value of such a product. This simplification and such a rule goes under the name of Wick's theorem, which we will present now.

We split the field $\phi(x)$ into a part with positive energy $\phi^+(x)$ and a part with negative energy $\phi^{(-)}x$:

$$\begin{aligned} \phi(x) &= \phi^+(x) + \phi^-(x) && (7.90) \\ \phi^+(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k e^{-ikx} \quad &; \quad \phi^-(x) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} a_k^\dagger e^{ikx} && (7.91) \end{aligned}$$

Notice that $\phi^{(+)}$ has annihilation operators a_k and a minus in the exponent $\exp(-ikx)$, while $\phi^{(-)}$ has creation operators $a_k^\dagger$ and a plus in the exponent $\exp(ikx)$, and not the other way! Then

$$\hat{N}\left[\phi(x_1)\phi(x_2)\right] = \phi^+(x_1)\phi^+(x_2) + \phi^-(x_2)\phi^+(x_1) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2) \quad (7.92)$$

For two fields we have

$$\begin{aligned} \hat{T}\phi(x_1)\phi(x_2) &= \left[\phi^+(x_1)\phi^+(x_2) + \phi^+(x_1)\phi^-(x_2) + \phi^-(x_1)\phi^+(x_2) + \phi^-(x_1)\phi^-(x_2)\right]\Theta(x_1^0 - x_2^0) \\ &\quad + \left[x_1 \leftrightarrow x_2\right]\Theta(x_2^0 - x_1^0) \\ &= \left(\hat{N}\left[\phi(x_1)\phi(x_2)\right] + \left[\phi^+(x_1), \phi^-(x_2)\right]\right)\Theta(x_1^0 - x_2^0) \\ &\quad + \left(x_1 \leftrightarrow x_2\right)\Theta(x_2^0 - x_1^0) \qquad (7.93) \end{aligned}$$

This notation is particularly useful since the vacuum expectation value of a normal ordered product is always zero

$$\langle 0|\hat{N}[\hat{\mathcal{O}}]|0\rangle = 0 \quad (7.94)$$

while the rest is easily computed:

$$\begin{aligned} \left[\phi^+(x_1), \phi^-(x_2)\right] &= \int \frac{d^3k_1}{(2\pi)^3 2\omega_{k_1}} \int \frac{d^3k_2}{(2\pi)^3 2\omega_{k_2}} \left[a_{k_1}, a_{k_2}^\dagger\right] e^{-ik_1x_1 + ik_2x_2} \\ &= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik(x_1 - x_2)} \qquad (7.95) \end{aligned}$$

This is a c-number, not an operator anymore, so

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \left[\phi^+(x_1), \phi^-(x_2)\right]\Theta(x_1^0 - x_2^0) + \left[\phi^+(x_2), \phi^-(x_1)\right]\Theta(x_2^0 - x_1^0) \quad (7.96)$$

The integral representation for the Heaviside step function is

$$\Theta(y) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi i} \frac{e^{i\omega y}}{\omega - i\epsilon} \tag{7.97}$$

since for $y > 0$ one can close the integral with the upper half circle at infinity with zero contribution ($\exp(-Im(\omega)y) \to 0$) so that the closed upper half-plane contributes exactly 1, while for $y < 0$ one needs to add the contribution of the lower half circle at infinity (now $\exp(+Im(\omega)y) \to 0$) but there is no pole in the lower half-plane giving zero.

Then

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi} \frac{-i}{2\omega_k(\omega - i\epsilon)} e^{i(\omega-\omega_k)(x_1^0 - x_2^0) + i\vec{k}(\vec{x}_1 - \vec{x}_2)} + (x_1 \leftrightarrow x_2) \tag{7.98}$$

defining a integration variables as

$$k^0 = -\omega + \omega_k \tag{7.99}$$

we get

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{2\omega_k(\omega_k - k^0 - i\epsilon)} \left(e^{-ik(x_1-x_2)} + e^{ik(x_1-x_2)}\right) \tag{7.100}$$

Then we change $k \to -k$ in the second integral and get

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-ie^{-ik(x_1-x_2)}}{2\omega_k} \left(\frac{1}{\omega_k - k^0 - i\epsilon} + \frac{1}{\omega_k + k^0 - i\epsilon}\right) \tag{7.101}$$

and using

$$\omega_k^2 = \vec{k}^2 + m^2 \tag{7.102}$$

we finally arrive at the known result

$$\langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{-ik(x_1-x_2)} \tag{7.103}$$

The correlation function of an odd number of free fields is always zero. Repeating the procedure for four fields we get (exercise)

$$\begin{aligned} \langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = \; & + \; \langle 0|\hat{T}\phi(x_1)\phi(x_2)|0\rangle\langle 0|\hat{T}\phi(x_3)\phi(x_4)|0\rangle \\ & + \; \langle 0|\hat{T}\phi(x_1)\phi(x_3)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(x_4)|0\rangle \\ & + \; \langle 0|\hat{T}\phi(x_1)\phi(x_4)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(x_3)|0\rangle \end{aligned} \tag{7.104}$$

The correlation function for free fields is thus just the sum of propagators with all possible permutations. This can be easily generalised for an arbitrary number of free fields. This is the Wick theorem.

We can now derive the Feynman rules. Take the ϕ^4 theory:

$$\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int} \tag{7.105}$$

$$\mathcal{L}_0 = \frac{1}{2}\partial\phi^2 - \frac{m^2}{2}\phi^2 \quad , \quad \mathcal{L}_{int} = -\frac{\lambda}{4!}\phi^4 \tag{7.106}$$

The Feynman rule for the propagator in x-space is given by (7.103) and so in p-space

$$\frac{i}{k^2 - m^2 + i\epsilon} \tag{7.107}$$

while we derive for the vertex

$$\begin{aligned}
& \langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\exp\left(i\int d^4z\mathcal{L}_{int}(\phi(z))\right)|0\rangle \\
\rightarrow \quad & -i\frac{\lambda}{4!}\int d^4z\langle 0|\hat{T}\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\phi^4(z)|0\rangle \\
= \quad & -i\lambda\int d^4z\langle 0|\hat{T}\phi(x_1)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_2)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_3)\phi(z)|0\rangle\langle 0|\hat{T}\phi(x_4)\phi(z)|0\rangle \\
= \quad & \prod_{i=1}^{4}\int\frac{d^4k_i}{(2\pi)^4}\frac{ie^{-ik_ix_i}}{k_i^2 - m^2 + i\epsilon}(2\pi)^4\delta^4(k_1 + \ldots + k_4)\,(-i\lambda)
\end{aligned} \tag{7.108}$$

As we see, the Feynman rule for the vertex in p-space for the ϕ^4 theory

$$-i\lambda \tag{7.109}$$

7.8 Exercise 6

(A)

- In the ϕ^4 theory compute the 6-point Green's function for connected diagrams up to order $\mathcal{O}(\lambda^2)$. How does it look in p-space?
- Draw all connected Feynman diagrams for the 8-point G.f. up to order $\mathcal{O}(\lambda^3)$. Use the Feynman rules and write down the same G.f. in p space.
- Compute the 2-point G.f. in p space up to $\mathcal{O}(\lambda)$. What goes wrong?

(B)

Consider a model with two real scalar fields ϕ and χ with the Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\chi)^2 - \frac{1}{2}m_\phi^2\phi^2 - \frac{1}{2}m_\chi^2\chi^2 - \frac{\lambda}{4}\phi^2\chi^2 \tag{7.110}$$

1. Derive the Feynman rules in p space.

2. Draw all connected Feynman diagrams for all possible 2-point Green functions to λ and 4 and 6 point G.f. to order λ^2.

3. Use the Feynman rules in p space for all the above cases.

8 * The generating functional (2h15min)

In this section we will rederive the Feynman diagrams and rules, but now, instead through the canonical quantisation), directly from the path integral. For this case we will generalise what we have already derived in the classical approximation. We will first consider a system of free field, and then introduce the interaction. We will see which Feynman diagrams we have to add to get quantum corrections, and how we should generalise the Feynman rules.

8.1 The example of a free field

Clearly this example is not interesting for its dynamics, since there is none. We will get acquainted with the methods and put the basis in the simplest case, the one we know how to solve exactly. We will then evolve perturbatively around this solution.

Let' s consider a free bosonic real field with zero spin. We have already encountered such a Lagrangian, the KG one:

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial\phi\right)^2 - \frac{m^2}{2}\phi^2 \tag{8.1}$$

Whatever else with a higher power of ϕ becomes interaction, the part which we know how to treat in 4 dimensions only perturbatively. This essentially follows from the fact that we are able to analytically perform only the Gauss integral, whatever is of higher power in the exponent is out of our reach. We have to put in a Gauss form the action with the Lagrangian plus the source term

$$\int d^4x \left[\frac{1}{2}\left(\partial\phi\right)^2 - \frac{m^2}{2}\phi^2 + J\phi\right] = \int d^4x \left[-\frac{1}{2}\phi\left(\partial^2 + m^2\right)\phi + J\phi\right] \tag{8.2}$$

Going to the right-hand-side we integrated by parts, taking into account that fields are negligible at infinity.

We integrate the exponent of this action over $\mathcal{D}\phi$, so we better get rid of the linear term in (8.2). We thus make a translation $\phi(x) \to \phi(x) + \phi_0(x)$ (the integral measure does not change under it, $\mathcal{D}\phi(x) \to \mathcal{D}\phi(x)$) and get

$$\int d^4x \left[-\frac{1}{2}\left(\phi + \phi_0\right)\left(\partial^2 + m^2\right)\left(\phi + \phi_0\right) + J\left(\phi + \phi_0\right)\right] \tag{8.3}$$

$\phi_0(x)$ can be arbitrary, so we choose it to be the solution of

Figure 5: The first approximation to the 2-point Green function.

$$\left(\partial^2 + m^2\right)\phi_0(x) = J(x) \tag{8.4}$$

What is left is thus

$$\int d^4x \left[-\frac{1}{2}\phi\left(\partial^2 + m^2\right)\phi + \frac{1}{2}J\phi_0\right] \tag{8.5}$$

$\phi_0(x)$ is essentially via (8.4) a functional of the source $J(x)$. In the previous sections we have already defined the Feynman propagator $\Delta(x)$ as a solution of

$$\left(\partial_x^2 + m^2\right) i\Delta(x-y) = -i\delta^4(x-y) \tag{8.6}$$

so that

$$\phi_0(x) = \int d^4y\, i\Delta(x-y)\, iJ(y) \tag{8.7}$$

We can evaluate the Feynman propagator, the most used form is through its Fourier transform:

$$i\Delta(z) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{-ikz}}{k^2 - m^2 + i\epsilon} \tag{8.8}$$

To evaluate the full generating functional we need to integrate over $\mathcal{D}\phi(x)$: whatever this integration gives, it is irrelevant because it gets canceled by the same factor in the denominator. What remains is (with the index 0 we denote the generating functional for the free field)

$$iW_0[J] = \frac{1}{2}\int d^4x \int d^4y\, iJ(x)\, i\Delta(x-y)\, iJ(y) \tag{8.9}$$

From here we can, using the definition (6.23), immediately check, that the only nonzero connected Green function for the free real scalar field is

$$G_0(x_1, x_2) = i\Delta(x_1 - x_2) \tag{8.10}$$

while all the others are zero. The free field can propagate undisturbed.

8.2 Interaction

So far we compute everything exactly for the case of a free field. Of course the interesting part stays in the interaction, which we will include now. We can do it for example by adding to the free Lagrangian the term (8.1)

$$\mathcal{L} = \mathcal{L}_0 - \frac{\lambda}{4!}\phi^4 \tag{8.11}$$

Before continuing a word on the mass dimensions of the various quantities. The coordinates have mass dimensions $[x] = -1$; since the action is dimensionless, we find that $[\mathcal{L}] = 4$ and $[\phi] = 1$, so that also the coupling constant λ is dimensionless (this is why we choose exactly such an interaction).

Already this apparently minimal addition makes the problem much harder, so that we are able to calculate the Green's functions only perturbatively. Let's assume that this extra interacting piece is small enough (i.e. $\lambda \ll 1$), so that $W[J]$ can be expanded in powers of λ.

Let's now see how the generating functional changes in the first order of powers of λ. We know already the result, since this is exactly the classical approximation. We will rederive it now once again, in a slightly different way. We have to do with functions and not operators, so we can separate the exponent part with interaction from the rest:

$$\begin{aligned}
\exp\left(iW[J]\right) &= \frac{\int \mathcal{D}\phi(x) \exp\left[i\int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 + J\phi\right)\right]}{\int \mathcal{D}\phi(x) \exp\left[i\int d^4x \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4\right)\right]} \\
&= \frac{\int \mathcal{D}\phi(x) \exp\left(-i\frac{\lambda}{4!}\int d^4x\phi^4\right) \exp\left[i\int d^4x \left(\mathcal{L}_0 + J\phi\right)\right]}{\left(\text{as above}\right)_{J=0}}
\end{aligned} \tag{8.12}$$

From the simplified example

$$\int dx f(x) e^{jx} = f\left(\frac{d}{dj}\right)\int dx e^{jx} \tag{8.13}$$

we guess an analogous form in the case of functionals:

$$\begin{aligned}
\exp\left(iW[J]\right) &= \frac{\exp\left(-i\frac{\lambda}{4!}\int d^4z \left(\frac{\delta}{\delta i J(z)}\right)^4\right)\int \mathcal{D}\phi(x) \exp\left[i\int d^4x \left(\mathcal{L}_0 + J\phi\right)\right]}{\left(\text{as above}\right)_{J=0}} \\
&= \frac{e^{\hat{O}} e^{iW_0[J]}}{\left(\text{as above}\right)_{J=0}} \to \frac{1 + \left(1 + \hat{O}\right)\left(e^{iW_0[J]} - 1\right)}{\left(\text{as above}\right)_{J=0}}
\end{aligned} \tag{8.14}$$

where we defined

$$\hat{O} = \int d^4z \left(-i\frac{\lambda}{4!}\right)\left(\frac{\delta}{\delta i J(z)}\right)^4 \tag{8.15}$$

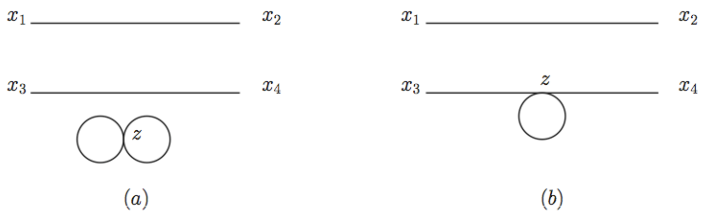


Figure 6: Two cases of non connected graphs.

in kept only terms up to order λ.

The denominator normalises $W[0] = 0$. In the following we will be interested only in Green's functions, so we will skip it.

Let's now consider the 4-point Green function: obviously we need to expand $\exp(iW_0[J])$ to the eighth order of the source J (four powers will be cleaned up by the four derivatives due to the interaction ϕ^4, the remaining four by the four derivatives from the definition of the 4-point Greens function from the generating functional (6.23)), which means to order $(iW_0[J])^4$. At first glance one could think that we can expand only up to the fourth derivative of the fourth power of $iW_0[J]$, and that we do not need to expand the term independent of $\hat{O}$. However, doing like that we would get also the unconnected diagrams, as for example those on fig. (6). The logarithm takes care of cancelling such unconnected diagrams, so we need to consistently expand the logarithm:

$$\begin{aligned} iW[J] &\rightarrow \log\left(1+\left(1+\hat{O}\right)\left(e^{iW_0[J]}-1\right)\right) \\ &= \left[\left(1+\hat{O}\right)\left(e^{iW_0[J]}-1\right)\right] - \frac{1}{2}\left[\left(1+\hat{O}\right)\left(e^{iW_0[J]}-1\right)\right]^2 \\ &+ \frac{1}{3}\left[\left(1+\hat{O}\right)\left(e^{iW_0[J]}-1\right)\right]^3 + \ldots = e^{-iW_0[J]}\hat{O}\left(e^{iW_0[J]}\right) \\ &\rightarrow \hat{O}\frac{(iW_0[J])^4}{4!} - (iW_0[J])\hat{O}\frac{(iW_0[J])^3}{3!} + \frac{(iW_0[J])^2}{2}\hat{O}\frac{(iW_0[J])^2}{2!} \end{aligned} \tag{8.16}$$

where we wrote down only the terms, which contribute to the final result.

We now take into account that third and higher derivatives of the functional $iW_0[J]$ vanish:

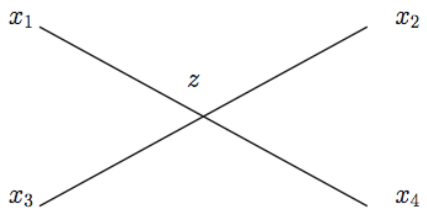


Figure 7: The first approximation to the 4-point Green function.

$$\begin{aligned}
\left(\frac{\delta}{\delta iJ(z)}\right)^4 \frac{(iW_0[J])^4}{4!} &= \left(\frac{\delta iW_0[J]}{\delta iJ(z)}\right)^4 + 6(iW_0[J])\left(\frac{\delta iW_0[J]}{\delta iJ(z)}\right)^2\left(\frac{\delta^2 iW_0[J]}{\delta(iJ(z))^2}\right) \\
&+ \frac{3}{2}(iW_0[J])^2\left(\frac{\delta^2 iW_0[J]}{\delta(iJ(z))^2}\right)^2 \\
\left(\frac{\delta}{\delta iJ(z)}\right)^4 \frac{(iW_0[J])^3}{3!} &= 6\left(\frac{\delta iW_0[J]}{\delta iJ(z)}\right)^2\left(\frac{\delta^2 iW_0[J]}{\delta(iJ(z))^2}\right) + 3(iW_0[J])\left(\frac{\delta^2 iW_0[J]}{\delta(iJ(z))^2}\right)^2 \\
\left(\frac{\delta}{\delta iJ(z)}\right)^4 \frac{(iW_0[J])^2}{2!} &= 3\left(\frac{\delta^2 iW_0[J]}{\delta(iJ(z))^2}\right)^2
\end{aligned} \tag{8.17}$$

If we sum up everything and take into account that

$$\frac{\delta iW_0[J]}{\delta iJ(z)} = \int d^4u\, iJ(u)\, i\Delta(u-z) \tag{8.18}$$

we see that the unconnected diagrams indeed cancel out. It remains

$$\begin{aligned}
iW[J] &\to \int d^4z \left(-i\frac{\lambda}{4!}\right)\left(\frac{\delta iW_0[J]}{\delta iJ(z)}\right)^4 \\
&= \int dx_1 \ldots \int dx_4 \frac{\lambda}{4!} J(x_1)\ldots J(x_4) \\
&\times \quad (-i)\int dz i\Delta(x_1-z)\ldots i\Delta(x_4-z)
\end{aligned} \tag{8.19}$$

The final result for the 4-point Green function is (up to order λ)

$$G(x_1,x_2,x_3,x_4) = (-i\lambda)\int d^4z i\Delta(x_1-z)i\Delta(x_2-z)i\Delta(x_3-z)i\Delta(x_4-z) \tag{8.20}$$

which we already know from (5.42), reminding that now $V'''' = \lambda$ and $V''' = 0$. For this reason we have, instead of the four graphs in fig. 3 only the first contribution, which is shown once again with the Feynman diagram on fig. 7

As we have already seen, we can systematise this procedure (Feynman rules for the theory ϕ^4 (8.11) in coordinate space):

1. first we draw all connected Feynman diagrams with given external points $x_1, \ldots, x_n$
2. the vertex is in the point z, to it belongs the term $-i\lambda$ (i because we always have $\exp(iS)$, $-\lambda$ because this is how the interaction term looks like in (8.11),4!, since this is exactly the right number of ways we can connect four external points to the vertex);
3. each link means a propagator $i\Delta(x_i - z)$;
4. we need to integrate over the whole spacetime z.

The next step is to Fourier transform all propagators on the right-hand side of the equation as in (8.8) and integrate over the coordinates of the interaction z.

$$\begin{aligned} G_1(x_1, x_2, x_3, x_4) &= \int \frac{d^4p_1}{(2\pi)^4} \int \frac{d^4p_2}{(2\pi)^4} \int \frac{d^4p_3}{(2\pi)^4} \int \frac{d^4p_4}{(2\pi)^4} \\ &\times (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4) e^{-i(p_1x_1 + p_2x_2 + p_3x_3 + p_4x_4)} \\ &\times (-i\lambda) \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \frac{i}{p_3^2 - m^2} \frac{i}{p_4^2 - m^2} \end{aligned} \tag{8.21}$$

In general we can define the n-point Green function in p space $G(p_1, \ldots, p_N)$ as

$$\begin{aligned} G_1(x_1, \ldots, x_N) &= \int \frac{d^4p_1}{(2\pi)^4} \cdots \int \frac{d^4p_N}{(2\pi)^4} e^{-i(p_1x_1 + \ldots + p_4x_4)} \\ &\times (2\pi)^4 \delta^4(p_1 + \ldots + p_N) G(p_1, \ldots, p_N) \end{aligned} \tag{8.22}$$

In our case this is

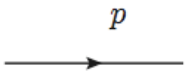
$$G(p_1, \ldots, p_4) = (-i\lambda) \frac{i}{p_1^2 - m^2} \frac{i}{p_2^2 - m^2} \frac{i}{p_3^2 - m^2} \frac{i}{p_4^2 - m^2} \tag{8.23}$$

The Feynman rules for the ϕ^4 theory are even simpler in p space:

1. draw all connected Feynman diagrams with given external particles $1, \ldots, n$
2. for every vertex write $-i\lambda$;
3. for each propagator take $i/(p^2 - m^2)$;
4. in every vertex the total 4-momentum is conserved;
5. the conservation of 4-momenta in all vertices does not determine all internal four-momenta; we have to integrate over all undetermined ones, for each of them we have then $\int d^4q/(2\pi)^4$;

6. the symmetry factor: if all 4! do not get canceled, one needs to take this into account.

Regarding the propagator and vertices, we denote them as



$$\frac{i}{p^2 - m^2}$$

$$-i\lambda$$

It is not strange that these rules are in accord with those obtained with the canonical quantisation.

8.3 Perturbation and loops

In the above case we computed the 4-point Green function to first order in powers of λ. What we get on top with this procedure are the quantum corrections, although we have not derived them yet. These are proportional to higher powers of λ on a given amplitude (Green function). We see this is the case from equation (6.12), taking into account our Lagrangian (8.11) and redefine the field as $\phi \to \hbar^{1/2}\phi$. In a given number of external particles (fixed number of derivatives over J) the quantities λ and $\hbar$ enter only through the combination $\hbar\lambda$, which proves what we said.

What type of diagrams give higher powers of λ in the ϕ^4 theory? For a diagram with E external legs, I internal propagators and L closed loops the following relation is always satisfied:

$$4 - E = -2I + 4L \tag{8.24}$$

which follows simply from dimensional analysis. In fact

$$[W] = M^0 \qquad , \qquad [J] = M^3 \tag{8.25}$$

and each functional derivative due to (5.36)

$$[\delta/\delta J] = M \tag{8.26}$$

so that the E-point Green function in x-space (6.23) has the dimension

$$[G^{(E)}(x)] = M^E \tag{8.27}$$

So the Green function in p-space (5.44)

$$[G^{(E)}(p)] = M^{4-3E} \tag{8.28}$$

Each loop gives an integration over the internal four-momenta (together $4L$ integrations), each propagator contributes two inverse mass dimensions (together $-2(I+E)$).

Due to the fact that 4 propagators exits from each vertex and that each internal propagator is shared by two vertices, we can similarly derive that

$$E + 2I = 4V_4 \tag{8.29}$$

where V_4 is the number of vertices. From here we get the relation

$$L = 1 + V_4 - E/2 \tag{8.30}$$

which tells us that at the fixed number of external legs the number of loops and the power of the coupling constant increase the same way.

So, let's repeat. To compute physical quantities (cross sections, decay widths), with n external (initial incoming and final outgoing) particles, we need the n-point Green function. This is illustrated with connected Feynman diagrams, while the Feynman rules relate to each diagram an analytic expression as function of the quantum numbers (momenta, spin, etc.) of the external particles.

8.4 Exercise 7

- The equivalence between the canonical formalism and the path integral is shown by the equation

$$\frac{\langle 0|\hat{T}F[\hat{\phi}]e^{i\int \mathcal{L}_{int}(\hat{\phi})}|0\rangle}{\langle 0|\hat{T}e^{i\int \mathcal{L}_{int}(\hat{\phi})}|0\rangle} = \frac{\int \mathcal{D}\phi(x)F[\phi]e^{i\int(\mathcal{L}_0(\phi)+\mathcal{L}_{int}(\phi))}}{\int \mathcal{D}\phi(x)e^{i\int(\mathcal{L}_0(\phi)+\mathcal{L}_{int}(\phi))}} \tag{8.31}$$

  where on the left-hand side $\hat{\phi}(x)$ are free field operators, which are expanded as usually with the creation and annihilation operators (canonical quantisation), while on the right-hand side $\phi(x)$ are simple functions.

  In the known ϕ^4 model show the equality (8.31) up to order λ for the case of 4-point Green function. Compare single terms in both cases.

9 Physical quantities (45 min)

Once we have a n-point Green function, the next step towards a physical quantity is the so called scattering or S matrix. The Green function is already almost the right thing, but surely not completely. The external particles obey the on-shell condition between energy and three-momentum, $p^2 - m^2 = 0$. If we impose this constraint, the external propagators of the Green function would diverge, since these are their poles. The correct prescription is however simple, one needs just to keep the residuum of this pole. The number of poles

tells us how many external particles are involved in the process, while the residuum is the amplitude: imagine we are interested in the scattering process of particles described the above Lagrangian ϕ^4, let's say a particle with four momentum p_1 hits another particle with four-momentum p_2, while the outgoing particles have four-momenta q_j, $j = 1, ..., n$. The amplitude for such a process is simply

$$\begin{aligned} iT(p_1, p_2; q_1, ...q_n) &= \lim_{p_i^2, q_j^2 \to m^2} \prod_{i=1}^{2} \left[-i\left(p_i^2 - m^2\right)\right] \prod_{j=1}^{n} \left[-i\left(q_j^2 - m^2\right)\right] \\ &\times \; G(p_1, p_2, q_1, ...q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - \ldots - q_n) \\ &= \; G_{amp}(p_1, p_2, q_1, ...q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - ... - q_n)\big|_{p_i^2, q_j^2 \to m^2} \\ &= \; iA(p_1, p_2, q_1, ..., q_n)(2\pi)^4 \delta^4(p_1 + p_2 - q_1 - \ldots - q_n) \end{aligned} \tag{9.1}$$

It is thus enough that for the computation of the Green functions we do not include the external propagators, those connected to external particles. We call such a Green function, G_{amp}, the amputated Green function. We see also that the total 4-momentum must be conserve. The difference between amputated Green function G_{amp} and the amplitude[4] A is in the fact that the four momenta of the external legs in the amputated Green function are arbitrary (subject only to the total conservation of four/momentum), while in the amplitude they satisfy the relation $p^2 \equiv p_0^2 - \overrightarrow{p}^2 = m^2$. When this relation is satisfied, we say that the external particle is on-shell: very far from the interaction point such particles behave in a good approximation as free and thus have in a good approximation a well defined four momentum (they are waves $\exp(-ip_i x_i)$ or $\exp(iq_j y_j)$).

Keep in mind that only external particles satisfy the on mass shell relation ($p^2 = m^2$). The internal particle of a Feynman diagram are virtual, their energy k^0 and three momentum $\overrightarrow{k}$ are not related in any way (they are independent quantities), so the internal propagators do not diverge (except in special cases, but this happens only when we integrate over the whole four/momentum of the virtual particle).

9.1 * The amplitude from the path integral

Equation (9.1) can be written more formally. First we define the so-called S-matrix

$$\hat{S} = \hat{N}\left[\exp\left(i\int dz\, \hat{\phi}(z)\left(\partial_z^2 + m^2\right)\frac{\delta}{\delta i J(z)}\right)\right](iW[J])\Big|_{J=0} \tag{9.2}$$

where the canonically quantised operator for the free real scalar field

$$\hat{\phi}(z) = \int \frac{d^3p}{(2\pi)^3 2p_0}\left(\hat{a}(p)e^{-ipz} + \hat{a}^\dagger(p)e^{ipz}\right) \tag{9.3}$$

satisfies the Klein-Gordon equation ($p^2 = m^2$, $p_0 > 0$)

[4] In a sloppy way we call with the same name ampliutude both the quantity T as A from the definition (9.1).

$$\left(\partial_z^2 + m^2\right)\hat{\phi}(z) = 0 \tag{9.4}$$

and the operator $\hat{N}$ is the normal ordered product. As we know, it acts on operators bringing all creation operators to the left of all creation operators.

The elements of the S-matrix are products in Hilbert space of initial states (kets) and final states (bras). We get the amplitude (9.1) as

$$iT(p_1, p_2; q_1, q_2) = \langle q_1 q_2 | \left(\hat{S} - \hat{I}\right) | p_1 p_2 \rangle \tag{9.5}$$

where $\hat{I}$ is the identity (particles that go out are the same and with the same momenta etc. as those that go in - nothing has happened).

9.2 The cross section

As we know already from quantum mechanics, the main information on the scattering process is in the cross section. When we have the amplitude, we can get the cross section similarly as in quantum mechanics. Let us remind this derivation. To simplify the problem, let us consider the scattering of two scalar particles into two scalar particles (the generalisation to a final state with more particles is direct, while we will comment later the cases of fermions or gauge bosons).

The amplitude for such a transition is

$$iT(k_1, k_2, p_3, p_4) = (2\pi)^4 \delta^4(k_1 + k_2 - p_3 - p_4) iA(k_1, k_2, p_3, p_4) \tag{9.6}$$

Strictly speaking, the initial particles are wave packets, i.e. distributions for different momenta with a sharp peak around p_1 or p_2 (such is the usual experimental situation). The initial state is described more precisely by

$$\int d\tilde{k}_1 \int d\tilde{k}_2 \, f_1(k_1) f_2(k_2) |k_1 k_2\rangle \tag{9.7}$$

where we used a short notation for the Lorentz invariant integration measure ($\omega^2 - \overrightarrow{k}^2 = m^2$)

$$d\tilde{k} \equiv \frac{d^3k}{(2\pi)^3 2\omega} \tag{9.8}$$

The square of the amplitude is then

$$\begin{aligned} & \int d\tilde{k}_1 \int d\tilde{k}_2 \int d\tilde{q}_1 \int d\tilde{q}_2 f_1(k_1) f_1^*(q_1) f_2(k_2) f_2^*(q_2) \\ \times \quad & (2\pi)^4 \delta^4(k_1 + k_2 - p_3 - p_4)(2\pi)^4 \delta^4(q_1 + q_2 - p_3 - p_4) \\ \times \quad & A(k_1, k_2, p_3, p_4) A^*(q_1, q_2, p_3, p_4) \end{aligned}$$

We rewrite the second δ-function in (9.9) as (using the first δ-function)

$$\begin{aligned}(2\pi)^4\delta^4(q_1+q_2-p_3-p_4) &= (2\pi)^4\delta^4(q_1+q_2-k_1-k_2) \\ &= \int d^4x e^{i(q_1+q_2-k_1-k_2)x} \end{aligned} \tag{9.9}$$

keeping in mind that the wave packets have a sharp peak at $p_1$and. p_2, so that we can write

$$\begin{aligned}&(2\pi)^4\delta^4(k_1+k_2-p_3-p_4)A(k_1,k_2,p_3,p_4)A^*(q_1,q_2,p_3,p_4) \\ &\approx (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\left|A(p_1,p_2,p_3,p_4)\right|^2 \end{aligned} \tag{9.10}$$

and define the Fourier transform of the distribution $f_j(k_j)$

$$\tilde{f}_j(x) = \int d\tilde{k}_j f_j(k_j) e^{-ik_j x} \tag{9.11}$$

so that the square of the amplitude becomes now (in the remaining δ-function we have already transformed k_i with p_i before)

$$\int d^4x \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\left|A(p_1,p_2,p_3,p_4)\right|^2 \tag{9.12}$$

In the final state we are not looking after states with infinitely sharply defined 4-momenta $\overrightarrow{p}_{3,4}$, but all states with momenta in the interval between $\overrightarrow{p}_{3,4}$ and $\overrightarrow{p}_{3,4}+d\overrightarrow{p}_{3,4}$, so we have to multiply the above expression with the Lorentz invariant number of such states

$$d\tilde{p}_3 d\tilde{p}_4 \tag{9.13}$$

All together the probability for transition for a unit volume and time is

$$\frac{dW}{VT} = \left|\tilde{f}_1(x)\right|^2 \left|\tilde{f}_2(x)\right|^2 \left|A(p_1,p_2,p_3,p_4)\right|^2 d\,Lips_2(p_1+p_2;p_3,p_4) \tag{9.14}$$

where the Lorentz invariant phase space for n particles with total 4-momentum P is defined generally as

$$d\,Lips_n(P;p_1,\ldots,p_n) = (2\pi)^4\delta^4(P-\sum_{i=1}^{n} p_i)\prod_{j=1}^{n}\frac{d^3p_j}{(2\pi)^3 2\omega_j} \tag{9.15}$$

The cross section $d\sigma$ is defined from

$$\frac{dW}{VT} = d\sigma j\rho \tag{9.16}$$

where the target density (let's be in the laboratory system, where the particles 1 have zero velocity) is

$$\rho = \left|\tilde{f}_1(x)\right|^2 2m \tag{9.17}$$

(we took the standard state normalisation $2p_0$), while the flux is made of particles 2:

$$j = \left|\tilde{f}_2(x)\right|^2 2|\overrightarrow{p}_2| \tag{9.18}$$

In general the seemingly non-invariant product $m|\overrightarrow{p}_2|$ can be written down in a Lorentz invariant way as

$$m|\overrightarrow{p}_2| = \left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2} \tag{9.19}$$

Finally the differential cross section is

$$d\sigma(p_1p_2 \to p_3p_4) = \frac{|A(p_1,p_2,p_3,p_4)|^2}{4\left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2}} d\,Lips_2(p_1+p_2;p_3,p_4) \tag{9.20}$$

This quantity is a Lorentz scalar, invariant of the choice of the coordinate system in which we measure.

The generalisation to scattering of two particles into $n+2$ particles is now logically unproblematic:

$$d\sigma(p_1p_2 \to p_3\dots p_{n+2}) = \frac{|A(p_1,p_2,p_3,\dots,p_{n+2})|^2}{4\left[(p_1p_2)^2 - p_1^2p_2^2\right]^{1/2}} d\,Lips_n(p_1+p_2;p_3,\dots,p_{n+2}) \tag{9.21}$$

9.3 Decay width

Usually we are not interested into the possibility that three or more initial particles scatter, since the probability for three particles to meet each other is usually negligibly small. There is however another possibility, i.e. a single initial particles decaying into final state. In this case we do not speak about cross section but the decay width. As we know, this changes in different inertial systems (particles which fly by live longer according to a static observer). What we usually mention is the decay width for a particle in its own inertial system. For it we can use almost the same equation as for the cross section (we skip the flux) and get

$$d\Gamma(P \to p_1\dots p_n) = \frac{|A(P,p_1,\dots,p_n)|^2}{2m} d\,Lips_n(P;p_1,\dots,p_n) \tag{9.22}$$

where in the system of the decaying particles we have of course $P = (m, \overrightarrow{0})$.

10 Quantum electrodynamics (QED) (2h 15 min)

10.1 The Lagrangian

This is the field theory which describes the electron and the photon., i.e. the electromagnetic interaction of a charged fermion. The Lagrangian, which describes it, has been already written down

$$\mathcal{L} = \bar{\psi}\, i\gamma^\mu \left(\partial_\mu - ieA_\mu\right)\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{10.1}$$

where the field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{10.2}$$

is found from the requirement of gauge invariance under local (spacetime dependent) phase U(1) transformation:

$$\psi \to e^{i\alpha(x)}\psi \ \ , \ \ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha(x) \tag{10.3}$$

10.2 Feynman rules

We can derive the Feynman rules similarly as in the case of a scalar field.

10.2.1 Propagators

Let's first consider the propagators. We get them in p space as the inverse of the quadratic Lagrangian term. In the case of a real scalar field the i times the action in the exponent of the path integral was

$$\begin{aligned} iS[\phi] &= i\int d^4z \mathcal{L}(\phi(z), \partial\phi(z)) = i\int d^4z \frac{1}{2}\left[(\partial\phi)^2 - m^2\phi^2\right] + \dots \\ &= -\frac{1}{2}\int d^4z\, \phi(z)\, i(\partial^2 + m^2)\phi(z) \\ &= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, \phi(-p)\ \left[(-i)(p^2 - m^2)\right]\phi(p) \end{aligned} \tag{10.4}$$

where we arrived tot he second line after integration by part, and to the third one by the Fourier transform

$$\phi(z) = \int \frac{d^4p}{(2\pi)^4}\phi(p)e^{-ipz} \tag{10.5}$$

The propagator of the real scalar field in p space is defined as the inverse of the square bracket in (10.4), i.e.

$$G_S(p) = \frac{i}{p^2 - m^2} \tag{10.6}$$

In the case of a complex scalar field, there would have been no $1/2$ at the beginning, but the result would have been the same as (10.6).

We can proceed in the same way for the case of fermions

$$G_F(p) = \frac{i}{\not{p} - m} \tag{10.7}$$

where

$$\not{p} \equiv \gamma^\mu p_\mu \tag{10.8}$$

A problem arises if we use the same procedure for the case of a vector field. In this case we get after the integration by parts and the transition to p space

$$i \int dz \left(-\frac{1}{4}\right) F_{\mu\nu} F^{\mu\nu} = -\frac{1}{2} \int \frac{d^4p}{(2\pi)^4} A^\mu(-p) \left[i \left(p^2 g_{\mu\nu} - p_\mu p_\nu \right) \right] A^\nu(p) \tag{10.9}$$

But the inverse of the square bracket does not exist, since the matrix $p^2 g_{\mu\nu} - p_\mu p_\nu$ has an eigenvector p^ν with zero eignevalue!

Although this may seem strange, we should have expected some problem since we never choose a gauge. So we now correct the mistake and without breaking Lorentz invariance add a term (4.9):

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} (\partial A)^2 \tag{10.10}$$

This can be understood even better in the path integral formulation. Without this extra term, which breaks gauge invariance, the integral looks

$$\int \mathcal{D} A_\mu e^{i S_{inv}[A]} \tag{10.11}$$

where S_{inv} is the invariant part of the action. This would be analogous to an integral of a periodic function of an angle α, for example $f(\cos(\alpha))$, from $-\infty$ to $+\infty$ instead from 0 to 2π. Said differently, the integrand (10.11) has too much symmetry. We have to integrate only over "one period", and so we choose the gauge, i.e. we exchange the above integral with for example

$$\int \mathcal{D} A_\mu \delta(\partial A) e^{i S_{inv}[A]} \tag{10.12}$$

We repeat the above calculation

$$
\begin{aligned}
& i\int dz\left[\left(-\frac{1}{4}\right)F_{\mu\nu}F^{\mu\nu}-\frac{1}{2\xi}(\partial A)^2\right] \\
= & -\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}A^\mu(-p)\left[i\left(p^2g_{\mu\nu}-(1-1/\xi)p_\mu p_\nu\right)\right]A^\nu(p) \qquad (10.13)
\end{aligned}
$$

We guess the inverse as

$$
G_V^{\nu\sigma}(p)=A(p^2)g^{\nu\sigma}+B(p^2)p^\nu p^\sigma \qquad (10.14)
$$

and of course require

$$
\left[i\left(p^2g_{\mu\nu}-(1-1/\xi)\,p_\mu p_\nu\right)\right]G_V^{\nu\sigma}(p)=g_\mu{}^\sigma \qquad (10.15)
$$

The final solution is

$$
G_V^{\nu\sigma}(p)=\frac{i}{p^2}\left(-g^{\nu\sigma}+(1-\xi)\frac{p^\nu p^\sigma}{p^2}\right) \qquad (10.16)
$$

A particularly useful and easy choice of the gauge is the so-called 't Hooft-Feynman gauge $\xi=1$, which gives asa very simple photon propagator

$$
G_V^{\nu\sigma}(p)=\frac{-ig^{\nu\sigma}}{p^2} \qquad (10.17)
$$

This is very often used, although the general choice (10.16) can sometimes be useful, since the parameter ξ must disappear at the end (a physical quantity cannot depend on the choice of gauge),, and so the general calculation represents a test.

10.2.2 Vertices

the next thing we have to determine are the Feynman rules for the vertices. We have to do only with one such vertex, since we have only one interaction term in the Lagrangian

$$
\mathcal{L}_{int}=e\bar{\psi}\not{A}\psi \qquad (10.18)
$$

We know how to determine the Feynman rule for this vertex. We are interested into a diagram, where the component ξ of a Dirac fermion with momentum p_1 emits the component η of a Dirac fermion with a momentum p_2 and the component μ of the photon with momentum q. We thus calculate the Green function in p space

$$
\int dxdydz\,G\left[\psi_\xi(x)\bar{\psi}_\eta(y)A_\mu(z)\right]e^{-ip_1x+ip_2y+iqz} \qquad (10.19)
$$

and then leave out all the three external propagators and the delta function. The result:

$$
ie(\gamma_\mu)_{\xi\eta} \qquad (10.20)
$$

10.2.3 Fermion loops

When using the Wick theorem we have often to interchange two fermion fields. Since they anti-commute (different from bosons which commute), we get during such operation an extra factor (-1). Let's see more precisely when this happens. It is enough to have a look to the only two possible cases to which all diagrams reduce to. In the first one the fermion fields are external, in the second they are internal. Because of the conservation of the fermion line (these are always continuous, which can be seen from the fact that fermion fields appear, due to Lorentz invariance, only quadratically in the Lagrangian) we cannot have a third case.

In the first case the part of the Green function we are interested in looks like

$$\langle 0|\hat{T}\psi_\xi(x)\bar{\psi}_\eta(y)\prod_{i=1}^{N} i\int dz_i e\bar{\psi}\not{A}\psi(z_i)|0\rangle \tag{10.21}$$

where all the vertices in between are connected by the fermion line. It is not difficult to convince oneself that there is no minus sign here.

The second case is the one with two internal vertices. We have thus a close loop

$$\langle 0|\hat{T}\prod_{i=1}^{N} i\int dz_i e\bar{\psi}\not{A}\psi(z_i)|0\rangle \tag{10.22}$$

Here it is clear that we get exactly a factor (-1).

Let's repeat: for every closed fermion loop we get an extra factor (-1). Of course the whole loop must be fermionic, from the beginning to the end.

10.2.4 External legs

The last thing we need to specify is what to add for an external particles. This is something new, which was not present in the case of a real scalar field. Let's try to guess. We get nothing of this kind for the case of a Green function, only in the case of the S-matrix, which has been written in (9.2) and which we want to generalise for the case of the vector field with spin 1 A_μ (the photon) and of the fermion particle of spin 1/2 (the electron).

In the case of the real scalar field we expanded the external asymptotically free field (which satisfies the KG equation) with creation and annihilation operators ($p_0 = \sqrt{m^2+\vec{p}^2}$ and m the mass of the scalar field)

$$\hat{\phi}(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}\left(a(p)e^{-ipx} + a^\dagger(p)e^{ipx}\right) \tag{10.23}$$

Similarly we do with the vector field, which satisfies the Maxwell equations in empty space ($p_0 = |\overrightarrow{p}|$, since the photon is massless)

$$\hat{A}^\mu(x) = \sum_{\lambda=1}^{2}\int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}\left(a_\lambda(p)\epsilon_\lambda^\mu(p)e^{-ipx} + a_\lambda^\dagger(p)\epsilon_\lambda^{*\mu}e^{ipx}\right) \tag{10.24}$$

where λ denotes one of the two possible transverse photon polarisations, $\epsilon_\lambda(p)$ is the polarisation vector which satisfies

$$p_\mu \epsilon^\mu_\lambda(p) = 0 \tag{10.25}$$

we get for the free fermion field (the same relation between the energy p_0 and the three momentum $\vec{p}$ is valid as in the previous case of a free scalar field), which satisfies the Dirac equation

$$\hat{\psi}(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(b_s(p) u_s(p) e^{-ipx} + d_s^\dagger(p) v_s(p) e^{ipx} \right) \tag{10.26}$$

where s denotes one of the two possible spin choices for the fermion, $b_s(p)$ and $d_s^\dagger(p)$ are the annihilation operator for the fermion (electron) and the creation operator for the antifermion (positron). Similarly

$$\bar{\hat{\psi}}(x) = \sum_{s=1}^{2} \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0} \left(d_s(p) \bar{v}_s(p) e^{-ipx} + b_s^\dagger(p) \bar{u}_s(p) e^{ipx} \right) \tag{10.27}$$

To find out which factors has to be used in the case of external fermions or gauge bosons we have to remember how we reached the amplitude (9.1) from the definition of the S-matrix (9.2) for the case of the scalar field.

Let's take the example of two particles with momenta p_1 and p_2 at the beginning, and two particles with momenta q_1 and q_2 at the end. The amplitude is by definition

$$\begin{aligned} A(p_1, p_2 \to q_1, q_2) \quad \times \quad & (2\pi)^4 \delta^4(p_1 + p_2 - q_1 - q_2) = \langle q_1, q_2 | \left(\hat{S} - \hat{I} \right) | p_1, p_2 \rangle \\ = \quad & \langle 0 | a(q_1) a(q_2) \left(\hat{S} - \hat{I} \right) a^\dagger(p_1) a^\dagger(p_2) | 0 \rangle \end{aligned} \tag{10.28}$$

The part of the scattering matrix we are interested in has the appropriate number of creation and annihilation operators:

$$\hat{S}-\hat{I} \;\rightarrow\; \frac{1}{4!}\,\hat{N}\left[\prod_{i=1}^{4}\int dz_i\hat{\phi}(z_i)i\left(\partial_{z_i}^2+m^2\right)\frac{\delta}{\delta iJ(z_i)}\right] iW[J]\Bigg|_{J=0} \tag{10.29}$$

$$= \frac{1}{4}\int dz_1\int dz_2\int dz_3\int dz_4 \tag{10.30}$$

$$\times \int\frac{d^3k_1}{(2\pi)^3 2k_{01}}a^\dagger(k_1)e^{ik_1z_1}\int\frac{d^3k_2}{(2\pi)^3 2k_{02}}a^\dagger(k_2)e^{ik_2z_2}$$

$$\times \int\frac{d^3k_3}{(2\pi)^3 2k_{03}}a(k_3)e^{-ik_3z_3}\int\frac{d^3k_4}{(2\pi)^3 2k_{04}}a(k_4)e^{-ik_4z_4}$$

$$\times\; i\left(\partial_{z_1}^2+m^2\right)\ldots i\left(\partial_{z_4}^2+m^2\right)G(z_1,z_2,z_3,z_4)$$

$$= \frac{1}{4}\int\frac{d^3k_1}{(2\pi)^3 2k_{01}}\ldots\int\frac{d^3k_4}{(2\pi)^3 2k_{04}}a^\dagger(k_1)a^\dagger(k_2)a(k_3)a(k_4)$$

$$\times\; (2\pi)^4\delta^4(k_1+k_2-k_3-k_4)G_{amp}(k_1,k_2,k_3,k_4) \tag{10.31}$$

where *amp* means amputated (i.e. the Green function without external legs). From here it follows

$$A(p_1p_2\rightarrow p_3p_4)=\frac{1}{i}G_{amp}(p_1,p_2,p_3,p_4)(2\pi)^4\delta^4(p_1+p_2-p_3-p_4) \tag{10.32}$$

which we already knew.

The above derivation helps in the generalisation to external particles with spin 1/2 and 1. In this case what gets generalised is of course also the form of the S matrix (the KG operator should be replaced for example with the Dirac operator, etc.), but on top of this it is clear that we get the following extra factors, if we compare (10.23) with (10.24), (10.26) and (10.27).

First let's see the case of an external photon with momentum q and polarisation λ. If it is in the initial state, we multiply the amplitude with

$$\epsilon_\lambda^\mu(q) \tag{10.33}$$

while if we find ti as a final state, we multiply with its complex conjugated value

$$\epsilon_\lambda^{\mu*}(q) \tag{10.34}$$

For a fermion with momentum p and spin s we use the following rules:

incoming particle:

$$u_s(p) \tag{10.35}$$

incoming antiparticle:

$$\bar{v}_s(p) \tag{10.36}$$

outgoing particle:

$$\bar{u}_s(p) \tag{10.37}$$

outgoing antiparticle:

$$v_s(p) \tag{10.38}$$

Each fermion line in the Feynman diagram has an arrow, which points in the case of a particle in the direction of the momentum (in the direction from initial state to the final state), while in the case of antiparticles the arrow points in the opposite direction than the momentum. We follow the rule to go in the Feynman diagram always in the opposite direction of the arrow (which shows the fermion number current): we start first with the outgoing particle, ($\bar{u}_s(p)$) or incoming antiparticle ($\bar{v}_s(p)$), and finish with the incoming particle ($u_s(p)$) or outgoing antiparticle ($v_s(p)$). So, as required by Lorentz invariance, we always have on the left spinors with a bar, and on right spinors without it.

10.2.5 Summary of Feynman rules for QED

Let's shortly summarise the rules:

- fermion propagator

$$\left(\frac{i}{\not{p} - m}\right)_{\xi\eta}$$

- photon propagator (covariant gauge)

$$\frac{i}{p^2}\left(-g^{\mu\nu} + (1-\xi)\frac{p^\mu p^\nu}{p^2}\right)$$

- vertex

$$ie(\gamma_\mu)_{\xi\eta}$$

- factor (-1) for each closed fermion loop
- incoming (outgoing) photon:

$$\epsilon^\mu_\lambda(q) \quad (\epsilon^{\mu *}_\lambda(q))$$

- incoming (outgoing) electron:

$$u_s(p) \quad (\bar{u}_s(p))$$

- incoming (outgoing) positron:

$$\bar{v}_s(p) \quad (v_s(p))$$

10.3 Exercise 8

- Compute the amplitude for the Compton scattering $e\gamma \to e\gamma$.
- Check that the amplitude vanishes if we replace $\epsilon_\mu(k) \to k_\mu$.
- Calculate the scattering cross section, average over the initial spins and polarisations, and sum up over the final ones. Use the above behaviour for $\epsilon_\mu(k) \to k_\mu$ to prove that the exchange

$$\sum_\lambda \epsilon_\lambda^\mu \epsilon_\lambda^{\nu *} \to -g^{\mu\nu}$$

  gives the correct result.
- Check that in the limit $\omega \to 0$ (initial photon energy) one gets the Thompson cross section ($\alpha \equiv e^2/(4\pi)$)

$$\sigma = \frac{8\pi\alpha^2}{3m^2}$$

10.4 The Ward-Takahashi equations

The original Lagrangian is invariant under the gauge transformation, but the need for choosing a gauge destroys it. Of course the physical results should not depend on the choice of gauge(α), since this is something like the choice of basis or coordinates. Because of this nontrivial requirement the Green function satisfy special equations, which we call Ward-Takahashi identities. These identities are automatic, it is enough to follow correctly the Feynman rules in calculating the Greens' functions, they are nevertheless useful since they represent a possible nontrivial test of the calculation.

Let's see how they look like. We can derive them form the definition of the generating functional. In the case of QED we have not even write it down, since we derived the Feynman rules directly. Let's do it now. We introduce the generating functional for both connected and unconnected diagrams

$$\begin{aligned} & Z\left[J_\mu, \eta, \bar{\eta}\right] = \exp\left(iW\left[J_\mu, \eta, \bar{\eta}\right]\right) \qquad (10.39) \\ = \quad & \frac{1}{N}\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp\left[i\int dz \left(\mathcal{L}(A,\psi,\bar{\psi}) + J^\mu A_\mu + \bar{\psi}\eta + \bar{\eta}\psi\right)\right] \end{aligned}$$

where the Lagrangian is divided into the gauge invariant piece and a pice which breaks gauge invariance

$$\mathcal{L} = \mathcal{L}_{inv} - \frac{1}{2\xi}\left(\partial A\right)^2 \tag{10.40}$$

while the factor N keeps the functional correctly normalised, $Z[0,0,0] = 1$ oz. $W[0,0,0] = 0$.

The sources J_μ are simple generalisations of the source in previous examples of a real scalar field, they are now 4 instead of a single one. A word more is needed for the fermion sources η and $\bar{\eta}$. Each of them has also 4 (Dirac) components, and we should be careful on their anti-commutation relations. The quantities η, $\bar{\eta}$, ψ, $\bar{\psi}$ are not operators, since we use the path integral, but they are Grassmanian numbers, which anti-commute among themselves. So for example

$$\frac{\delta}{\delta\bar{\eta}(x)}\int dz\bar{\eta}(z)\psi(z) = \psi(x) \tag{10.41}$$

as we expect also in the boson case, but we get another minus sign for

$$\frac{\delta}{\delta\eta(x)}\int dz\bar{\psi}(z)\eta(z) = -\bar{\psi}(x) \tag{10.42}$$

since the derivative operator (Grassman) needed to overtake the Grassmanian $\bar{\psi}$.

In the generating functional (10.39) let's transform the fields as

$$A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{10.43}$$

$$\psi \rightarrow \psi' = e^{i\alpha}\psi \tag{10.44}$$

$$\bar{\psi} \rightarrow \bar{\psi}' = \bar{\psi}e^{-i\alpha} \tag{10.45}$$

and expand all to order α. The integrand in the exponent changes first by

$$-\frac{1}{e\xi}(\partial A)\partial^2\alpha + J^\mu\frac{1}{e}\partial_\mu\alpha - i\alpha\bar{\psi}\eta + i\alpha\bar{\eta}\psi \tag{10.46}$$

and then, after after multiple integration by parts and the expansion of the exponent we get

$$\int \mathcal{D}A\mathcal{D}\psi\mathcal{D}\bar{\psi}\left(-\frac{1}{\xi}\partial^2\partial A - \partial J - ie\bar{\psi}\eta + ie\bar{\eta}\psi\right)e^{i\int dz\left(\mathcal{L}(A,\psi,\bar{\psi})+JA+\bar{\psi}\eta+\bar{\eta}\psi\right)} = 0 \tag{10.47}$$

With the known trick we exchange the fields in the integrand with functional derivatives over the sources and keep into mind Grassmanian particularities (10.42):

$$-\frac{1}{\xi}\partial_x^2\partial_x^\mu\frac{\delta Z}{\delta iJ^\mu(x)} - \partial^\mu J_\mu(x)Z - ie\eta_\vartheta(x)\frac{\delta Z}{\delta i\eta_\vartheta(x)} + ie\bar{\eta}_\vartheta(x)\frac{\delta Z}{\delta i\bar{\eta}_\vartheta(x)} = 0 \tag{10.48}$$

With the index ϑ we remind the reader, that we have to sum over this Dirac index. Now let's take a derivative over the whole expression

$$\frac{\delta^2}{\delta i\bar{\eta}_\zeta(x_1)\delta i\eta_\rho(x_2)} \tag{10.49}$$

and put all sources to zero: $J_\mu = \eta = \bar{\eta} = 0$. The second term in (10.48) vanishes, what remains is

$$\begin{aligned} - & \frac{1}{\xi}\partial_x^2\partial_x^\mu \left.\frac{\delta^3 Z}{\delta i J^\mu(x)\delta i\bar{\eta}_\zeta(x_1)\delta i\eta_\rho(x_2)}\right|_0 \\ - & e\delta(x-x_2)\left.\frac{\delta^2 Z}{\delta i\bar{\eta}_\zeta(x_1)\delta i\eta_\rho(x)}\right|_0 - e\delta(x-x_1)\left.\frac{\delta^2 Z}{\delta i\eta_\rho(x_2)\delta i\bar{\eta}_\zeta(x)}\right|_0 = 0 \end{aligned} \tag{10.50}$$

which is nothing else than

$$\begin{aligned} & \frac{1}{\xi}\partial_x^2\partial_x^\mu\langle 0|\hat{T}A_\mu(x)\psi_\zeta(x_1)\bar{\psi}_\rho(x_2)|0\rangle \\ + & e\delta(x-x_2)\langle 0|\hat{T}\psi_\zeta(x_1)\bar{\psi}_\rho(x_2)|0\rangle - e\delta(x-x_1)\langle 0|\hat{T}\psi_\zeta(x_1)\bar{\psi}_\rho(x_2)|0\rangle = 0 \end{aligned} \tag{10.51}$$

We interpret the above expectation values of the time ordered product as on the left-hand side of eq. (8.31) and take into account the generalised definition of the time ordered product (7.55) for fermions (with a minus)

$$\begin{aligned} \langle 0\left|\hat{T}\psi(x)\bar{\psi}(y)\right|0\rangle & \equiv \langle 0\left|\psi(x)\bar{\psi}(y)\right|0\rangle\Theta(x^0-y^0) \\ & - \langle 0\left|\bar{\psi}(y)\psi(x)\right|0\rangle\Theta(y^0-x^0) \end{aligned} \tag{10.52}$$

as we would expect to be for (anti-commuting) Grassmanian variables. The Fourier transform of the three point G.f. is

$$\begin{aligned} & \langle 0|\hat{T}A_\mu(x)\psi_\zeta(x_1)\bar{\psi}_\rho(x_2)|0\rangle \\ = & \int\frac{d^4q}{(2\pi)^4}e^{iqx}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1x_1}\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x_2}(2\pi)^4\delta^4(p_1-p_2-q) \\ \times & G^{(2)}_{\mu\mu'}(q)S_{\zeta\zeta'}(p_1)G^{(3)}_{amp}\left[A^{\mu'}(q)\psi_{\zeta'}(p_1)\bar{\psi}_{\rho'}(p_2)\right]S_{\rho'\rho}(p_2) \end{aligned} \tag{10.53}$$

where $G^{(3)}_{amp}$ is the three-point amputated G.f., i.e. without propagators at the external legs, while $G^{(2)}$ and S are the photon and electron propagators, all in p space:

$$\langle 0|\hat{T}A_\alpha(x)A_\beta(y)|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}G^{(2)}_{\alpha\beta}(p) \tag{10.54}$$

$$\langle 0|\hat{T}\psi_\zeta(x)\bar{\psi}_\rho(y)|0\rangle = \int\frac{d^4p}{(2\pi)^4}e^{-ip(x-y)}S_{\zeta\rho}(p) \tag{10.55}$$

Using (10.16) we get

$$\begin{aligned} & \frac{1}{\xi}\partial_x^2\partial_x^\mu\langle 0|\hat{T}A_\mu(x)\psi_\zeta(x_1)\bar{\psi}_\rho(x_2)|0\rangle \qquad (10.56)\\ = & \int\frac{d^4q}{(2\pi)^4}e^{iqx}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1x_1}\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x_2}(2\pi)^4\delta^4(p_1-p_2-q)\\ \times & \ (-1)q_\mu S_{\zeta\zeta'}(p_1)G^{(3)}_{amp}\left[A^\mu(q)\psi_{\zeta'}(p_1)\bar{\psi}_{\rho'}(p_2)\right]S_{\rho'\rho}(p_2)\end{aligned}$$

The second term in (10.51) is then

$$e\int\frac{d^4q}{(2\pi)^4}e^{iq(x-x_2)}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1x_1}\int\frac{d^4p_2'}{(2\pi)^4}e^{ip_2'x_2}(2\pi)^4\delta^4(p_1-p_2')S_{\zeta\rho}(p_1) \qquad (10.57)$$

and get after the redefinition $p_2' = p_2 + q$

$$e\int\frac{d^4q}{(2\pi)^4}e^{iqx}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1x_1}\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x_2}(2\pi)^4\delta^4(p_1-p_2-q)S_{\zeta\rho}(p_1) \qquad (10.58)$$

Similarly the third term is first

$$-e\int\frac{d^4q}{(2\pi)^4}e^{iq(x-x_1)}\int\frac{d^4p_1'}{(2\pi)^4}e^{-ip_1'x_1}\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x_2}(2\pi)^4\delta^4(p_1'-p_2)S_{\zeta\rho}(p_2) \qquad (10.59)$$

and after redefinition $p_1' = p_1 - q$

$$-e\int\frac{d^4q}{(2\pi)^4}e^{iqx}\int\frac{d^4p_1}{(2\pi)^4}e^{-ip_1x_1}\int\frac{d^4p_2}{(2\pi)^4}e^{ip_2x_2}(2\pi)^4\delta^4(p_1-p_2-q)S_{\zeta\rho}(p_2) \qquad (10.60)$$

After summing (10.56), (10.58) and (10.60) and skipping all common factors

$$-q_\mu S_{\zeta\zeta'}(p_1)G^{(3)}_{amp}\left[A^\mu(q)\psi_{\zeta'}(p_1)\bar{\psi}_{\rho'}(p_2)\right]S_{\rho'\rho}(p_2) + eS_{\zeta\rho}(p_1) - eS_{\zeta\rho}(p_2) = 0 \qquad (10.61)$$

Multiply everything with

$$\left(S^{-1}(p_1)\right)_{\alpha\zeta}\left(S^{-1}(p_2)\right)_{\rho\beta} \qquad (10.62)$$

(implicitly we sum up over all repeating Dirac indices) and finally get

$$-q_\mu G^{(3)}_{amp}\left[A^\mu(q)\psi_\alpha(p_1)\bar{\psi}_\beta(p_2)\right] + e\left(S^{-1}(p_2)\right)_{\alpha\beta} - e\left(S^{-1}(p_1)\right)_{\alpha\beta} = 0 \qquad (10.63)$$

This is a Ward identity, which is valid at all orders in perturbation theory. Let's check that it is valid at the lowest order. Keeping into account (10.20) and (10.7):

$$-q_\mu ie\left(\gamma^\mu\right)_{\alpha\beta} + e\frac{\not{p}_2 - m}{i} - e\frac{\not{p}_1 - m}{i} = 0 \tag{10.64}$$

which is indeed true due to momentum conservation $q = p_1 - p_2$.

During the derivation we assumed (10.56), which we know to be true at the leading order. It turns out that it is always true, although we will not prove it.

11 Some new quantities (45 min)

So far we introduced the generating functional, the Green functions, the amputated G.f., both in x and p space, the amplitude, the S-matrix. From the practical point of view, the amputated G.f. in p space, which becomes the amplitude once the external particles are put on-shell, seems the most useful.

There are some more basic quantities, from which one can easily compute the G.f.

11.1 Generating functional for connected diagrams

In the previous section we already introduced the generating functional for both connected and unconnected Feynman diagrams $Z[J]$:

$$Z[J] = \exp\left(iW[J]\right) \tag{11.1}$$

In calculations we always used only the generating functional for connected diagrams $W[J]$ and via

$$G^{(n)}(x_1, \ldots, x_n) = \left.\frac{\delta^n iW[J]}{\delta iJ(x_1)\ldots\delta iJ(x_n)}\right|_{J=0} \tag{11.2}$$

computed the connected Greens' functions.

11.2 1-point irreducible (1-PI) vertices

We divided the diagrams into connected and unconnected. The connected ones were all those, which could be divided into two parts only by cutting at least one line in the diagram (in an unconnected diagram we can divide it without cutting any line).

Let's now define even more special diagrams: those which can be divided into two parts by cutting at least two lines. Such F. diagrams describe 1-particle irreducible (1-PI) vertices or G.f. n-point 1-PI vertices in p space will then be those basic blocks which we mentioned before. We get them through the Legendre transformation

$$W[J] = \Gamma[\phi] + \int dx J(x)\phi(x) \tag{11.3}$$

As is $W[J]$ the generator of G.f. of all connected diagrams, so is $\Gamma[\phi]$ the generator of all 1-particle irreducible vertices. We will check this on some examples.

The quantities J and ϕ are independent, so (by definition)

$$\frac{\delta W[J]}{\delta J(x)} = \phi(x) \tag{11.4}$$

$$\frac{\delta \Gamma[\phi]}{\delta \phi(x)} = -J(x) \tag{11.5}$$

If we take the limit $J(x) \to 0$, we obtain the vacuum expectation value of the 1-point Green's function

$$\phi_{cl}(x) = \left.\frac{\delta W[J]}{\delta J(x)}\right|_{J=0} = \langle 0|\hat{T}\phi(x)|0\rangle \tag{11.6}$$

which is the solution to the equation of motion

$$\frac{\delta \Gamma[\phi]}{\delta \phi_{cl}(x)} = 0 \tag{11.7}$$

In the limit of constant fields this is the vacuum expectation value $\langle\phi\rangle$.

Generically

$$G(x_1, \ldots, x_n) \equiv \frac{\delta^n i W[J]}{\delta i J(x_1) \ldots \delta i J(x_n)} \tag{11.8}$$

$$\Gamma(x_1, \ldots, x_n) \equiv \frac{\delta^n \Gamma[\phi]}{\delta \phi(x_1) \ldots \delta \phi(x_n)} \tag{11.9}$$

The second derivatives are thus

$$G(x,y) \equiv \frac{\delta^2 i W[J]}{\delta i J(x) \delta i J(y)} = -i\frac{\delta \phi(x)}{\delta J(y)} = -i\frac{\delta \phi(y)}{\delta J(x)} \tag{11.10}$$

$$\Gamma(x,y) \equiv \frac{\delta^2 \Gamma[\phi]}{\delta \phi(x) \delta \phi(y)} = -\frac{\delta J(x)}{\delta \phi(y)} = -\frac{\delta J(y)}{\delta \phi(x)} \tag{11.11}$$

and are inverses of each other.

$$\int dy G(x,y)\Gamma(y,z) = \int dy (-i)\frac{\delta \phi(x)}{\delta J(y)}(-1)\frac{\delta J(y)}{\delta \phi(z)} = i\delta(x-z) \tag{11.12}$$

If we take a functional derivative

$$\frac{\delta}{\delta i J(v)} = -i \int du \, \frac{\delta \phi(u)}{\delta J(v)} \frac{\delta}{\delta \phi(u)} = \int du \, G(v,u) \frac{\delta}{\delta \phi(u)} \tag{11.13}$$

we get

$$\int dy G(x,y,v)\Gamma(y,z) + \int dy \int du\, G(x,y)G(v,u)\Gamma(u,y,z) = 0 \tag{11.14}$$

Let's multiply with $G(z,w)$ and integrate over $\int dz$:

$$iG(x,w,v) + \int dy \int dz \int du\, G(x,y)G(w,z)G(v,u)\Gamma(y,z,u) = 0 \tag{11.15}$$

In the limit $J \to 0$ the quantity $\Gamma(y,z,u)$ is obviously nothing else than (up to an irrelevant phase) the amputated 3-point G.f. in x space.

We can get also the inverse (in some sense) of the above equation:

$$\Gamma(y,z,u) = \int dx \int dw \int dv\, \Gamma(y,x)\Gamma(z,w)\Gamma(u,v)G(x,w,v) \tag{11.16}$$

Let's now take another derivative over $iJ(t)$. Using the equality (11.13) and taking repeatedly into account (11.15) we get after some change of variable names

$$\begin{aligned} G(x_1,x_2,x_3,x_4) &= \int dy_1 \int dy_2 \int dy_3 \int dy_4 \\ &\times\ G(x_1,y_1)G(x_2,y_2)G(x_3,y_3)G(x_4,y_4) \\ &\times\ G_{amp}(y_1,y_2,y_3,y_4) \end{aligned} \tag{11.17}$$

where the 4-point amputated G.f. is related to the 1-PI one via

$$\begin{aligned} -iG_{amp}(y_1,y_2,y_3,y_4) &= \Gamma(y_1,y_2,y_3,y_4) \\ &+\ i\int dz \int dz'\, \Gamma(y_1,y_2,z)G(z,z')\Gamma(z',y_3,y_4) \\ &+\ i\int dz \int dz'\, \Gamma(y_1,y_3,z)G(z,z')\Gamma(z',y_2,y_4) \\ &+\ i\int dz \int dz'\, \Gamma(y_1,y_4,z)G(z,z')\Gamma(z',y_2,y_3) \end{aligned} \tag{11.18}$$

It is clear that the 4-point G.f. (11.17) can be divided into two parts cutting a single line: we can cut one of the external propagators $G(x_i,y_i)$, or, in diagrams of the last three terms in (11.18), cutting the internal propagator $G(z,z')$.

Similarly we can continue to an arbitrary G.f. $G^{(n)}$, which can be always made out of basic blocks, 1-point irreducible vertices $\Gamma^{(i)}$, $i = 3,\ldots,n$ and propagators $G^{(2)}$.

All this can be easily translated into p-space. As usual we define these quantities as

$$
\begin{aligned}
G(x_1,\ldots,x_n) &= \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1x_1} \ldots \int \frac{d^4p_n}{(2\pi)^4} e^{-ip_nx_n} \\
&\times \tilde{G}^{(2)}(p_1)\ldots\tilde{G}^{(2)}(p_n) \\
&\times (2\pi)^4\delta^4(p_1+\ldots+p_n)\,\tilde{G}_{amp}(p_1,\ldots,p_n) \qquad (11.19)\\
\Gamma(x_1,\ldots,x_n) &= \int \frac{d^4p_1}{(2\pi)^4} e^{-ip_1x_1} \ldots \int \frac{d^4p_n}{(2\pi)^4} e^{-ip_nx_n} \\
&\times (2\pi)^4\delta^4(p_1+\ldots+p_n)\,\tilde{\Gamma}(p_1,\ldots,p_n) \qquad (11.20)
\end{aligned}
$$

which gives for the above 3- and 4-point amputated G.f.

$$
\begin{aligned}
\tilde{G}_{amp}(p_1,p_2,p_3) &= i\tilde{\Gamma}(p_1,p_2,p_3) \qquad (11.21)\\
\tilde{G}_{amp}(p_1,p_2,p_3,p_4) &= i\tilde{\Gamma}(p_1,p_2,p_3,p_4) \qquad (11.22)\\
&+ i\tilde{\Gamma}(p_1,p_2,-p_1-p_2)\,\tilde{G}(p_1+p_2)\,i\Gamma(-p_3-p_4,p_3,p_4) \\
&+ i\Gamma(p_1,p_3,-p_1-p_3)\,\tilde{G}(p_1+p_3)\,i\tilde{\Gamma}(-p_2-p_4,p_2,p_4) \\
&+ i\tilde{\Gamma}(p_1,p_4,-p_1-p_4)\,\tilde{G}(p_1+p_4)\,i\tilde{\Gamma}(-p_2-p_4,p_2,p_3)
\end{aligned}
$$

where one automatically takes into account that the sum of all momenta is conserved.

11.3 Exercise 9

- Compute the amplitude for the process $e^+e^- \to \mu^+\mu^-$ in QED at tree order.
- Write down all possible amplitudes for polarised fermions. In the calculation take the approximation $m_e = m_\mu = 0$.

11.4 Exercise 10

- Calculate the differential cross section $d\sigma/d\Omega$ and the total cross section σ for unpolarised fermions in the process of the previous exercise.

12 ∞ (6 h)

We have already encountered infinite integrals. They did not appear at the leading order (in tree level Feynman diagrams), but only in corrections (loops).

12.1 Regularisation

The first step towards taming infinities is its regularisation, i.e. a redefinition of the theory, so that everything is finite, and the original theory and infinities are recovered

back only after some limit. The most used regularisation is the dimensional regularisation, which conserves gauge invariance. Essentially the infinite integral in 4-dimensions gets generalised into an integral in d dimensions, where d is not necessarily an integer. The integrals can be evaluated for arbitrary d, and the results have usually poles for integer dimensions. From here the problems when we take the limit $d \to 4$ i.e. $\epsilon \equiv 4 - d \to 0$.

Let's see it more precisely for the case of ϕ^4. 2-point 1-PI G.f. is up to 1-loop order

$$\Gamma^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + \frac{1}{i}\frac{1}{2}(-i\lambda)\int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{i}{k^2 - m^2} \tag{12.1}$$

The integral is infinite for $\epsilon = 0$, since for large k it is essentially $\int_{-\infty}^{+\infty} k dk$. But this is no more true for an arbitrary non-integer ϵ. Let's have a look at Peskin's book:

$$\int \frac{d^d k}{(2\pi)^d} \frac{1}{(k^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \Delta^{-n+d/2} \tag{12.2}$$

Let's use it for our case, use the relations

$$\begin{aligned} \Gamma(x+1) &= x\Gamma(x) & (12.3)\\ \lim_{\epsilon \to 0} \Gamma(\epsilon/2) &= 2/\epsilon - \gamma + \mathcal{O}(\epsilon) & (12.4) \end{aligned}$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, and get up to order $\mathcal{O}(\epsilon^0)$

$$\Gamma^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + \frac{\lambda m^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln m^2\right) \tag{12.5}$$

The expression is clearly divergent in the physically sensible limit of 4 dimensions.

Similarly we get for the 4-point 1-PI G.f. up to 1-loop

$$\Gamma^{(4)}(p_i, m^2, \lambda, \epsilon) = -\lambda + I(s, m^2, \lambda, \epsilon) + I(t, m^2, \lambda, \epsilon) + I(u, m^2, \lambda, \epsilon) \tag{12.6}$$

where

$$I(p^2, m^2, \lambda, \epsilon) = \frac{1}{i}\frac{1}{2}(-i\lambda)^2 \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \frac{i}{k^2 - m^2} \frac{i}{(k+p)^2 - m^2} \tag{12.7}$$

and where we introduced the Mandelstam variables (remember that $p_1 + p_2 + p_3 + p_4 = 0$)

$$\begin{aligned} s &\equiv (p_1 + p_2)^2 = (p_3 + p_4)^2 & (12.8)\\ t &\equiv (p_1 + p_3)^2 = (p_2 + p_4)^2 & (12.9)\\ u &\equiv (p_1 + p_4)^2 = (p_2 + p_3)^2 & (12.10) \end{aligned}$$

for which

$$s + t + u = 4m^2 \tag{12.11}$$

(in general the right-hand side is equal to $m_1^2 + m_2^2 + m_3^2 + m_4^2$).

We again make use of Peskin's book. First we use

$$\frac{1}{A_1 \dots A_n} = \int_0^1 dx_1 \dots \int_0^1 dx_n \, \delta\left(\sum_{i=1}^n x_i - 1\right) \frac{(n-1)!}{(x_1 A_1 + \dots + x_n A_n)^n} \tag{12.12}$$

to get

$$I(p^2, m^2, \lambda, \epsilon) = \frac{\lambda^2}{2i} \int \frac{d^{4-\epsilon}k}{(2\pi)^{4-\epsilon}} \int_0^1 dx \frac{1}{[k^2 - m^2 + (p^2 + 2pk)x]^2} \tag{12.13}$$

Eith a change of variables

$$k' = k + xp \tag{12.14}$$

we get rid of the linear term in k, so that we can use again eq. (12.2). The result is

$$I(p^2, m^2, \lambda, \epsilon) = \frac{\lambda^2}{2(4\pi)^2} \left[\frac{2}{\epsilon} - \gamma + \ln 4\pi - \int_0^1 dx \ln\left(m^2 - p^2 x(1-x)\right)\right] \tag{12.15}$$

12.2 Renormalisation

We regularised the two- and four-point 1-PI G.f., but they still remain infinite in the limit $\epsilon \to 0$. Before performing such a limit we redefine the parameters of our model: m^2 and λ. We are allowed to do it, after all the physically measurable quantities are amplitudes or 1-PI G.f., and not necessarily the parameters of the Lagrangian. Som we write

$$m^2 = m_R^2 - \delta m^2 \tag{12.16}$$

$$\lambda = \lambda_R \mu^\epsilon Z_\lambda \tag{12.17}$$

With the index R we denote renormalised, i.e. finite, quantities. Of course we hide all the dangerous terms $1/\epsilon$ in δm^2, i.e. $\delta Z_\lambda = Z_\lambda - 1$, but in principle also something finite can be added. There are infinite possible choices, and the physical quantities again should not depend on that.

As we said, there is still a lot of choice to be done in defining (12.16) and (12.17). m_R and λ_R should of course be finite. We get rid of infinities if we choose for example the following renormalisation constraints:

$$\Gamma^{(2)}(m_R^2, m^2, \lambda, \epsilon) = 0 \tag{12.18}$$

$$\Gamma^{(4)}(0, m^2, \lambda, \epsilon) = -\lambda_R \mu^\epsilon \tag{12.19}$$

Formally δm^2 and δZ_λ are of higher order in coupling constant λ_R than m_R^2 or 1. This means that

$$\delta m^2 = -\frac{\lambda_R m_R^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi + 1 - \ln\frac{m_R^2}{\mu^2}\right) \tag{12.20}$$

$$\delta Z_\lambda = \frac{3\lambda_R}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi - \ln\frac{m_R^2}{\mu^2}\right) \tag{12.21}$$

Let's define now the renormalisation functions $\Gamma_R^{(n)}$, which are functions of the renormalised parameters as (as we will see later, this definition can be slightly changed in more general settings, but in our ϕ^4 example at 1-loop it is ok)

$$\Gamma_R^{(n)}(p_i, m_R^2, \lambda_R) = \Gamma^{(n)}(p_i, m^2, \lambda, \epsilon) \tag{12.22}$$

So the renormalised 1-PI G.f. (we came back to four dimensions in the meantime, i.e. we took the limit $\epsilon \to 0$)

$$\Gamma_R^{(2)}(p^2, m_R^2, \lambda_R) = p^2 - m_R^2 \tag{12.23}$$

i.e.

$$\begin{aligned}\Gamma_R^{(4)}(p_i, m_R^2, \lambda_R) = -\lambda_R &+ I_R(s, m_R^2, \lambda_R, m_R^2) \\ &+ I_R(t, m_R^2, \lambda_R, m_R^2) + I_R(u, m_R^2, \lambda_R, m_R^2)\end{aligned} \tag{12.24}$$

where we used

$$I_R(p^2, m_1^2, \lambda, m_2^2) = -\frac{\lambda^2}{2(4\pi)^2}\int_0^1 dx \ln\frac{m_1^2 - p^2 x(1-x)}{m_2^2} \tag{12.25}$$

12.3 A further complication: field renormalisation

We saw that the corrections to the two- and 4-point 1-PI Green functions are of the form

$$\Gamma^{(2)}(p^2, m^2, \lambda, \epsilon) = p^2 - m^2 + I^{(2)}(p^2, m^2, \lambda, \epsilon) \tag{12.26}$$

$$\Gamma^{(4)}(p_i, m^2, \lambda, \epsilon) = -\lambda + I^{(4)}(p_i, m^2, \lambda, \epsilon) \tag{12.27}$$

In the ϕ^4 case to 1-loop it was enough to renormalise the mass and the coupling constant. At two loops or in slightly more complicated theories it turns out that also $I^{(2)}$ is a function of p^2 and in fact proportional to $1/\epsilon$. In this case we have to renormalise also the field

$$\phi = Z_\phi^{1/2}\phi_R \tag{12.28}$$

This must be done even in the case with finite $I^{(2)}(p^2)$ (but infinite in the $\epsilon \to 0$ limit). This can be seen in the following way. Let's define m^2_{POL} as

$$\Gamma^{(2)}(m^2_{POL}, m^2, \lambda, \epsilon) = 0 \tag{12.29}$$

In our above examples we had $m^2_R = m^2_{POL}$, but this is not necessarily always true. The residuum of the pole of the propagator is not 1 in general, but

$$\left.\frac{d\Gamma^{(2)}}{dp^2}(p^2, m^2, \lambda, \epsilon)\right|_{p^2=m^2_{POL}} = Z_\phi^{-1} \neq 1 \tag{12.30}$$

For such cases the equation for the amplitude is invalid. In fact we derived it for the correctly normalised ($Z_\phi = 1$) fields, for which the KG operator transforms the external legs into a δ function. Let's call such a field ϕ_R and its source J_R. The source J of the non-canonically normalised field ϕ can be defined as

$$J_R \phi_R = J\phi \tag{12.31}$$

from where

$$J = Z_\phi^{-1/2} J_R \tag{12.32}$$

Schematically the amplitude is

$$\begin{aligned} A^{(n)} &\sim \left[i\left(\partial^2 + m^2\right)\right]^n \left(\frac{\delta}{\delta J_R}\right)^n W[J_R]\Big|_{J_R=0} \\ &\to Z_\phi^{-n/2} \left[i(\partial^2 + m^2)\right]^n G^{(n)} \end{aligned} \tag{12.33}$$

Taking into account

$$G^{(n)} \sim \left[G^{(2)}\right]^n G^{(n)}_{amp} \tag{12.34}$$

and

$$-i(\partial^2 + m^2) \left\{ \begin{array}{c} G^{(2)} \\ G^{(2)}_R \end{array} \right\} \sim \left\{ \begin{array}{c} Z_\phi \\ 1 \end{array} \right\} \tag{12.35}$$

eq. (12.33) becomes

$$A^{(n)} \sim Z_\phi^{n/2} G^{(n)}_{amp} \tag{12.36}$$

On another side

$$G^{(n)} \sim \langle \phi^n \rangle = Z_\phi^{n/2} G^{(n)}_R \tag{12.37}$$

from which it follows that (12.33) can be written

$$A^{(n)} \sim G^{(n)}_{amp,R} \tag{12.38}$$

as we expected.

The renormalised 1-PI G.f. is defined after comparison (12.33)-(12.38) as

$$\Gamma^{(n)}_R(p_i, m^2_R, \lambda_R) = Z^{n/2}_\phi(m^2, \lambda, \epsilon)\Gamma^{(n)}(p_i, m^2, \lambda, \epsilon) \tag{12.39}$$

The renormalisation constraints are

$$\begin{aligned}
\Gamma^{(2)}_R(m^2_R, m^2_R, \lambda_R) &= 0 && (12.40)\\
\frac{d\Gamma^{(2)}_R}{dp^2}(m^2_R, m^2_R, \lambda_R) &= 1 && (12.41)\\
\Gamma^{(4)}_R(0, m^2_R, \lambda_R) &= = -\lambda_R && (12.42)
\end{aligned}$$

from which we can compute δm^2, δZ_λ and $\delta Z_\phi = Z_\phi - 1$.

12.4 Counter-terms

We can exchange the bare quantities m^2, λ, ϕ with the renormalised m^2_R, λ_R, ϕ_R can be done from the start. The bare Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 \tag{12.43}$$

can be replaced by a sum of the renormalised Lagrangian

$$\mathcal{L}_R = \frac{1}{2}(\partial\phi_R)^2 - \frac{1}{2}m^2_R\phi^2_R - \frac{\lambda_R\mu^\epsilon}{4!}\phi^4_R \tag{12.44}$$

and counter-terms (CT)

$$\mathcal{L}_{CT} = \frac{1}{2}\delta Z_\phi(\partial\phi_R)^2 - \frac{1}{2}\left(\delta Z_\phi m^2_R - Z_\phi\delta m^2\right)\phi^2_R - \frac{\lambda_R\mu^\epsilon}{4!}\left(Z_\lambda Z^2_\phi - 1\right)\phi^4_R \tag{12.45}$$

We consider these new terms as real Lagrangian terms, derive from them the Feynman rules and take them into account in the Feynman diagrams. Clearly they are formally of higher power of the coupling constant λ_R. We can expand them in powers of the coupling constant (the coefficients can of course be singular in $1/\epsilon$)

$$\begin{aligned}
\delta m^2 &= \delta m^2_1\lambda_R + \delta m^2_2\lambda^2_R + \dots && (12.46)\\
\delta Z_\lambda &= \delta Z_{\lambda 1}\lambda_R + \delta Z_{\lambda 2}\lambda^2_R + \dots && (12.47)\\
\delta Z_\phi &= \delta Z_{\phi 1}\lambda_R + \delta Z_{\phi 2}\lambda^2_R + \dots && (12.48)
\end{aligned}$$

and should thus be formally taken into account at the right power of λ_R.

12.5 Different renormalisation conditions (schemes)

The renormalisation conditions above are not unique. We can choose them differently, and indeed this is what we often do. Let's now shortly describe the so called $\overline{MS}$ scheme, which is used especially in QCD.

We do not have here any constraints in special kinematic points as before, but we determine the counter-terms simply from the requirement of getting rid of all terms $2/\epsilon - \gamma + \ln 4\pi$. The name $\overline{MS}$ means minimal subtraction, the bar added is because on top of the divergent terms we included also the above finite terms.

In the previous ϕ^4 case would this mean

$$\begin{aligned} \delta m^2 &= -\frac{\lambda_R m_R^2}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) && (12.49)\\ \delta Z_\lambda &= \frac{3\lambda_R}{2(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) && (12.50)\\ \delta Z_\phi &= 0 && (12.51) \end{aligned}$$

Of course the renormalised 1-PI G.f. look different than before

$$\begin{aligned} \Gamma_R^{(2)}(p^2, m_R^2, \lambda_R, \mu) &= p^2 - m_R^2 + \frac{\lambda_R m_R^2}{2(4\pi)^2}\left(1 - \ln\frac{m_R^2}{\mu^2}\right) && (12.52)\\ \Gamma_R^{(4)}(p_i, m_R^2, \lambda_R, \mu) &= -\lambda_R + I_R(s, m_R^2, \lambda_R, \mu^2) && (12.53)\\ &+ I_R(t, m_R^2, \lambda_R, \mu^2) + I_R(u, m_R^2, \lambda_R, \mu^2) \end{aligned}$$

However we have to remember that now m_R and λ_R are something completely different from before (different numbers).

12.6 Exercise 11

- To the ϕ^4 Lagrangian add a fermion and the terms

$$\bar{\psi}\left(i\partial\!\!\!/ - m_f\right)\psi - y\phi\bar{\psi}\psi \qquad (12.54)$$

- Compute a correction to the vertex $\bar{\psi}\psi\phi$ at one loop and renormalise y in the $\overline{MS}$ scheme.
- Calculate the vertex ϕ^3 at one loop and show that it diverges.
- From here it follows that we should have added already from the beginning a term $M\phi^3$, and the renormalisation of M would cancel the divergence of the vertex ϕ^3, which appears at one loop, similarly as the renormalisation of y cancels the divergence of the vertex $\bar{\psi}\psi\phi$.
- What happens in the $m_f \to 0$ limit? Why the vertex ϕ^3 in this case does not get generated at one loop for $M = 0$? Which is the symmetry that forbids it?

12.7 Renormalisation group equations

The above expressions are a bit surprising at first sight, since it looks like the physical quantities depends on an extra arbitrary parameter μ, which entered into the result via dimensional regularisation and the requirement of dimensionless renormalised coupling constant. This would have usually[5] happened even in theories without any mass parameter: renormalisation itself generates a dimensionfull factor, which we call dimensional transmutation. This way or another, the dependence over the unknown μ is only apparent. We will show that the renormalised parameters (m_R^2, λ_R in the $\overline{MS}$ scheme) depend also on μ, so that the total effect gets neutralised and the physical quantities are independent of μ.

To check this is of course important, but we will get another bonus. We will in fact show that we can, using this trick, resum up an infinite series of dangerous large logarithms, which show up in some kinematical limits, and so save the perturbation expansion.

Let's start recognising that all bare parameters are independent on μ.

$$0 = \mu \frac{d\lambda}{d\mu} = \mu \frac{d}{d\mu} \left(\lambda_R \mu^\epsilon Z_\lambda(\lambda_R, \epsilon) \right) \tag{12.55}$$

From here we get the equation

$$\epsilon \lambda_R + \mu \frac{d\lambda_R}{d\mu} \left(1 + \lambda_R \frac{\partial \ln Z_\lambda(\lambda_R, \epsilon)}{\partial \lambda_R} \right) = 0 \tag{12.56}$$

The ansatz for the solution is an expansion in positive powers of the renormalised coupling constant (higher orders become important only and the level of two loops or later)

$$\mu \frac{d\lambda_R}{d\mu} = A\lambda_R + B\lambda_R^2 + \dots \tag{12.57}$$

Taking into account (12.50) we get

$$\mu \frac{d\lambda_R}{d\mu} = -\epsilon \lambda_R + \frac{3\lambda_R^2}{(4\pi)^2} \tag{12.58}$$

Now we can safely take the limit $\epsilon \to 0$ and get an equation, which describes the change of the renormalised coupling constant with the scale μ.

$$\mu \frac{d\lambda_R}{d\mu} = \frac{3\lambda_R^2}{(4\pi)^2} \quad (\equiv \beta(\lambda_R)) \tag{12.59}$$

The expansion in powers of λ_R on the righthand-side is called the β function. The equation can be integrated

$$\lambda_R(\mu) = \frac{\lambda_R(\mu_0)}{1 - \frac{3\lambda_R(\mu_0)}{(4\pi)^2} \ln \frac{\mu}{\mu_0}} \tag{12.60}$$

[5] except in conformal theories

Increasing the scale we arrive at a singularity, since the beta function is positive. This means that the coupling constant gets increased with the scale at which we measure it. Of course the approximation breaks down before we reach the singularity, called the Landau pole, since λ_R becomes large enough so that we cannot stop the expansion at one loop level.

The same can be done with the mass:

$$0 = \mu \frac{dm^2}{d\mu} = \mu \frac{d}{d\mu} \left(m_R^2 - \delta m^2(m_R^2, \lambda_R, \epsilon) \right) \tag{12.61}$$

from where

$$\mu \frac{dm_R^2}{d\mu} + \left(m_R^2 \mu \frac{d\lambda_R}{d\mu} + \lambda_R \mu \frac{dm_R^2}{d\mu} \right) \frac{1}{2(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi \right) = 0 \tag{12.62}$$

Similarly as before we expand

$$\mu \frac{dm_R^2}{d\mu} = C\lambda_R + \dots \tag{12.63}$$

and using (12.58) we get at first order

$$\mu \frac{dm_R^2}{d\mu} = \frac{\lambda_R}{(4\pi)^2} m_R^2 \tag{12.64}$$

the solution of which is

$$m_R^2(\mu) = \frac{m_R^2(\mu_0)}{\left(1 - \frac{3\lambda_R(\mu_0)}{(4\pi)^2} \ln \frac{\mu}{\mu_0} \right)^{1/3}} \tag{12.65}$$

12.8 An explicit example

Imagine we have an accelerator of particles ϕ, which we postulate can be described by our old friend, the ϕ^4 Lagrangian. We are interested in elastic scattering of two particles (this means that the number and type of particles at the beginning and at the end are the same).

The incoming particles have momenta in the centre-of-mass system (let it be equal to the laboratory system)

$$p_1^\mu = (E, 0, 0, p) \tag{12.66}$$
$$p_2^\mu = (E, 0, 0, -p) \tag{12.67}$$

while the final particles have

$$p_3^\mu = -(E, p\sin\theta, 0, p\cos\theta) \tag{12.68}$$
$$p_4^\mu = -(E, -p\sin\theta, 0, -p\cos\theta) \tag{12.69}$$

where

$$E^2 - p^2 = m^2 \tag{12.70}$$

and m^2 is the pole of the propagator. At tree order this is just the mass parameter of the Lagrangian, while at the one loop level this is m_R^2 defined via (12.23) in our case, or via (12.29) in general. This is also the mass felt by gravity in the Newton's law.

Imagine we can somehow measure the scattering cross section σ_{exp} in the limit where the initial particles have a very small momentum (this is usually hard since it essentially mean that the initial particles do not move, so there is no scattering; we will imagine it as a kind of limit). We choose this definition only to get simpler equations, in the real case we measure in a different kinematical point. In this convenient limiting ($p \to 0$) kinematical phase space point the Mandelstam variables are

$$s \to 4m_R^2 \tag{12.71}$$
$$t \to 0 \tag{12.72}$$
$$u \to 0 \tag{12.73}$$

At tree order the amplitude

$$\Gamma^{(4)}(p_i, m, \lambda) = -\lambda \tag{12.74}$$

is independent of momenta. Since we measure the cross section, we have to evaluate the phase space etc. from formula (9.20). The result is

$$\sigma = \frac{\left|\Gamma^{(4)}\right|^2}{64\pi E^2} = \frac{\lambda^2}{64\pi E^2} \tag{12.75}$$

The coupling constant λ is determined from the measured σ_{exp} in the limit (12.71)-(12.73):

$$\sigma_{exp} = \frac{\lambda^2}{64\pi m^2} \tag{12.76}$$

So in general

$$\sigma = \sigma_{exp} \left(\frac{m}{E}\right)^2 \tag{12.77}$$

Imagine now that we are not happy with the precision of comparison between theory and experiment. In this case we can use the results of the next approximation, i.e. in the one loop approximation. For the renormalisation constraints we choose (12.40)-(12.42), so that the expression for the 4-point Green function is given in (12.24). This is in the kinematical point we measure (12.71)-(12.73) equal to (we have to evaluate the integrals of type (12.25))

$$\Gamma^{(4)}_{exp} = -\lambda_R + \frac{\lambda_R^2}{(4\pi)^2} \tag{12.78}$$

When squaring $\Gamma^{(4)}$we should not take into account the largest power λ_R^4, since this is already of higher order (if we calculated up to two loops, we would have also the order λ^3 in (12.78), which would, multiplied with the tree order, give also the same power), so that

$$\sigma_{exp} = \frac{1}{64\pi m_R^2}\left(\lambda_R^2 - 2\frac{\lambda_R^3}{(4\pi)^2}\right) \tag{12.79}$$

Through this equation we calculate the value of λ_R. If it is small enough, i.e. if

$$\frac{\lambda_R}{(4\pi)^2} \ll 1 \tag{12.80}$$

the perturbative expansion of the Greens' functions converge and the computation is consistent.

We calculate the scattering cross section in an arbitrary kinematical point using (9.20) and (12.24), where however now λ_R is numerically fit through (12.79).

What if, instead of (12.42), we had used a different constraint, for example in the limit (12.71)-(12.73)

$$\Gamma^{(4)}_R \to -\lambda'_R \tag{12.81}$$

Then

$$\sigma_{exp} = \frac{\lambda_R'^2}{64\pi m_R^2} \tag{12.82}$$

As we see, the numerical value of λ'_R is different from λ_R, the final result however is, the same, up to corrections formally of higher order in powers of the coupling constant. Numerically the physical result (the scattering cross section as a function of momenta) can slightly depend on the choice of the renormalisation scheme, but the difference is, if the convergence is good (12.80), small enough .

What if we use instead the $\overline{MS}$ scheme? To avoid extra complications, let's use this scheme only for the coupling constant, while let the mass be still the propagator pole:

$$\Gamma^{(2)}_R = p^2 - m_R^2 \tag{12.83}$$

$$\begin{aligned}\Gamma^{(4)}_R &= -\lambda_R(\mu) + I_R(s, m_R^2, \lambda_R(\mu), \mu^2) \\ &+ I_R(t, m_R^2, \lambda_R(\mu), \mu^2) + I_R(u, m_R^2, \lambda_R(\mu), \mu^2)\end{aligned} \tag{12.84}$$

The coupling constant $\lambda_R(\mu)$ is now, differently from previous one, the running one, dependent on the parameter μ through the renormalisation group equation (12.59). We

should not worry about it. Using the same procedure as before we get first in the limit (12.71)-(12.73)

$$\Gamma^{(4)}_{exp} \to -\lambda_R(\mu) + \frac{\lambda_R^2(\mu)}{(4\pi)^2}\left(1 + 3\ln\frac{\mu}{m_R}\right) \tag{12.85}$$

At $\mu = m_R$ we have the same expression as before, see (12.78)

$$\Gamma^{(4)}_{exp} = -\lambda_R(m_R) + \frac{\lambda_R^2(m_R)}{(4\pi)^2} \tag{12.86}$$

so $\lambda_R(m_R)$ is numerically equal to λ_R, which is got through eq. (12.79). When we integrate (12.59), we take into account the following boundary condition:

$$\lambda_R(\mu) = \frac{\lambda_R(m_R)}{1 - \frac{3\lambda_R(m_R)}{(4\pi)^2}\ln\frac{\mu}{m_R}} \tag{12.87}$$

The careful reader can now ask, if the last expression can be maintained in this form, or if it should be expanded in powers of $\lambda_R(m_R)$, as usual. But here there is a difference, since we have here also $\ln(\mu/m_R)$, which could be in principle large. If it is small enough

$$\frac{\lambda_R(m_R)}{(4\pi)^2}\ln\frac{\mu}{m_R} \ll 1 \tag{12.88}$$

one can safely expand to quadratic order, insert it into (12.84), and get

$$\begin{aligned}\Gamma^{(4)}(p_i, m_R^2, \lambda_R) = -\lambda_R \;&+\; I_R(s, m_R^2, \lambda_R, m_R^2) \\ &+\; I_R(t, m_R^2, \lambda_R, m_R^2) + I_R(u, m_R^2, \lambda_R, m_R^2)\end{aligned} \tag{12.89}$$

totally the same as (12.24). In this case there is no difference between the original scheme and the $\overline{MS}$ scheme.

Once get a difference however, when (12.88) is not satisfied, but (12.80) is. This is obviously possible only at very (exponentially) large ratios μ/m_R. In this case the solution of the RG equation helped to sum up all powers

$$\left(\frac{\lambda_R(m_R)}{(4\pi)^2}\ln\frac{\mu}{m_R}\right)^n \tag{12.90}$$

which appear at the level of n-th loop. Of course these n-th loops for $n > 1$ has not been calculated, but the solution of the RG equation allows us to resum these dominant terms in this limit.

The careful reader will obviously be again confused. Why should we use at all these large (or small) ratios μ/m_R, after all we said we are allowed to (at least in principle) use any such ratio. Such a reason could be for example the behaviour of the cross section at very large energies, for example when $s/m_R^2 \to \infty$. In the case (12.24) the correction to tree order$-\lambda_R$ would be of the form

$$\frac{\lambda_R^2}{(4\pi)^2}\ln\frac{s}{m_R^2} \tag{12.91}$$

and this number is not necessarily less than one. Higher terms would contribute higher powers

$$\left(\frac{\lambda_R^2}{(4\pi)^2}\ln\frac{s}{m_R^2}\right)^n \tag{12.92}$$

and the expansion would not converge.

This problem is solved with the running coupling constant (12.87) and the use of (12.85) for $\mu \approx E$. Then the ratios

$$\left(\frac{\lambda_R(m_R)}{(4\pi)^2}\ln\frac{s}{\mu^2}\right)^n \tag{12.93}$$

which appear in (12.85) and eventually at higher orders, are small enough not to spoil the convergence. Of course we included all these contributions of higher orders into the running coupling constant

$$\lambda_R(E) = \frac{\lambda_R(m_R)}{1 - \frac{3\lambda_R(m_R)}{(4\pi)^2}\ln\frac{E}{m_R}} \tag{12.94}$$

The whole procedure converges under condition

$$\frac{\lambda_R(E)}{(4\pi)^2} \ll 1 \tag{12.95}$$

which is not true if we are too close to the Landau pole.

We see that resummation saved the perturbative expansion in the case of very large (or small) energies. This is why the scheme with running coupling constant is useful.

12.9 Exercise 12

- Compute the differential unpolarised cross section for $e^-\mu^- \to e^-\mu^-$. Make use of the results for $e^+e^- \to \mu^+\mu^-$. In a similar way use the known results for Compton scattering $e\gamma \to e\gamma$ to calculate the cross section for annihilation $e^+e^- \to \gamma\gamma$. Comment the "crossing" symmetry.

12.10 Exercise 13

- Consider the system with interaction $\lambda\sigma\phi_1\phi_2$. If the corresponding masses satisfy $m_\sigma > m_1 + m_2$ calculate the decay width Γ_σ at tree level. Show that at the first nonzero order in λ the optical theorem is valid

$$m_\sigma \Gamma_\sigma = Im\left(\Gamma^{(2)}_{\sigma\sigma}(m_\sigma^2)\right)$$

where $\Gamma^{(2)}_{\sigma\sigma}(p^2)$ is the 2-point 1-PI G.f. for σ.

13 Quantum electrodynamics up to one loop (1h30min)

In this section we will use the method of counter-terms, all quantities will be renormalised, but for simplicity we will write them without the sign R.

The tree level renormalised Lagrangian for QED is then (we choose the Feynman gauge $\xi = 1$)

$$\mathcal{L} = \bar{\psi} i \partial\!\!\!/ \psi - m\bar{\psi}\psi + e\mu^{\epsilon/2}\bar{\psi} A\!\!\!/ \psi - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial A)^2 \tag{13.1}$$

while the counter-terms are of the form

$$\mathcal{L}_{CT} = \delta Z_2 \bar{\psi} i \partial\!\!\!/ \psi - \delta m \bar{\psi}\psi + \delta Z_1 e\mu^{\epsilon/2}\bar{\psi} A\!\!\!/ \psi - \frac{\delta Z_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \tag{13.2}$$

We consider these new terms as small perturbations. They are formally of higher order in powers of the coupling constant (δZ_i and δm are order e), so we consider them as interaction. We write for them the following Feynman rules

- counter-term for the fermion propagator

$$i\left(\delta Z_2 p\!\!\!/ - \delta m\right) \tag{13.3}$$

- counter-term for the photon propagator

$$-i\delta Z_3\left(p^2 g_{\mu\nu} - p_\mu p_\nu\right) \tag{13.4}$$

- vertex counter-term

$$i\delta Z_1 e\mu^{\epsilon/2}\gamma_\mu \tag{13.5}$$

We check the first one by considering the whole quadratic part of the fermion

$$(1+\delta Z_2)\bar{\psi} i\partial\!\!\!/\psi - (m+\delta m)\bar{\psi}\psi$$

Clearly the propagator is in this case

$$\frac{i}{(1+\delta Z_2)\not{p} - (m+\delta m)}$$

If we now expand, we get (take into account that $(M_1 M_2)^{-1} = M_2^{-1} M_1^{-1}$)

$$\begin{aligned}\frac{i}{(\not{p}-m)+(\delta Z_2 \not{p} - \delta m)} &= \frac{i}{(\not{p}-m)[1+(\not{p}-m)^{-1}(\delta Z_2 \not{p} - \delta m)]} \\ &= \frac{i}{\not{p}-m} + \frac{i}{\not{p}-m}\left[i\left(\delta Z_2 \not{p} - \delta m\right)\right]\frac{i}{\not{p}-m} + \ldots\end{aligned}$$

which coincides with the above rule.

It si not difficult to check also the complete quadratic photon term

$$-\frac{1+\delta Z_3}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 - \frac{1}{2}(\partial A)^2$$

gives the following propagator

$$\frac{-i}{(1+\delta Z_3)p^2}\left(g_{\mu\nu} + \delta Z_3 \frac{p_\mu p_\nu}{p^2}\right)$$

which can be expanded

$$\frac{-ig_{\mu\nu}}{p^2} + \frac{-ig_{\mu\alpha}}{p^2}\left[-i\delta Z_3\left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right)\right]\frac{-ig_{\beta\nu}}{p^2} + \ldots$$

as with the above rules.

The vertex correction is obvious.

13.1 Electron propagator

The one loop contribution to it is usually denoted as

$$\frac{i}{\not{p}-m}\left(-i\Sigma(p)\right)\frac{i}{\not{p}-m}$$

which gives for the 2-point 1-PI G.f. for the electron

$$\Gamma^{(2)}_\psi(p) = \not{p} - m + \delta Z_2 \not{p} - \delta m - \Sigma(p) \tag{13.6}$$

Now let's calculate:

$$-i\Sigma(p) = \left(ie\mu^{\epsilon/2}\gamma^\alpha\right)\int \frac{d^d k}{(2\pi)^d}\frac{i}{\not{k}-m}\frac{-ig_{\alpha\beta}}{(k-p)^2}\left(ie\mu^{\epsilon/2}\gamma^\beta\right) \tag{13.7}$$

As common we use the identity

$$\frac{1}{\not{k} - m} = \frac{\not{k} + m}{k^2 - m^2} \tag{13.8}$$

The definition for the γ matrices

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{13.9}$$

is valid in arbitrary (also non-integer) dimensions, so ($g^\alpha{}_\alpha = d$)

$$\begin{aligned} \gamma^\alpha \gamma_\alpha &= d & (13.10)\\ \gamma^\alpha \gamma^\mu \gamma_\alpha &= -(d-2)\gamma^\mu & (13.11) \end{aligned}$$

We first get

$$\Sigma(p) = -ie^2\mu^\epsilon \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)\not{k} + dm}{(k^2 - m^2)(k-p)^2} \tag{13.12}$$

and withe the usual trick

$$\Sigma(p) = -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)\not{k} + dm}{\left[k^2 - m^2(1-x) + (p^2 - 2kp)x\right]^2} \tag{13.13}$$

We introduce a new variable $k' = k - xp$ (and skip the prime)

$$\begin{aligned} \Sigma(p) &= -ie^2\mu^\epsilon \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{-(d-2)x\not{p} + dm}{\left[k^2 - m^2(1-x) + p^2x(1-x)\right]^2}\\ &= \frac{e^2\mu^\epsilon}{(4\pi)^d}\Gamma(2 - d/2) \int_0^1 dx \frac{-(d-2)x\not{p} + dm}{\left[m^2(1-x) - p^2x(1-x)\right]^{2-d/2}}\\ &= -\frac{e^2}{(4\pi)^2}(\not{p} - 4m)\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) + \mathcal{O}(1) \end{aligned} \tag{13.14}$$

Compare with (13.6) and get in the $\overline{MS}$ scheme

$$\begin{aligned} \delta Z_2 &= -\frac{e^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) & (13.15)\\ \delta m &= -4m\frac{e^2}{(4\pi)^2}\left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) & (13.16) \end{aligned}$$

13.2 Photon propagator

All contributions at one loop order (with counter-terms included) can be written as

$$\frac{-ig_{\mu\nu}}{p^2}+\frac{-ig_{\mu\alpha}}{p^2}\left[i\Pi^{\alpha\beta}(p)-i\delta Z_3\left(p^2g^{\alpha\beta}-p^\alpha p^\beta\right)\right]\frac{-ig_{\beta\nu}}{p^2}+\ldots \tag{13.17}$$

where (remember the (-1) factor for the fermion loop!)

$$\begin{aligned} i\Pi^{\alpha\beta}(p) &= (-1)\int\frac{d^dk}{(2\pi)^d}Tr\left(\frac{i}{\not k+\not p-m}ie\mu^{\epsilon/2}\gamma^\alpha\frac{i}{\not k-m}ie\mu^{\epsilon/2}\gamma^\beta\right) \\ &= -e^2\mu^\epsilon\int\frac{d^dk}{(2\pi)^d}\frac{Tr\left[\left(\not k+\not p+m\right)\gamma^\alpha\left(\not k+m\right)\gamma^\beta\right]}{\left[(k+p)^2-m^2\right]\left[k^2-m^2\right]} \end{aligned} \tag{13.18}$$

We need to generalise the dimensionality of the γ matrices for d dimensions. We can define completely consistently

$$Tr\left(1\right) = f(d) \tag{13.19}$$

$$Tr\left(\gamma^\alpha\gamma^\beta\right) = f(d)g^{\alpha\beta} \tag{13.20}$$

$$Tr\left(\gamma^\alpha\gamma^\beta\gamma^\mu\gamma^\nu\right) = f(d)\left(g^{\alpha\beta}g^{\mu\nu}-g^{\alpha\mu}g^{\beta\nu}+g^{\alpha\nu}g^{\beta\mu}\right) \tag{13.21}$$

with the only constraint that $f(4)=4$.

We get

$$\begin{aligned} i\Pi^{\alpha\beta}(p) &= -e^2\mu^\epsilon f(d)\int\frac{d^dk}{(2\pi)^d}\frac{2k^\alpha k^\beta+k^\alpha p^\beta+p^\alpha k^\beta+\left(m^2-k(k+p)\right)g^{\alpha\beta}}{\left[(k+p)^2-m^2\right]\left[k^2-m^2\right]} \\ &= -e^2\mu^\epsilon f(d)\int_0^1dx\int\frac{d^dk}{(2\pi)^d}\frac{2k^\alpha k^\beta+k^\alpha p^\beta+p^\alpha k^\beta+\left(m^2-k(k+p)\right)g^{\alpha\beta}}{\left[k^2-m^2+(p^2+2kp)x\right]^2} \end{aligned}$$

We again introduce a new variable $k'=k+xp$ (and again we skip the prime)

$$\begin{aligned} i\Pi^{\alpha\beta}(p) &= -e^2\mu^\epsilon f(d)\int_0^1dx\int\frac{d^dk}{(2\pi)^d}\frac{1}{\left[k^2-m^2+p^2x(1-x)\right]^2} \\ &\times \left[-2x(1-x)p^\alpha p^\beta+\left((2/d-1)k^2+m^2+p^2x(1-x)\right)g^{\alpha\beta}\right] \end{aligned} \tag{13.22}$$

where we used the relation

$$k^\alpha k^\beta\to\frac{k^2}{d}g^{\alpha\beta} \tag{13.23}$$

which is of course valid under the integration.

Let's now check if the photon gets a mass due to one loop corrections. This would be catastrophic, since it would mean the breaking of gauge invariance. As we will see, nothing of this happens, which confirms our belief that dimensional regularisation explicitly conserves gauge invariance.

We would potentially get a mass term in the $p \to 0$ limit of the above expression. It follows

$$\begin{aligned}\Pi^{\alpha\beta}(0) &= -e^2\mu^\epsilon f(d) g^{\alpha\beta} \qquad \int_0^1 dx \int \frac{d^dk}{(2\pi)^d} \frac{m^2 + (2/d-1)k^2}{\left(k^2-m^2\right)^2} \\ &= -e^2\mu^\epsilon f(d) g^{\alpha\beta} \qquad \left[m^2 \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(2-d/2)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-d/2} \right. \\ &\qquad \left. + (2/d-1) \frac{-i}{(4\pi)^{d/2}} \frac{d}{2} \frac{\Gamma(2-d/2-1)}{\Gamma(2)} \left(\frac{1}{m^2}\right)^{2-d/2-1} \right]\end{aligned} \tag{13.24}$$

Using (12.3) the result is exactly

$$i\Pi^{\alpha\beta}(0) = 0 \tag{13.25}$$

as we expected (and hoped).

Now let's see also the divergent pieces. In the proof of a massles photon we showed that

$$\int \frac{d^dk}{(2\pi)^d} \frac{m^2 + (2/d-1)k^2}{(k^2-m^2)^2} = 0 \tag{13.26}$$

From here it follows

$$\int \frac{d^dk}{(2\pi)^d} \frac{m^2 + (2/d-1)k^2}{\left[k^2 - m^2 + p^2x(1-x)\right]^2} = \int \frac{d^dk}{(2\pi)^d} \frac{p^2x(1-x)}{\left[k^2 - m^2 + p^2x(1-x)\right]^2} \tag{13.27}$$

and then

$$i\Pi^{\alpha\beta}(p) = -\frac{2e^2\mu^\epsilon f(d) i}{(4\pi)^{2-\epsilon/2}} \left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \frac{x(1-x)}{\left[m^2 - p^2x(1-x)\right]^{\epsilon/2}} \tag{13.28}$$

The limit $\epsilon \to 0$ gives

$$i\Pi^{\alpha\beta}(p) = -i\frac{4}{3}\frac{e^2}{(4\pi)^2} \left(p^2 g^{\alpha\beta} - p^\alpha p^\beta\right) \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) + \mathcal{O}(1) \tag{13.29}$$

We require that δZ_3 in (13.17) exactly cancels the divergent pieces:

$$\delta Z_3 = -\frac{4}{3}\frac{e^2}{(4\pi)^2} \left(\frac{2}{\epsilon} - \gamma + \ln 4\pi\right) \tag{13.30}$$

13.3 Exercise 14

- In the $\overline{MS}$ scheme show that in QED $Z_1 = Z_2$ at the one loop level, which follows from the requirement of gauge invariance

$$D_\alpha = \partial_\alpha - ieA_\alpha = \partial_\alpha - ie_R A_{R\alpha}$$

13.4 Exercise 15

- Calculate the anomalous magnetic moment of the electron in QED at one loop. Compare the result with the latest measurements and theoretical predictions in PDG (to be found in [6]).

References

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[6] http://inspirehep.net/

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