SYMMETRIES IN HEP

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Abstract

These lecture notes are written for a 5 hours short introductory course and exercises given at the Sarajevo School of High Energy Physics in May 2013.

Contents

1 Introduction

There are a number of books on symmetries in physics. What we will learn is described in standard field theory introductory books like for example Ryder's [1] or Peskin and Schroeder's [2], Maybe a bit more involved are Weinberg's books [4, 5]. For more one can try on INSPIRE [6].

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Throughout these lectures we will stick to the conventions $c = 1$ and $\hbar = 1$. This is nothing else than a specific choice of units. Masses and energies have for example the same unit, which we typically take as GeV (gigaelectronvolt=10^9 eV), time and position as GeV^{-1}. The conversion of these units into the classical ones is very simple: we multiply the quantity we want to convert with the right powers of $\hbar$ and c.

1.1 Motivation and aims

It is assumed that the reader knows the basics of quantum mechanics. What we will try to give are the basics of two aspects of symmetries particularly used in quantum field theory: space-time symmetries that lead to relativity through Lorentz transformations, and internal symmetries like isospin for the global case or gauge symmetries.

2 Relativity and the Lorentz group (1h)

2.1 Lorentz transformations

The essence of relativity are the Lorentz transformations. These will be carried out as rotations in 4-dimensional space (the time x^0 and spatial coordinates x^i, in short x^α). Latin indices will denote spatial coordinates and run from 1 to 3, while greek indices will run from 0 to 3.

Let's remind first how we describe rotations in a plane (2-dimensional space). It is simply given by

$$\begin{pmatrix} x' \\ y\prime \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.1}$$

The rotation matrix can be written (check it!) in an apparently weird way as

$$O = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} = \exp\left(i\alpha T\right) \tag{2.2}$$

where we call the matrix

$$T = -i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.3}$$

the generator of rotations in a 2-dimensional space. Matrices (2.2) form the Lie group SO(2). It is a Lie group because the parameter α is continuous, while SO(d) in general means orthogonal ($OO^T = O^T O = I$) rotations in d-dimensional space with unit determinant ($detO = 1$).

This form is particularly useful for generalizations. In d-dimensional space the vector $x = (x^1, ..., x^d)^T$ gets rotated by the group element O ($d \times d$ matrix). All these elements form the group SO(d), each single element is uniquely determined by the $d(d-1)/2$ angles $\alpha^{ab} = -\alpha^{ba}$ through

$$O = \exp\left(\frac{i}{2}\alpha^{ab}T_{ab}\right) \tag{2.4}$$

and the same number of rotation generators (the Kronecker δ is 1 if the two indices coincide and 0 otherwise)

$$(T_{ab})^{kl} = -i\left(\delta_a^k \delta_b^l - \delta_b^k \delta_a^l\right) \tag{2.5}$$

Let's remind the reader that the two indices a and b denote the generator (we could denote them differently, for example sequentially from 1 do $d(d-1)/2$, $T_{12} \to T_1$, $T_{13} \to T_2$, etc.), while the indices k and l denote the element of the given matrix.

In the case of SO(2) we have a single element, T_{12}, which has been written so far without indices for simplicity.

The representation of generators (2.5) is just one of infinitely many possibilities. In fact we can represent generators in general as $n \times n$ matrices, being the representation (2.5) the lowest[2] dimensional (d) and called the fundamental representation of the group SO(d). In general the generators (of arbitrary allowed representations) of the group SO(d) satisfy the algebra, defined by the commutator

$$[T_{ab}, T_{cd}] \equiv T_{ab}T_{cd} - T_{cd}T_{ab} = i\left(\delta_{ac}T_{bd} + \delta_{bd}T_{ac} - \delta_{bc}T_{ad} - \delta_{ad}T_{bc}\right) \tag{2.6}$$

The consistency of this definition can be checked by changing 1) $a \leftrightarrow b$, 2) $c \leftrightarrow d$, 3) $a \leftrightarrow c$ and $b \leftrightarrow d$ at the same time, taking into account that $T_{ab} = -T_{ba}$ and $\delta_{ab} = +\delta_{ba}$. It is not hard to check that the fundamental representation (2.5) indeed satisfies the definition (2.6).

If we allowed the determinant of the orthogonal matrices to be also -1, we would talk about the group O(d) instead of SO(d). In three dimensions this means that we include also mirroring around an arbitrary plane ($x^i \to -x^i$ for a single i) or inversion ($x^i \to -x^i$ for all i).

Let's go back to Lorentz transformations. The space-time is indeed 4-dimensional, but of a special kind. This can be seen for example in the form of the invariants. Remember that the distance between two elements in the 4-dimensional space-time is given by $c^2(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2$, and not by the sum of squares of differences, as it would be in the usual Euclidean space. This means that we get the product of two vectors in Minkowski space if we add between them a matrix - the metric tensor, which takes care of these extra minuses. If for example $a^\mu = (a^0, a^i)$ and $b^\mu = (b^0, b^i)$, then the product is given by

[2] An exception for $d < 4$ is given by the spinorial representations, see next section.

$$a^0b^0 - a^1b^1 - a^2b^2 - a^3b^3 = a^\mu g_{\mu\nu} b^\nu \tag{2.7}$$

where the metric tensor is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.8}$$

We will keep the usual sum convention for two identical Lorentz indices (without the need for an explicit sum sign), being one index up and the other down (never both up or both down). The metric tensor can raise or lower such an index. The equation above can be thus written in many different but equivalent ways:

$$a^\mu g_{\mu\nu} b^\nu = a^\mu b_\mu = a_\mu g^{\mu\nu} b_\nu = a_\mu b^\mu \tag{2.9}$$

where we defined

$$a_\mu \equiv g_{\mu\nu} a^\nu = (a_0, a_1, a_2, a_3) = (a^0, -a^1, -a^2, -a^3) \tag{2.10}$$

and the inverse of the metric tensor

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{2.11}$$

which satisfies

$$\left(gg^{-1}\right)_\mu{}^\nu = g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu{}^\nu \tag{2.12}$$

$$\left(g^{-1}g\right)^\mu{}_\nu = g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu \tag{2.13}$$

We see from here that $g_\mu{}^\nu = \delta_\mu{}^\nu$.

We can generalize now the previous definitions valid for SO(d) to SO(d_+,d_-), where we have a diagonal metric tensor with d_+ elements 1 and d_- elements -1 (in our case we will be interested in the case SO(1,3)). All the definitions so far are still ok provided we systematically substitute all δ's with g's and interpret all matrix products with an intermediate g. The definition of the commutator in the algebra SO(1,3) is thus for example

$$[T_{\alpha\beta}, T_{\mu\nu}] = i\left(g_{\alpha\mu} T_{\beta\nu} + g_{\beta\nu} T_{\alpha\mu} - g_{\beta\mu} T_{\alpha\nu} - g_{\alpha\nu} T_{\beta\mu}\right) \tag{2.14}$$

while we have to understand the definition of the group element (2.4), i.e. a matrix in the 4-dimensional Minkowski space, as

$$\Lambda^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \frac{\alpha^{\alpha_1\beta_1}}{2}(iT_{\alpha_1\beta_1})^{\mu}{}_{\nu} + \frac{1}{2}\frac{\alpha^{\alpha_1\beta_1}}{2}\frac{\alpha^{\alpha_2\beta_2}}{2}(iT_{\alpha_1\beta_1})^{\mu}{}_{\lambda}(iT_{\alpha_2\beta_2})^{\lambda}{}_{\nu} + ... \tag{2.15}$$

The different matrices can be obtained from (2.5)

$$(iT_{\alpha\beta})^{\mu}{}_{\nu} = (iT_{\alpha\beta})^{\mu\lambda}\, g_{\lambda\nu} = \left(\delta^{\mu}_{\alpha}\delta^{\lambda}_{\beta} - \delta^{\lambda}_{\alpha}\delta^{\mu}_{\beta}\right) g_{\lambda\nu} \tag{2.16}$$

Now it is not hard to check, that (2.15) is really a Lorentz transformation

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} \tag{2.17}$$

Let's see for example, that the rotation given by $\alpha^{01} = \alpha$ is nothing else than the Lorentz transformation (boost) in the x direction:

$$\Lambda^{\mu}{}_{\nu} = \left[\delta + \alpha\,(iT_{01}.g) + \frac{\alpha^2}{2}\,(iT_{01}.g)^2 + ...\right]^{\mu}_{\ \nu} \tag{2.18}$$

From (2.16) it follows

$$iT_{01}.g = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.19}$$

and finally after summing the whole infinite series

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\alpha & -\sinh\alpha & 0 & 0 \\ -\sinh\alpha & \cosh\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.20}$$

Introducing a new variable $\cosh\alpha = 1/\sqrt{1-v^2}$, we get exactly a Lorentz boost (v is the velocity) in the x direction.

It is good to remember that due to the antisymmetry of the generators one gets from (2.15)

$$\Lambda_{\mu}{}^{\sigma} = \left(\Lambda^{-1}\right)^{\sigma}{}_{\mu} \tag{2.21}$$

so that

$$\Lambda^{\mu}{}_{\nu}\Lambda_{\mu}{}^{\sigma} = \delta^{\sigma}{}_{\nu} \tag{2.22}$$

Due to that vector products are Lorentz scalars:

$$a'^{\mu}b'_{\mu} = \Lambda^{\mu}{}_{\nu}a^{\nu}\Lambda_{\mu}{}^{\sigma}b_{\sigma} = a^{\nu}b_{\nu} \tag{2.23}$$

An arbitrary tensor gets transformed under Lorentz transformations similarly as

$$T'^{\alpha_1\ldots\alpha_m}_{\beta_1\ldots\beta_n} = \Lambda^{\alpha_1}{}_{\mu_1}\ldots\Lambda^{\alpha_m}{}_{\mu_m}\Lambda_{\beta_1}{}^{\nu_1}\ldots\Lambda_{\beta_n}{}^{\nu_n}T^{\mu_1\ldots\mu_m}_{\nu_1\ldots\nu_n} \tag{2.24}$$

2.2 Spinors

At first sight it could seem that the fundamental representation for the generators (2.5) is also the simplest one, and that transformations of all higher tensors can be obtained by (2.24). It turns out this is not true, so that there is an even simpler representation of the Lorentz group, from which we can derive even the transformation matrix $\Lambda^{\mu}{}_{\nu}$ which is used in (2.17) and (2.24).

This follows from the following derivation (let's limit ourselves to four dimensions): imagine there exist four 4×4 matrices γ^{μ}, which satisfy the Dirac algebra

$$\{\gamma^{\mu}, \gamma^{\nu}\} \equiv \gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu} \tag{2.25}$$

Then it is possible to show that the matrices

$$\Sigma^{\mu\nu} = \frac{1}{4i}\left[\gamma^{\mu}, \gamma^{\nu}\right] \tag{2.26}$$

satisfy the commutation relations (2.14).

Such matrices are given explicitly for example (in the so-called chiral representation) by

$$\gamma^{\mu} = \begin{pmatrix} 0 & \sigma^{\mu} \\ \bar{\sigma}^{\mu} & 0 \end{pmatrix} \tag{2.27}$$

where $\sigma^{\mu} = (1, \sigma^{i})$, $\bar{\sigma}^{\mu} = (1, -\sigma^{i})$ and σ^{i} are the Pauli matrices. The spinorial representation Ψ of the Lorentz group is the one that transforms as

$$\Psi' = \Lambda_{1/2}\Psi = \exp\left(\frac{i}{2}\alpha^{\mu\nu}\Sigma_{\mu\nu}\right)\Psi \tag{2.28}$$

In our case (4d) this is the 4-dimensional Dirac spinor. Since space-time is even dimensional, these Lorentz generators in the spinorial representation are block diagonal. The irreducible spinorial representations of the Lorentz group are thus two-dimensional. In the basis (2.27) they are the two Weyl spinors ψ_L and ψ_R:

$$\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \tag{2.29}$$

Under parity the spinor transforms as

$$\Psi \to \gamma^0 \Psi \tag{2.30}$$

i.e.

$$\psi_L \leftrightarrow \psi_R \tag{2.31}$$

As promised, we can write the Lorentz transformation matrices in the fundamental representation $\Lambda^\mu{}_\nu$ from the matrices of Lorentz transformations in the spinorial representation $\Lambda_{1/2}$:

$$\Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} = \Lambda^\mu{}_\nu \gamma^\nu \tag{2.32}$$

For later use let's introduce also the 4×4 matrices

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.33}$$

and

$$\bar{\Psi} \equiv \Psi^\dagger \gamma^0 \tag{2.34}$$

which transforms as

$$\bar{\Psi}' = \bar{\Psi} \Lambda_{1/2}^{-1} \tag{2.35}$$

We can now write down all the 16 possible bilinear combinations of Dirac spinors

$$\begin{aligned}
S &= \bar{\Psi}\Psi & (2.36)\\
P &= \bar{\Psi}\gamma^5\Psi & (2.37)\\
V^\mu &= \bar{\Psi}\gamma^\mu\Psi & (2.38)\\
A^\mu &= \bar{\Psi}\gamma^\mu\gamma^5\Psi & (2.39)\\
T^{\mu\nu} &= \bar{\Psi}\left[\gamma^\mu, \gamma^\nu\right]\Psi & (2.40)
\end{aligned}$$

The notation (S = scalar, P = pseudoscalar, V = vector, A = axial vector, T = tensor) reminds us how the quantities transform under Lorentz

$$\begin{aligned}
(S', P') &= (S, P) & (2.41)\\
(V'^\mu, A'^\mu) &= \Lambda^\mu{}_\nu \left(V^\nu, A^\nu\right) & (2.42)\\
T^{\mu\nu} &= \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta T^{\alpha\beta} & (2.43)
\end{aligned}$$

and parity

$$S \rightarrow +S \tag{2.44}$$
$$P \rightarrow -P \tag{2.45}$$
$$V^\mu \rightarrow (-1)^{\delta_{\mu 0}+1} V^\mu \tag{2.46}$$
$$A^\mu \rightarrow (-1)^{\delta_{\mu 0}} A^\mu \tag{2.47}$$
$$T^{\mu\nu} \rightarrow (-1)^{\delta_{\mu 0}+\delta_{\nu 0}} T^{\mu\nu} \tag{2.48}$$

2.3 All representations of the Lorentz group

It is instructive to rewrite the generators of SO(1,3) as

$$T_{0a} = K_a \quad , \quad T_{ab} = \epsilon_{abc} J_c \tag{2.49}$$

where the Levi-Civita tensor ϵ_{abc} is antisymmetric under the exchange of arbitrary two indices and $\epsilon_{123} = 1$. In the generators K we recognize the generators of Lorentz boosts, while generators J represent rotations in 3-dimensional space (angular momenta!). The commutation rules (2.14) can be now written

$$[K_a, K_b] = i\epsilon_{abc} J_c \tag{2.50}$$
$$[J_a, K_b] = i\epsilon_{abc} K_c \tag{2.51}$$
$$[J_a, J_b] = -i\epsilon_{abc} J_c \tag{2.52}$$

Although this looks less compact as before, it gives a better physical insight. The last equation is nothing else than the commutation rules for the operators of the angular momentum. We can define new linear combinations through

$$A_a = \frac{1}{2}(J_a + iK_a) \tag{2.53}$$
$$B_a = \frac{1}{2}(J_a - iK_a) \tag{2.54}$$

so that the commutation relations simplify to

$$[A_a, A_b] = i\epsilon_{abc} A_c \tag{2.55}$$
$$[A_a, B_b] = 0 \tag{2.56}$$
$$[B_a, B_b] = i\epsilon_{abc} B_c \tag{2.57}$$

These are commutation relations for rotations in 2-dim complex space. We talk about the group SU(2), which has three generators (a general SU(n), i.e. rotations - not orthogonal anymore, but unitary matrices - in n-dim complex space, has $n^2 - 1$ generators). A and B are thus generators of two SU(2) groups.

SO(4) group generators get thus divided into generators of two different and independent rotations, two SU(2) (one with generators A, the other one with generators B). The group SO(4) is locally equivalent to the group SU(2)×SU(2).

The fields which describe elementary particles, transform as irreducible representations of the Lorentz group, and can be characterized by two multiples of 1/2, i.e. with one spin for each SU(2). The simplest irreducible representation is thus the Lorentz scalar $(0,0)$. we then have the two types of spinors. We denote the above SU(2) groups with the indices L (left) and R (right), so that we can have two types of elementary (Weyl) spinors $\psi_L \sim (1/2, 0)$ and $\psi_R \sim (0, 1/2)$, which we call the left-handed and the right-handed spinor. With a single one of them we can describe only massless fermions, while we need both of them (i.e. a Dirac spinor) for a massive fermion. It gets transformed under Lorentz transformations through $\Lambda_{1/2}$, see (2.28). Now we understand the subscript 1/2, we have to do with spin 1/2. Another representation which is very often used in field theory is the one that describes the vector boson $A^\mu \sim (1/2, 1/2)$, which is partially made out of spin 0 $(= 1/2 - 1/2)$and partially of spin 1 $(= 1/2 + 1/2)$.

Exercise

1. Check that $\Sigma_{\mu\nu} = c\,[\gamma_\mu, \gamma_\nu]$ satisfy the commutation rules for SO(d_+, d_-) and determine c.

2. Show that $\Sigma_{\mu\nu}$ are block diagonal in the chiral representation of the γ matrices.

3. Show equation (2.32).

4. Check equations (2.50)-(2.52), i.e. check the proportionality factor in the definition of angular momentum operator from T_{ab}.

5. Show that the Dirac equation is Lorentz covariant.

3 Equations of motions, internal symmetries (1h)

3.1 The action

The equation of motion can be derived by the variational principle, more precisely by the principle of extremal (minimal) action. Let's imagine that we have a Lagrangian density (from now on we will call it simply Lagrangian), which is a scalar under Lorentz (and other, see later) transformations. It is a function of the fields ϕ and their first derivatives $\partial_\mu\phi$, i.e. $\mathcal{L}(\phi, \partial_\mu\phi)$.

The action is defined as a space-time integral of the Lagrangian

$$S[\phi] = \int d^4x \mathcal{L}(\phi, \partial_\mu\phi) \tag{3.1}$$

The principle of extremal action states that we get the equations of motion by requiring that the action does not change if we infinitesimally change the fields $\phi \to \phi' = \phi + \delta\phi$

$$\begin{aligned} S[\phi'] - S[\phi] &= \int d^4x \left[\mathcal{L}(\phi', \partial_\mu \phi') - \mathcal{L}(\phi, \partial_\mu \phi)\right] \\ &= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu \phi)\right] \\ &= \int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta\phi\right) + \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right)\right)\delta\phi\right] = 0 \end{aligned} \tag{3.2}$$

In coming to the last line we took into account that the difference of derivatives is the derivative of a difference $\delta(\partial_\mu \phi) = \partial_\mu \delta\phi$ and used integration by parts.

The first term is a total derivative. By Gauss' theorem its integral depends only on the field value at the boundary (at infinity). If we limit ourselves to fields and their derivatives which drops off fast enough at infinity, this term does not contribute. The action is thus extremal for fields that satisfy the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0 \tag{3.3}$$

What type of Lagrangians we use? As we said, they are Lorentz scalars. We would like also that they reproduce the known equations of motion for free fields (no interaction), i.e. the Klein-Gordon equation (spin 0), the Dirac equation (spin 1/2) and the Maxwell equations (spin 1).

It is not hard to check that the Lagrangian for a free real scalar field is

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2}m^2\phi^2 \tag{3.4}$$

where the normalization to 1/2 of the first term (with derivatives) is from historical reasons similar as in quantum mechanics:

$$L = \frac{1}{2}m\left(\frac{d\overrightarrow{x}}{dt}\right)^2 \tag{3.5}$$

In fact remember that what correspond to time t and coordinate $\overrightarrow{x}(t)$ in quantum mechanics are the vector $x^\mu = (t, \overrightarrow{x})$ and field $\phi_i(x^\mu)$, respectively, in field theory.

For the free complex field we have

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi \tag{3.6}$$

The normalization here is chosen so that a complex field can be expressed by two real scalar fields $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$.

Similarly we get for fermions

$$\mathcal{L} = \bar{\psi}\left(i\gamma^\mu \partial_\mu - m\right)\psi \tag{3.7}$$

where for the derivation of the Euler-Lagrange equations we have to consider ψ and $\bar{\psi}$ as two independent fields (similarly as ϕ and ϕ^* in (3.6)).

Finally we get Maxwell's equations from

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \mathcal{L}_{gf} \tag{3.8}$$

where the last term depends on the gauge fixing. For a Lorentz gauge fixing

$$\mathcal{L}_{gf} = -\frac{1}{2\xi} \left(\partial_\mu A^\mu\right)^2 \tag{3.9}$$

with ξ the Lagrange multiplier.

The factor $-1/4$ in (3.8) is found, if we require the same normalization as in (3.4) for the time derivatives of the spatial components A^i. For the time components A^0 we do not have such a term, which confirms that all four components of A^μ are not physical degrees of freedom, this is why we need the gauge fixing term (for example (3.9)).

3.2 Noether's theorem

Let's go back to (3.2). Let's assume now that the field ϕ satisfy the equations of motion, and let's concentrate to internal symmetries of the Lagrangian, i.e. to such variations $\delta\phi$, which keep the Lagrangian invariant. The first term of the integrand vanishes everywhere in spacetime and not only after the integration or at infinity (as was true for an arbitrary $\delta\phi$ so far)

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi \right) = 0 \tag{3.10}$$

If we have more fields, we need to add up single contributions. We will be interested in continuous transformations, described by a Lie group (T_a are generators)

$$\phi' = e^{i\alpha^a T_a} \phi \tag{3.11}$$

In this case $\delta\phi = i\alpha^a T_a \phi$. The number of conserved currents we get are equal to the number of generators:

$$\partial_\mu j_a^\mu = 0 \ \ , \ \ j_a^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} i T_a \phi \tag{3.12}$$

These relations are correct only after taking into account the equation of motion, i.e. on-shell.

Let's check it in the known case of the Lagrangian for a free complex field (3.6). The symmetry here is the phase U(1):

$$\phi' = e^{i\alpha} \phi \ \ , \ \ \phi^{*\prime} = e^{-i\alpha} \phi^* \tag{3.13}$$

under which the Lagrangian (3.6) is invariant. The current is

$$j^\mu = i \left(\phi \partial^\mu \phi^* - \phi^* \partial^\mu \phi \right) \tag{3.14}$$

Its divergence is

$$\partial_\mu j^\mu = i\left(\phi\partial^2\phi^* - \phi^*\partial^2\phi\right) = i\left(-\phi m^2\phi^* + \phi^* m^2\phi\right) = 0 \tag{3.15}$$

where we imposed the equations of motion

$$\left(\partial^2 + m^2\right)\phi = 0 \tag{3.16}$$

We say that the current (3.12) is conserved, because the quantity

$$Q_a = \int d^3x j_a^0 \tag{3.17}$$

which we call the charge, does not change with time:

$$\frac{dQ_a}{dt} = \int d^3x \partial_0 j_a^0 = \int d^3x \partial_\mu j_a^\mu = 0 \tag{3.18}$$

As usual we limited ourselves to fields that decrease fast enough at infinity, and used the Gauss' theorem (the volume integral of a divergence is equal to the boundary integral). The current (3.12) and the charge (3.17) are called Noether's current and Noether's charge, respectively. The Noether's theorem says that such currents are conserved.

3.3 Gauge invariance

So far we assumed transformations (3.11) with constant parameters α_a. Now we will consider the very important case of these parameters being functions of space-time coordinates, $\alpha_a = \alpha_a(x)$. We will call them gauge transformations.

The motivation for them comes from the gauge invariance of Maxwell's equations. If we want to couple the fermion

$$\mathcal{L} = \bar\psi i\gamma^\mu\partial_\mu\psi \tag{3.19}$$

with the electromagnetic (EM) field, we use the well known quantum mechanical recipe

$$i\partial_\mu \to p_\mu \to p_\mu + eA_\mu \to i\partial_\mu + eA_\mu \to iD_\mu \tag{3.20}$$

In the process we defined the covariant derivative

$$D_\mu\psi = \left(\partial_\mu - ieA_\mu\right)\psi \tag{3.21}$$

The Lagrangian

$$\mathcal{L}_1 = \bar\psi i\gamma^\mu D_\mu\psi \tag{3.22}$$

is invariant under the gauge transformations

$$\psi \to e^{i\alpha(x)}\psi \ , \ \ A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha \tag{3.23}$$

In fact we defined the covariant derivative of the field to transform under gauge transformations exactly as the field itself

$$(D_\mu \psi)' = e^{i\alpha(x)} D_\mu \psi \tag{3.24}$$

Similarly is the kinetic term for the electromagnetic field

$$\mathcal{L}_2 = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.25}$$

invariant under gauge transformations, being already the EM tensor invariant itself.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.26}$$

$$F'_{\mu\nu} = F_{\mu\nu} \tag{3.27}$$

The sum (3.22) and (3.25) describes the Lagrangian for a fermion and the EM fields.

Transformations (3.23) form, as we know, the group U(1), this is the phase change of the fermion. This can be generalized to other groups, which has been first done by Yang and Mills in the fifties. They wrote down for the first time a Lagrangian invariant under SU(2) gauge transformations.

In an even more general way we would like to find out the Lagrangians invariant under

$$\psi' = U\psi \ \ , \ \ U = U(\alpha(x)) \ \ , \ \ U^\dagger = U^{-1} \tag{3.28}$$

where U are unitary matrices, elements of the group SU(N). Terms of the form $\bar{\psi}\psi$ are invariant because of the unitarity of the transformation matrix U. Problems arise however when we encounter derivatives:

$$\partial_\mu \psi' = U \partial_\mu \psi + (\partial_\mu U)\, \psi \tag{3.29}$$

so that because of the second term the kinetic term in the Lagrangian is not invariant

$$\bar{\psi}' i\gamma^\mu \partial_\mu \psi' \neq \bar{\psi} i\gamma^\mu \partial_\mu \psi \tag{3.30}$$

Similarly as we did before with the photon field A_μ we want now to transform the usual derivative into a covariant one. Our group now is SU(N) and we have thus $N^2 - 1$ gauge bosons, the same number as the group generators. We define first the matrix

$$A_\mu \equiv A^a_\mu T^a \tag{3.31}$$

and let the generators T^a be in the same representation as ψ. If ψ is for example in the fundamental representation, then in the case of SU(2) the generators T^a are properly normalized Pauli matrices $\sigma^a/2$, while in the case SU(3) they are Gell-Mann matrices. In any case the convention will be that the generators in the fundamental representation are normalized as

$$Tr\left(T^a T^b\right) = \frac{1}{2}\delta^{ab} \tag{3.32}$$

(a different choice for the norm just redefines the coupling constant g). Let's now use the same trick as before, i.e. define the covariant derivative

$$D_\mu \psi \equiv (\partial_\mu - igA_\mu)\,\psi \tag{3.33}$$

which behaves under gauge transformations

$$\psi' = e^{i\alpha_a(x)T^a}\psi \ \ , \ \ A'_\mu = UA_\mu U^\dagger + \frac{c}{g}\partial_\mu (U)\,U^\dagger \tag{3.34}$$

as in (3.24). We guessed the change of the vector field (actually of the matrix (3.31)). Requiring (3.24) for $\alpha(x) \equiv \alpha_a(x)T^a$ we can determine the constant $c = -i$.

How does it look now the kinetic term for the gauge boson? We can use a similar ansatz as before, but with a mandatory trace since we deal with matrices now:

$$\mathcal{L} = -\frac{1}{4c_2} Tr\,(F_{\mu\nu}F^{\mu\nu}) \tag{3.35}$$

which however does not tell us much until we define $F_{\mu\nu}$ for a general group SU(N). Trying with (3.26) proves inefficient, so we can try with the following ansatz (taking into account the antisymmetry under interchange of indices)

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + c_1 g\,[A_\mu, A_\nu] \tag{3.36}$$

Requiring

$$F'_{\mu\nu} = UF_{\mu\nu}U^\dagger \tag{3.37}$$

i.e. the invariance of (3.35) gives $c_1 = -i$. The last unknown constant c_2 comes just from normalization. We would like to have the same normalization of the kinetic term for every gauge field A^a_μ as we had it for the electromagnetic field. In this convention c_2 is just the normalization factor of the generators (in a given representation)

$$Tr\left(T^a T^b\right) = c_2 \delta^{ab} \tag{3.38}$$

and thus equals $1/2$ in the case of fundamental representation.

Two comments.

First, we can use the same general formulae for SU(N) also for the simplest case of electromagnetism, with a single generator $T^a = q$ being a constant, which does not need to be equally normalized for different fields. This means that each field ψ_q has its own q. In this way we can interpret e as for example the electron charge (with $q = 1$), so that the up quark has $q = -2/3$, down quark $q = 1/3$, etc. The product qe is thus the U(1) charge of the field ψ_q. This is the difference between the abelian U(1) and the nonabelian SU(N) groups. While in the abelian case every field has its own charge, the charge of a nonabelian field depends only on its representation.

Second, as we defined the covariant derivative for fermionic fields, we could define it for bosonic fields as well:

$$D_\mu \phi \equiv (\partial_\mu - igA_\mu)\,\phi \tag{3.39}$$

The invariant kinetic term is now

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi \tag{3.40}$$

Exercise

1. Show that Noether's theorem for translations and rotations lead to the conservation of energy, momentum and angular momentum.
2. Calculate the value of c_1 from equation (3.36), checking if $F'_{\mu\nu} = UF_{\mu\nu}U^\dagger$. Check also that the same tensor can be written as $D_\mu A_\nu - D_\nu A_\mu$ or $[D_\mu, D_\nu]$.
3. Write down an invariant Lagrangian for scalars in the adjoint representation of the gauge symmetry SU(N).
4. Write down explicitly the kinetic terms of all the fields of the standard model.

4 The vacuum expectation value of a field (1h)

4.1 Negative m^2

So far we always assumed that m^2 in (3.4) were a positive number. This made sense, since the equation of motion for the field was the Klein-Gordan, which solutions are periodic in this case. If we have an interaction though, this is no more true. A negative m^2 points in this case to an instability (an exponentially increasing or decreasing function), which is a consequence to the fact, that we were expanding around a maximum and not a minimum of the action. If we write in fact

$$\mathcal{L} = \frac{1}{2}\,(\partial\phi)^2 - V(\phi) \tag{4.1}$$

then the Hamiltonian density is

$$\begin{aligned} \mathcal{H} &\equiv \Pi\,\partial_0\phi - \mathcal{L} = \frac{1}{2}\Pi^2 + \frac{1}{2}\,(\nabla\phi)^2 + V(\phi) && (4.2) \\ \Pi &\equiv \frac{\partial\mathcal{L}}{\partial\,(\partial_0\phi\,)} = \partial_0\phi && (4.3) \end{aligned}$$

The first two terms are positive definite, so the system has a minimal energy for constant solutions of the equation

$$\frac{\partial V(\phi)}{\partial\phi} = 0 \tag{4.4}$$

As an example let's take the known case ϕ^4

$$V(\phi) = \frac{m^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{4.5}$$

The equation of motion

$$m^2\phi + \frac{\lambda}{6}\phi^3 = 0 \tag{4.6}$$

has a single solution for $m^2 > 0$

$$\phi_0 = 0 \tag{4.7}$$

but three solutions for $m^2 < 0$

$$\phi_0 = 0 \ , \ \ \phi_\pm = \pm\sqrt{\frac{-6m^2}{\lambda}} \tag{4.8}$$

The first one corresponds to a maximum, while the other two are minima, which can be checked explicitly by calculating the second derivative of the potential

$$\frac{\partial^2 V(\phi)}{\partial\phi^2} = m^2 + \frac{\lambda}{2}\phi^2 \tag{4.9}$$

in the extrema (4.8):

$$\frac{\partial^2 V}{\partial\phi^2}(\phi_0) = m^2 < 0 \ , \ \ \frac{\partial^2 V}{\partial\phi^2}(\phi_\pm) = -2m^2 > 0 \tag{4.10}$$

The correct procedure is now to expand around the right minimum (the vacuum), which we take for example at:

$$\phi(x) = \phi_+ + \varphi(x) \tag{4.11}$$

We insert this to the original Lagrangian and get

$$\begin{aligned} \mathcal{L} &= \frac{1}{2}(\partial\varphi)^2 - V(\phi_+ + \varphi) \\ &= \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m_\varphi^2\varphi^2 - \frac{\mu}{3!}\varphi^3 - \frac{\lambda}{4!}\varphi^4 \end{aligned} \tag{4.12}$$

where

$$m_\varphi^2 = -2m^2 \ , \ \ \mu = \lambda\phi_+ \tag{4.13}$$

We explicitly see that the new mass m_φ^2 is positive. It is only now that we can freely calculate the Green's functions, i.e. evaluate the Feynman diagrams. The full original ϕ is thus a constant (solution to the classical equation of motion) plus quantum corrections (φ - creation and annihilation operators, etc).

It is now good to remember that the original Lagrangian had a discrete symmetry

$$\phi \to -\phi \tag{4.14}$$

In the case of a negative m^2 we had to choose a vacuum (there were two), ϕ_+ or ϕ_-, and thus break this symmetry. We say, that the symmetry has been spontaneously broken (the solutions to the equations of motions break the symmetry of the Lagrangian). This is denoted by a non-zero field vacuum expectation value (vev), and in this case this vev $\langle\phi\rangle$ is an order parameter:

$$\langle\phi\rangle = \phi_+ \neq 0 \tag{4.15}$$

The Feynman rules, the calculation of Green's functions etc. can be used only for fields with vanishing vev $\langle\varphi\rangle = 0$. The recipe is thus the following: first solve the equations of motion, then expand your fields around the vacuum (minimum), and only then use the whole machinery of Feynman etc.

4.2 The Nambu-Goldstone theorem

In the example above we spontaneously broke the discrete symmetry Z_2 (4.14). This is a group with only two elements, $+1$ and -1. For this reason we had only two possible vacua, ϕ_+ and ϕ_- in the case of negative m^2. This can be generalized to higher discrete or even continuous, i.e. Lie, groups. The Lie groups are the ones we will concentrate on now. Imagine to have a Lagrangian, symmetric under SO(N). We will consider a field in the fundamental representation, i.e. a vector

$$\vec{\phi} = \begin{pmatrix} \phi_1 \\ . \\ . \\ . \\ \phi_N \end{pmatrix} \tag{4.16}$$

The Lagrangian we will study is

$$\mathcal{L} = \frac{1}{2}\partial^\mu\vec{\phi}\partial_\mu\vec{\phi} - V(\vec{\phi}^2) \tag{4.17}$$

with the SO(N) invariant potential

$$V = -\frac{m^2}{2}\vec{\phi}^2 + \frac{\lambda^2}{4}(\vec{\phi}^2)^2 \tag{4.18}$$

We consider a positive m^2, which, as we saw, leads to spontaneous symmetry breaking, here of SO(N). This same SO(N) symmetry can be now used to rotate the vev into the direction of, let's say, ϕ_1. In a sense this is the choice of our coordinates. Now we plug

$$\vec{\phi} = \langle\vec{\phi}\rangle + \vec{\varphi} = \begin{pmatrix} v + \varphi_1 \\ \varphi_2 \\ . \\ . \\ \varphi_N \end{pmatrix} \tag{4.19}$$

into (4.17) getting

$$\begin{aligned} V(\langle\vec{\phi}\rangle + \vec{\varphi}) &= V(\langle\vec{\phi}\rangle) + \varphi_1 v(-m^2 + \lambda^2 v^2) \\ &+ \frac{1}{2}\varphi_1^2(-m^2 + 3\lambda^2 v^2) + \frac{1}{2}(\varphi_2^2 + \ldots + \varphi_N^2)(-m^2 + \lambda^2 v^2) \\ &+ \lambda^2 v \varphi_1 \vec{\varphi}^2 + \frac{\lambda^2}{4}(\vec{\varphi}^2)^2 \end{aligned} \tag{4.20}$$

The equation of motion is the term proportional to φ_1 and determines the vev

$$\langle\phi_1\rangle = v = m/\lambda \tag{4.21}$$

so that we remain with

$$V = \frac{1}{2}(2m^2)\varphi_1^2 + \lambda m \varphi_1 \vec{\varphi}^2 + \frac{\lambda^2}{4}(\vec{\varphi}^2)^2 \tag{4.22}$$

We see that there is only one massive state at this point, i.e. φ_1 with a now positive mass square $2m^2$, and $N-1$ massless states $\varphi_2, \ldots, \varphi_N$. This is a consequence of the spontaneous breaking of a continuous global symmetry. Nambu and Goldstone have studied such cases in general and proved a general theorem. The Nambu-Goldstone theorem tells us, that the number of massless fields (the so-called Nambu-Goldstone bosons) is equal to the number of broken generators.

It is easy to check that this counting works in our example. A N-dimensional vector breaks SO(N) into SO(N-1). At the start there were $N(N-1)/2$ generators, after the breaking only $(N-1)(N-2)/2$. The difference is $N-1$, and this is exactly the number of massless modes.

4.3 The Higgs mechanism

There is a further case we would like to consider. So far we assumed only global symmetries, i.e. the ones with only constant rotations. What if they become space-time dependent, i.e. what if the symmetry in question is a gauge one?

For simplicity let's consider the simplest example, the SO(2), or equivalently, the U(1) case. The two-component vector gets replaced by a complex scalar field. As we learned, a global symmetry gets promoted to a local one by changing the normal derivative with the covariant one, and adding the kinetic terms for the gauge bosons:

$$\mathcal{L} = (D_\mu\phi)^*(D^\mu\phi) - V(|\phi|) - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{4.23}$$

After symmetry breaking we expand as usually our complex field around the minimum

$$\phi = \frac{1}{\sqrt{2}}(v + \varphi_1 + i\varphi_2) \tag{4.24}$$

where v solves the equation of motion

$$\frac{dV}{dv} = 0 \tag{4.25}$$

at a minimum.

The covariant derivative becomes

$$D_\mu\phi = (\partial_\mu - ieA_\mu)\phi \to \frac{1}{\sqrt{2}}\left((\partial_\mu\varphi_1 + eA_\mu\varphi_2) + i(\partial_\mu\varphi_2 - evA_\mu - eA_\mu\varphi_1)\right) \tag{4.26}$$

so that

$$|D_\mu\phi|^2 \to \frac{1}{2}(\partial\varphi_1)^2 + \frac{1}{2}(\partial\varphi_2)^2 - evA_\mu\partial^\mu\varphi_2 + \frac{1}{2}e^2v^2A^2 + \text{interaction} \tag{4.27}$$

The term $-evA_\mu\partial^\mu\varphi_2$ is a mixing between ϕ_2 and the vector field. We should get rid of it, and this has been cleverly done long ago by 't Hooft. Remember that (4.17) is not well defined without the so called gauge fixing term. This term is usually taken as

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial A)^2 \tag{4.28}$$

(with ξ an arbitrary number, which must drop out from the final physical results) but in the case of symmetry breaking a better choice is

$$\mathcal{L}_{gf} = -\frac{1}{2\xi}(\partial A + \xi ev\varphi_2)^2 \tag{4.29}$$

After expanding one gets

$$\mathcal{L}_{gf} \to -\frac{1}{2\xi}(\partial A)^2 - ev\partial A\varphi_2 - \frac{\xi e^2v^2}{2}\varphi_2^2 \tag{4.30}$$

The third term in (4.27) and the second in (4.30) combine into a total derivative

$$-ev\partial^\mu(A_\mu\varphi_2) \tag{4.31}$$

and can be thus safely ignored.

The term $\frac{1}{2}e^2v^2A^2$ in (4.27) represents the mass term for the gauge boson. We have finally got a theory with a consistent massive vector field. Gauge symmetry in fact forbids the gauge boson to have a mass, but a spontaneously broken gauge symmetry automatically gives a massive gauge boson.

A final comment for the last term in (4.30). Apparently it is a mass term for φ_2, with a ξ-dependent mass. This is a signal that the field φ_2 is not physical. In fact it is not an asymptotic state and it must be considered only as a virtual state in the Feynman diagrams. The original ϕ_2 state has been eaten by the longitudinal polarization of the massive gauge boson. A massive gauge bosons has 3 degrees of freedom, while the massless one has only two. The counting of degrees of freedom now fits.

Exercise

1. Take the so-called σ model ($m^2 > 0$)

$$\mathcal{L} = \frac{1}{2}(\partial \overrightarrow{\phi})^2 + \frac{\mu^2}{2}\overrightarrow{\phi}^2 - \frac{\lambda^2}{4}\overrightarrow{\phi}^4 \tag{4.32}$$

which is invariant under SO(4) rotations, $\overrightarrow{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$. Find the minimum of the potential (use the symmetry O(4), so that only $\langle\phi_4\rangle \neq 0$). Expand the Lagrangian in new fields (let $\phi_{1,2,3}$ remain the same, $\phi_4 \to \langle\phi_4\rangle + \sigma$). What are the masses of the fields? What is the remained symmetry of the system after the breaking of SO(4)?

Derive the Feynman rules. At leading order λ^2 calculate the amplitude

$$\phi_i\phi_j \quad \to \quad \phi_k\phi_l$$

How does it look the last amplitude in the limit $m_\sigma \to 0$ but m/λ fixed? How big it is when the initial momenta go to zero?

2. Consider the SM Higgs and calculate the masses of the gauge bosons after spontaneous symmetry breaking.

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