$B \rightarrow V\gamma$ at NNLO in SCET

Ben Pecjak

DESY

Ljubljana Seminar

September 20, 2007

Based on work with Ahmed Ali (DESY) and Christoph Greub (Bern)
Outline

- Motivation
- QCD Factorization and SCET as applied to $B \rightarrow V\gamma$
- Matching calculations and hard-scattering kernels
- Numerical results and comparison with experiment
Radiative $B$ decays

**Example:** $b \to s \gamma$ transition

$B \to X_s \gamma = \text{inclusive radiative decay}$

$B \to K^* \gamma = \text{exclusive radiative decay}$

FCNCs are loop suppressed in Standard Model

$\quad B \to X_s \gamma$ "standard candle" for new physics
Motivation for studying $B \rightarrow V\gamma$

Inclusive $B \rightarrow X_s\gamma$ has received much attention

- Calculable using OPE and heavy-quark expansion
- Branching fraction known to NNLO in perturbation theory (Misiak +16 others 2006)
- Also known to NNLO with cut on photon energy (Becher, Neubert 2006)

Exclusive $B \rightarrow V\gamma$ decays ($V = K^*, \rho, \omega, \ldots$) also useful

- Exclusive $b \rightarrow d\gamma$ ($V = \rho, \omega$) will be well measured at LHC
- Provide independent checks on shape of unitarity triangle
- Calculable in QCD factorization approach
Idea of QCD factorization

\[ 65856 = 12 \times 56 \times 98 \]
QCD factorization and today’s talk

QCD Factorization:
Branching fraction obtained as a series in \((\alpha_s, \frac{\Lambda_{\text{QCD}}}{m_b}) \ll 1\)

\[
B(\bar{B} \rightarrow V\gamma) = B^{\text{LO}} \left| t^I \zeta_{V\perp} + t^{\Pi} \phi_{\perp}^V \phi_B^V + O \left( \frac{\Lambda_{\text{QCD}}}{m_b} \right) \right|^2
\]

- \(\zeta_{V\perp}\) and \(\phi_B^V, \phi^V\) are non-perturbative (but universal)
- Hard-scattering kernels \(t^I\) and \(t^{\Pi}\) are perturbative

\[
t^I = 1 + O(\alpha_s) + \ldots \quad t^{\Pi} = O(\alpha_s) + \ldots
\]

- The kernels \(t^I, t^{\Pi}\) known at NLO \((O(\alpha_s))\) for some time

Today’s talk: Hard-scattering kernels at NNLO \((O(\alpha_s^2))\)
Why higher-orders?

QCD factorization is limited by:

- Hadronic uncertainties, especially in $\zeta_{V\perp}$
- Power corrections (although hard to quantify)

Why bother with higher-order perturbative corrections?

1) Practical reasons

- In some ratios hadronic uncertainties tend to cancel
- NLO for branching fractions is LO for CP asymmetries

2) Theoretical reasons

- Check factorization at NNLO
- Study connection between QCDF and SCET (next slide)
What is SCET and why use it?

**What**: SCET = soft-collinear effective theory
(Bauer, Pirjol, Fleming, Stewart 2000)

**Why**: Allows to discuss factorization in EFT language
  - hard-scattering kernels = matching (Wilson) coefficients
  - non-perturbative functions = hadronic matrix elements

Advantages of SCET approach:
  - Mass scales $m_b^2 \gg m_b \Lambda_{\text{QCD}} \gg \Lambda_{\text{QCD}}^2$ are clearly separated
  - RG evolution of matching functions “resums” large logs
  - All-orders factorization proof possible
(Becher, Hill, Neubert 2005)
Outline

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- Matching calculations and hard-scattering kernels
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Outline of factorization discussion

1) QCD factorization
   - Introduce effective weak Hamiltonian
   - QCD factorization formula for matrix elements of weak Hamiltonian

2) Derivation of QCD factorization formula with SCET
Effective weak Hamiltonian

QCD effects at $\mu \sim m_b$ described by effective weak Hamiltonian:

$$\mathcal{H}_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{pq}^* V_{pb} \left[ C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{8} C_i Q_i \right]; \quad (q = s, d)$$

- The $C_i$ are Wilson coefficients depending on $M_W, M_Z, m_t$
- The $Q_i$ are operators built from QCD and photon fields

The Wilson coefficients are known in RG-improved perturbation theory to NNLO ($\alpha_s^2$) (Csakon, Gorbahn, Haisch, Misiak, others ...)

This part is the same for inclusive and exclusive decays
Operators in effective weak Hamiltonian

Example: \( b \rightarrow s \gamma \)

\[
H_{\text{eff}} = \frac{G_F}{\sqrt{2}} \sum_{p=u,c} V_{ps}^* V_{pb} \left[ C_1 Q_1^p + C_2 Q_2^p + \sum_{i=3}^{8} C_i Q_i \right]
\]

Most important operators for \( b \rightarrow s \gamma \):

\[
Q_1^p = (\bar{s} \ p)_{V-A} (\bar{p} \ b)_{V-A} \quad (p = u, c)
\]

\[
Q_7 = -\frac{e m_b(\mu)}{8\pi^2} \bar{s} \sigma^{\mu\nu} [1 + \gamma_5] b F_{\mu\nu}
\]

\[
Q_8 = -\frac{g m_b(\mu)}{8\pi^2} \bar{s} \sigma^{\mu\nu} [1 + \gamma_5] T^a b G_{\mu\nu}^a
\]

For \( b \rightarrow d \gamma \) replace \( s \rightarrow d \)
Hadronic matrix elements of weak Hamiltonian

Branching fraction:

\[ B(B \to V\gamma) = \frac{\tau_B m_B}{4\pi} \left( 1 - \frac{m_V^2}{m_B^2} \right) |A|^2 \]

Amplitude:

\[ \mathcal{A} \sim \langle V\gamma|\mathcal{H}_{\text{eff}}|\bar{B}\rangle \sim \sum_i \langle V\gamma|Q_i|\bar{B}\rangle \]

**Main challenge:** evaluate \( \langle V\gamma|Q_i|\bar{B}\rangle = \text{hadronic matrix elements} \)

- QCD factorization is a method for doing this
QCD factorization formula I

Hadronic matrix elements factorize in heavy-quark limit

\[
\langle V_\gamma | Q_i | \bar{B} \rangle = F^{B \rightarrow V \perp}_{I} T_i^I + \int d\omega \, du \, \phi^B_\perp(\omega) \, \phi^V_\perp(u) \, T_{II}^i(\omega, u) + O\left(\frac{\Lambda_{QCD}}{m_b}\right)
\]

Non-perturbative pieces: (from QCD sum rules, lattice)

- \( F^{B \rightarrow V \perp} \) is a hadronic form factor in QCD
- \( \phi^B_\perp, \phi^V_\perp \) are light-cone distributions amplitudes (LCDAs)

Perturbative pieces: (as a series in \( \alpha_s \))

- \( T_i^I \) are “vertex corrections”
- \( T_{II}^i \) are “hard spectator corrections”
The QCD form factor itself obeys a factorization formula (Beneke, Feldmann 2001)

\[ F_{B \rightarrow V} = T_{V}^I \zeta_{V} + \int d\omega \, du \, \phi^B_+(\omega) \, \phi^V_+ (u) \, T_{V}^{II} (\omega, u) + \mathcal{O} \left( \frac{\Lambda_{\text{QCD}}}{m_b} \right) \]

- the soft function $\zeta_{V}$ is purely non-perturbative
- **Equivalent** forms of factorization formula

\[ \langle V \gamma | Q_i | \bar{B} \rangle = F_{B \rightarrow V} \, T_i^I + \int d\omega \, du \, \phi^B_+(\omega) \, \phi^V_+ (u) \, T_i^{II} (\omega, u) \]

\[ \langle V \gamma | Q_i | \bar{B} \rangle = \zeta_{V} \, t_i^I + \int d\omega \, du \, \phi^B_+(\omega) \, \phi^V_+ (u) \, t_i^{II} (\omega, u) \]

Second form more useful for SCET
The hard-scattering kernels and factorization

Interested in hadronic matrix elements

\[ \langle Q_i \rangle_{\text{had}} = \langle V \gamma | Q_i | \bar{B} \rangle \]

Instead calculate partonic matrix elements (Feynman Diagrams)

\[ \langle Q_i \rangle_{\text{part}} = \langle (q\bar{q}') \gamma | Q_i | (b\bar{q}') \rangle \]

If the partonic matrix element satisfies (to all orders)

\[ \langle Q_i \rangle_{\text{part}} = \zeta_{V_\perp,\text{part}} t_i^I + \int d\omega \, du \, \phi^B_{+,(\omega)} \phi^V_{\perp,\text{part}}(u) t_i^{II}(\omega, u) \]

Then assume

\[ \langle Q_i \rangle_{\text{had}} = \zeta_{V_\perp} t_i^I + \int d\omega \, du \, \phi^B_+(\omega) \phi^V(u) t_i^{II}(\omega, u) \]
Vertex corrections

These are virtual corrections to matrix elements in $B \rightarrow X_s \gamma$

- QCD graphs almost completely known at NNLO
- Can obtain $t^I$ to same accuracy
Hard spectator corrections

Complications:
1) Integrals depend on two different perturbative scales:

\[
(2p \cdot p_b \sim m_b^2) \quad \gg \quad (2p \cdot k \sim m_b \Lambda_{\text{QCD}} \sim (1.5 \text{ GeV})^2)
\]

"hard" \quad "hard collinear"

- Large perturbative logs of form \( \ln(m_b/\Lambda_{\text{QCD}}) \) in \( t^\ll{i} \)
  \( \Rightarrow \) Need resummation

2) Individual graphs can contribute to both \( t^\ll{i} \) and \( t^\ll{ii} \)

Dealing with both points easiest in SCET
SCET factorization formula

In soft-collinear effective theory:

\[
\langle V_\gamma | Q_i | \bar{B} \rangle = \Delta_i C^A \zeta_{V\perp} + \int d\omega \, du \, \phi^B_+(\omega) \phi^V_\perp(u) \, t_i^{\Pi}(\omega, u)
\]

Spectator term is subfactorized:

\[
t_i^{\Pi}(u, \omega) = \int_0^1 d\tau \Delta_i C^{B_1}(\tau) j_\perp(\tau, u, \omega) \equiv \Delta_i C^{B_1} \ast j_\perp
\]

- \( \Delta_i C^i \) contain physics at the hard scale \( m^2_b \)
- \( j_\perp \) contains physics at the jet scale \( m_b \Lambda_{\text{QCD}} \)
- \( \zeta_{V\perp}, \phi^i \) are matrix elements of SCET operators
  \( \Rightarrow \) Distinguish vertex and spectator terms at operator level
  \( \Rightarrow \) Resum logs with RG evolution
Two-step matching: $\text{QCD} \rightarrow \text{SCET}_I \rightarrow \text{SCET}_{II}$

$$m_b \gg \sqrt{m_b \Lambda_{\text{QCD}}} \gg \Lambda_{\text{QCD}}$$

QCD $\rightarrow$ SCET$_I$  
SCET$_I$ $\rightarrow$ SCET$_{II}$

- $\mu_h \sim m_b$
- $\mu_i \sim \sqrt{m_b \Lambda_{\text{QCD}}}$
- $\mu_f \gtrsim \Lambda_{\text{QCD}}$

match $\rightarrow$ RG evol. $\rightarrow$ match $\rightarrow$ RG evol.

$$\Delta_i C^{B1}(\mu_h) \times U_1(\mu_h, \mu_i) \times j_\perp(\mu_i) \times U_{II}(\mu_i, \mu_f) \times \{\phi_i(\mu_f)\}$$

- $\Delta_i C^{B1}(\mu_h)$ and $j_\perp(\mu_i)$ are free of large logarithms
- RG evolution of matching functions is resummation
First matching step: QCD → SCET₁

Match the operators $Q_{i}$ onto SCET₁:

$$Q_{i} \rightarrow \Delta_{i}C^{A}J^{A} + \Delta_{i}C^{B1} \ast J^{B1} + \Delta_{i}C^{B2} \ast J^{B2} + \mathcal{O}\left(\frac{\Lambda_{QCD}}{m_{b}}\right)$$

$J^{i}$ are current operators in SCET₁:

$$J^{A} = \bar{\chi}_{hc} \ell_{\perp} (1 + \gamma_{5})h_{v}$$
$$J^{B1} = \bar{\chi}_{hc} \ell_{\perp} \bar{A}_{hc\perp} (1 + \gamma_{5})h_{v}$$
$$J^{B2} = \bar{\chi}_{hc} \bar{A}_{hc\perp} \ell_{\perp} (1 + \gamma_{5})h_{v}$$

Important points about “SCET₁”

- Fluctuations at $m_{b}$ are integrated out and encoded in the $\Delta_{i}C^{i}$
- Matrix elements of SCET₁ operators depend on the hard-collinear scale $m_{b}\Lambda_{QCD}$ and hadronic scale $\Lambda_{QCD}$
Example: QCD $\rightarrow$ SCET$_I$ for $Q_8 \sim \bar{b}G^{\mu\nu}s$
Integrating out $m_b \Lambda_{QCD}$: SCET\textsubscript{I} $\rightarrow$ SCET\textsubscript{II}

Hard-collinear (intermediate) scale:

$$m_b \Lambda_{QCD} \sim (1.5 \text{ GeV})^2 = \text{perturbative}$$

Would like to integrate this out (SCET\textsubscript{I} $\rightarrow$ SCET\textsubscript{II})

$$J^i \rightarrow j_i(m_b \Lambda_{QCD}) \ast O^{i,\text{SCET II}}(\Lambda_{QCD})$$

- For $J^{Bi}$ will do this and define hard-spectator term
- For $J^A$ can’t do this because convolution diverges

But: $ \langle V_\gamma | J^A | \bar{B} \rangle \sim \zeta V_\perp$

- $J^A$ maps onto vertex term
The vertex corrections in SCET

From previous slide

\[ Q_i \rightarrow \Delta_i C^A J^A + \ldots \]

Matrix element of \( J^A \) defines the soft function:

\[ \langle V_\gamma | J^A | \bar{B} \rangle \sim \zeta_{V_\perp} \]

Therefore

\[ \langle V_\gamma | Q_i | \bar{B} \rangle = \Delta_i C^A \zeta_{V_\perp} + \ldots \]

and

\[ \Delta_i C^A = t_i^I \]

SCET matching coefficient = hard-scattering kernel
SCET matrix element = non-perturbative function
Second matching step: SCET$_I \rightarrow$ SCET$_II$

\[ Q_i \rightarrow \cdots + \Delta_i C^{B1} \star J^{B1} + \Delta_i C^{B2} \star J^{B2} \]

Can further match $J^{Bi}$ onto 4-quark operators in SCET$_II$

\[ J^{B1} \rightarrow \int du \int d\omega j_\perp \left( \tau, u, \frac{m_b \omega}{\mu^2} \right) O^{B1}(u, \omega) \]

\[ J^{B2} \rightarrow \int du \int d\omega j_\parallel \left( \tau, u, \frac{m_b \omega}{\mu^2} \right) O^{B2}(u, \omega) \]

Important points about “SCET$_II$”

- Scale $m_b \Lambda_{QCD}$ is integrated out and encoded in jet functions $j_i$
- Matrix elements of the SCET$_II$ operators depend $\Lambda_{QCD}$
- The matrix elements of the SCET$_II$ operators factorize into soft and collinear parts (no $\mathcal{L}_{eff}^{S+C}$)
\[ \Phi_{\alpha\beta}^B(\tilde{\omega}) = \int \frac{dt}{2\pi} e^{it\tilde{\omega}} \langle 0 | \bar{q}_s^\prime(tn_-) [tn_-, 0] h_{v\alpha}(0) | \bar{B} \rangle \]

\[ \Phi_{\gamma\delta}^V(u) = \int \frac{ds}{2\pi} e^{-isu_{n_+}p} \langle V(p) | \bar{\xi}_{c,\delta}(sn_+) [sn_+, 0] \xi_{c,\gamma}'(0) | 0 \rangle \]
The hard spectator term in SCET

Hadronic matrix elements of SCET II operators:

\[ \langle V \gamma | O^{B1} | \bar{B} \rangle \sim \phi^B_+ (\omega) \phi^V_\perp (u) \]
\[ \langle V \gamma | O^{B2} | \bar{B} \rangle = 0 \]

- \( \phi^B_+ \) is a matrix element of soft (HQET) operator
- \( \phi^V_\perp \) is a matrix element of a collinear operator
- Proving that soft and collinear sectors factorize is complicated (Becher, Hill, Neubert 2005)

Put together to define spectator term in SCET

\[ \langle V \gamma | (\Delta_i C^{B1} \star J^{B1}) | \bar{B} \rangle = (\Delta_i C^{B1} \star j_\perp) \star \phi^B_\perp \phi^V_\perp \]
\[ \equiv t^I_i \star \phi^B_\perp \phi^V_\perp \]
RG evolution from $\mu_h$ to $\mu_i$

\[
\Delta_i C^{B1}(\mu_h) \times U_1(\mu_h, \mu_i) \times j_\perp(\mu_i) \times U_{II}(\mu_i, \mu_f) \times \{\phi_i(\mu_f)\}
\]

The RG-improved hard coefficients read (Becher, Hill, Lee, Neubert 2004)

\[
\Delta_i C^{B1}(u, \mu_i) = \left(\frac{m_b}{\mu_h}\right)^{a(\mu_h, \mu_i)} e^{S(\mu_h, \mu_i)} \int_0^1 dv \ U_\perp(u, v, \mu_h, \mu_i) \Delta_i C^{B1}(v, \mu_h)
\]

The evolution factor $U_\perp$ obeys integro-differential equation

\[
\mu \frac{d}{d\mu} U_\perp(u, v, \mu_h, \mu) = \int_0^1 dy \ \gamma_\perp(y, u) U_\perp(y, v, \mu_h, \mu_i)
\]

The solution is found **numerically**
RG evolution from $\mu_f$ to $\mu_i$

$$\Delta_i C^{B1}(\mu_h) \times U_I(\mu_h, \mu_i) \times j_\perp(\mu_i) \times U_{II}(\mu_i, \mu_f) \times \{ \phi^B_+(\mu_f) \phi^V_\perp(\mu_f) \}$$

The evolution factor $U_{II}(\mu_i, \mu_f)$ is product of:

- Evolution of $\phi^V_\perp$ (Brodsky-Lepage kernel)
- Evolution of $\phi^B_+$ (Neubert, Lange 2003)

Effect of this resummation on branching ratios is small

- No details today (no time)
SCET factorization summary

SCET factorization formula:
\[
\langle V \gamma | Q_i | \bar{B} \rangle = \Delta_i C^A \zeta_{\perp} + (\Delta C^{B1} \ast j_{\perp}) \ast \phi^{V} \ast \phi^{B}
\]

- Physics at hard scale $m_b$ in the $\Delta C^i$
- Physics at hard-collinear scale $m_b \Lambda_{QCD}$ in $j_{\perp}$
- SCET matching coefficients are the hard-scattering kernels
  \[
  t^I_i = \Delta_i C^A \\
  t^{II}_i = \Delta_i C^{B1} \ast j_{\perp}
  \]
- $\zeta_{\perp}, \phi^{V}_{\perp}, \phi^{B}_{\perp}$ are matrix elements in SCET
- RG evolution is resummation
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Matching Calculations I: Vertex Corrections

\[ \langle V_\gamma | Q_i | \bar{B} \rangle = \Delta_i C^A \zeta_{V_\perp} + \left( \Delta C^{B_1} \ast j_\perp \right) \ast \phi^V \ast \phi^B \]
Strategy for the matching calculations

\[ Q_i \rightarrow \Delta_i C^A J^A + \Delta_i C^{B_1} \star J^{B_1} + \Delta_i C^{B_2} \star J^{B_2} \]

To find matching coefficients:

- Equate renormalized Green’s functions in QCD and SCET

Matching coefficients independent of external states.

Simplest:

- Partonic matrix elements (Feynman diagrams)
- Use dim. reg. (scaleless integrals vanish)
- On-shell (many loop diagrams in SCET vanish in dim. reg.)
- To extract \( \Delta_i C^A \) choose states that don’t overlap with \( J^{B_i} \)
Vertex corrections

\[ Q_i \rightarrow \Delta_i C^A J^A + \Delta_i C^B J^B + \ldots \]

\[ J^A = \bar{\chi}_{hc} \ell_\perp (1 + \gamma_5) h_v; \quad J^B = \bar{\chi}_{hc} \ell_\perp A_{hc\perp} (1 + \gamma_5) h_v \]

Observation: The \textit{B}-type current has an extra gluon field \( A_{hc\perp} \)

- To match onto \( J^A \) use partonic states with \textbf{no external gluons}

\[ \langle Q_i \rangle \equiv \langle q(p)\gamma(q)|Q_i|b(p_b) \rangle \]

- For on-shell matching the only kinematic invariants \( \sim m_b^2 \)

\[ p^2 = m_b^2; \quad p^2 = 0; \quad 2p \cdot p_b = m_b^2 \]
Matching conditions for vertex corrections

Matching condition:

\[ \langle Q_i \rangle_{\text{QCD,ren}} = \langle Q_i \rangle_{\text{SCET,ren}} \]

QCD matrix element:
- \( \langle Q_i \rangle_{\text{QCD,ren}} \) are the virtual corrections to inclusive \( B \rightarrow X_s \gamma \)
- Can take them from known calculations

SCET matrix element:
- For on-shell matching no hard-collinear scale
- \( \text{SCET}_1 \) loop integrals are scaleless and vanish in dim. reg.

\[ \Rightarrow \langle Q_i \rangle_{\text{SCET,ren}} = \Delta_i C^A Z_J \langle J^A_{\text{tree}} \rangle \]
- \( Z_J \) is the renormalization factor for the \( J^A \) operator
Matching conditions for vertex corrections: II

Write the QCD amplitude as

$$\langle Q_i \rangle_{\text{QCD,ren}} \equiv D_i \langle Q_7, \text{tree} \rangle$$

In dim. reg. ($d = 4 - 2\epsilon$) the matching condition is

$$\Delta_i C^A(m_b, m_c, \mu) = \Delta_7 C^{A(0)} \lim_{\epsilon \to 0} Z_J^{-1}(\epsilon, m_b, \mu) D_i(\epsilon, m_b, m_c, \mu)$$

▷ Tree level coefficient

$$\Delta_7 C^{A(0)} = -\frac{e \bar{m}_b 2E_\gamma}{4\pi^2} \approx -\frac{e \bar{m}_b m_b}{4\pi^2}$$

▷ The renormalization factor $Z_J$ determined order by order by requiring that $\Delta_i C^A$ finite as $\epsilon \to 0$
Results for vertex corrections

The coefficients are obtained as a series in $\alpha_s$:

$$\Delta_i C^A = \Delta_7 C^{A(0)} \left[ \delta_{i7} + \frac{\alpha_s(\mu)}{4\pi} \Delta_i C^{A(1)} + \left( \frac{\alpha_s(\mu)}{4\pi} \right)^2 \Delta_i C^{A(2)} \right]$$

- For $Q_7$ and $Q_8$ we obtained exact results to NNLO ($\alpha_s^2$)
- For $Q_1$ we obtained the NNLO results only in large-$\beta_0$ limit

Checks on results:

- No IR poles in matching coefficients (factorization)
- $J^A$ also appears in SCET treatment of $B \to X_s \gamma$ with cut on $E_\gamma$
  ⇒ Checked $Q_7$ with Becher and Neubert 2006 (results agree)
The coefficients for $Q_7$ and $Q_8$ to NNLO

\begin{align*}
(L = \ln \mu/m_b) \\
\Delta_7 C^{A(1)} &= C_F \left[ -2L^2 - 7L - 6.8225 \right], \\
\Delta_7 C^{A(2)} &= C_F^2 \left( 2L^4 + 14L^3 + 38.1449L^2 + 56.14711L + 7.8159 \right) \\
&\quad + C_F C_A \left( -4.8889L^3 - 33.9758L^2 - 92.3415L - 83.8866 \right) \\
&\quad + C_F n_l \left( 0.8889L^3 + 6.8889L^2 + 19.9050L + 23.8254 \right) \\
&\quad + C_F n_h \left( -1.3333L^2 + 2.8889L - 0.810288 \right) \\
\Delta_8 C^{A(1)} &= C_F \left[ 2.6667L + 1.4734 + 2.0944i \right], \\
\Delta_8 C^{A(2)} &= -C_F^2 \left[ 5.3333L^3 + 32.2802L^2 + 50.9612L + 1.8875 \\
&\quad + i(4.1888L^2 + 31.4159L + 29.8299) \right] \\
&\quad + C_F C_A \left[ 15.1111L^2 + 31.6617L + 2.3833 + i(23.7365L + 28.0745) \right] \\
&\quad - C_F n_l \left[ 1.7778L^2 + 4.0386L + 1.7170 + i(2.7925L + 4.4215) \right] \\
&\quad + C_F n_h \left[ 1.7778L^2 - 2.0741L + 0.8829 \right]
\end{align*}
The coefficients for $Q_1$ to NNLO

$$\Delta_1 C^{A(1)} = \frac{m_b}{m_b} C_F \left[ -3.8519 L + r^{(1)} \left( \frac{m_c^2}{m_b^2} \right) \right]$$

$$\Delta_1 C^{A(2)} = -\frac{3\beta_0}{2} \frac{m_b}{m_b} C_F \left[ 2.47L^2 + l^{(2)} \left( \frac{m_c^2}{m_b^2} \right) L + r^{(2)} \left( \frac{m_c^2}{m_b^2} \right) \right]$$

- $r^{(i)}$ and $l^{(2)}$ calculated as an expansion in $m_c^2/m_b^2$
- NNLO result is only known in large-$\beta_0$ limit ($n_f \to -3\beta_0/2$)
- Deviations from large-$\beta_0$ limit can be important (discussed later)
Matching Calculations II: 
Hard spectator corrections

$$\langle V\gamma |Q_i| \bar{B} \rangle = \Delta_i C^A \zeta_{V\perp} + \left( \Delta C^{B_1} \star j_{\perp} \right) \star \phi^V_{\perp} \star \phi^B_{\perp}$$

Plan:

- Review lowest-order ($\alpha_s$) results
- Explain structure of $\alpha_s^2$ results
- Explain our calculation of $\alpha_s^2$ corrections from $Q_8$
The structure of $t^{\Pi}$ at $\mathcal{O}(\alpha_s)$

The leading contributions are $\mathcal{O}(\alpha_s)$:

$$t_i^{\Pi(0)}(u, \omega) = \int_0^1 d\tau \Delta_i C^{B1(0)}(\tau) j_{\perp}^{(0)}(\tau, u, \omega)$$

The hard coefficients $\Delta_i C^{B1}$ depend on the operator

$$\Delta_7 C^{B1(0)}(\tau) = \frac{e m_b}{4\pi^2}, \quad \Delta_8 C^{B1(0)}(\tau) = \frac{1 - \tau}{\tau} \frac{1}{3} \frac{e m_b}{4\pi^2}$$

$$\Delta_1 C^{B1(0)}(\tau) = \frac{1}{3} \frac{e m_b}{4\pi^2} f \left( \frac{m_c^2}{m_b^2}, \tau \right)$$

The jet function $j_{\perp}$ is universal

$$j_{\perp}^{(0)}(\tau, u, \omega) = -\frac{4\pi C_F \alpha_s}{N_c} \frac{1}{m_b \omega \bar{u}} \delta(\tau - u)$$
The structure of \( t^{\Pi} \) at \( \mathcal{O}(\alpha_s^2) \)

The \( \mathcal{O}(\alpha_s^2) \) corrections take the form

\[
t_i^{\Pi(1)}(u, \omega) = \Delta_i C^{B1(1)} \ast j_\perp^{(0)} + \Delta_i C^{B1(0)} \ast j_\perp^{(1)}
\]

Status of \( \mathcal{O}(\alpha_s^2) \) corrections:

- The one-loop jet function \( j_\perp^{(1)} \) known  
  (Beneke and Yang, Becher and Hill 2004)
- The one-loop hard coefficient \( \Delta_7 C^{B1(1)} \) known 
  (Becher, Hill, Neubert 2005)
- The hard coefficient \( \Delta_8 C^{B1(1)} \) known (our work)
- \( \Delta_1 C^{B1(1)} \) remains unknown (requires two loops)
The calculation of $t_8^{II}$ at $\mathcal{O}(\alpha_s^2)$

$$t_8^{II(1)} = \Delta_8 C^{B1(1)} \ast j_{\perp}^{(0)} + \Delta_8 C^{B1(0)} \ast j_{\perp}^{(1)}$$

Calculate $t_8^{II}$ directly in QCD factorization, but:

- Evaluate partonic matrix elements using “method of regions” (Beneke and Smirnov 1997)
- Show $t_8^{II}$ depends on only hard and hard-collinear regions. The correspondence with SCET is:

$$A_{h,\text{fin}}^{(1)} = \Delta_8 C^{B1(1)} \ast j_{\perp}^{(0)}$$

$$A_{hc,\text{fin}}^{(1)} = \Delta_8 C^{B1(0)} \ast j_{\perp}^{(1)}$$

Since $j_{\perp}^{(0)}$ is a $\delta$-function, get $\Delta_8 C^{B1(1)}(\tau)$ from the convolution
One loop graphs for $Q_8$

($\times = $ photon emission)
Calculating $t_8^{II}$ in QCD factorization

Isolate the one-loop graphs whose Dirac structure matches $O^{B1}$

$$A_8^{(1)} = A^{(1)} \left( \ell_{\perp} \gamma_{\nu_{\perp}} \otimes \gamma_{\perp}^{\nu} \right) + \ldots$$

UV renormalized amplitude:

$$A^{(1)} \rightarrow A^{(1)} + A^{(1)}_{c.t.} = A^{(1)} + \left( Z^{(1)}_{\alpha} + Z^{(1)}_{m} + Z^{(1)}_{88} - \frac{u}{\bar{u}} \frac{Z^{(1)}_{87}}{Q_d} \right) A^{(0)}$$

Extract $t_8^{II}$ from renormalized amplitude:

$$\phi b\bar{q}'(0) * t_8^{II(1)} * \phi q\bar{q}'(0) = A^{(1)} + A^{(1)}_{c.t.} - \phi b\bar{q}'(0) * t_8^{II(0)} * \phi q\bar{q}'(0) - \phi b\bar{q}'(0) * t_8^{II(0)} * \phi q\bar{q}'(1)$$

IR poles in amplitude subtracted by renormalized LCDAs:

- $t_8^{II}$ is free of IR physics
- Factorization
Results for $t_{8}^{II}$

$$\Delta_8 C^{B1(1)} \times j_{\perp}^{(0)} = A_{h, \text{fin}}^{(1)} = \frac{\alpha_s}{4\pi} [C_F h_F + C_A h_A] t_{8}^{II(0)}$$

$$(L_{hc} = \ln(m_b \omega/\mu^2))$$

$$h_A = \left(4 - 2 \ln u - \frac{2 \ln u}{u^2} + 2 \ln \bar{u}\right)L_{hc} + i\pi \left(1 - \frac{1}{u} - \frac{2 \ln u}{u^2}\right) + 2 - \frac{\pi^2}{3} + \frac{3}{\bar{u}}$$

$$- \frac{\ln \bar{u}}{u} + \left(-1 + \frac{2}{u} - \frac{1}{u^2}\right) \ln u + \left(1 - \frac{3}{2u} - \frac{1}{2(2 - \bar{u})}\right) \ln \bar{u} \ln u - \ln^2 \bar{u}$$

$$+ \left(1 + \frac{1}{u^2}\right) \ln^2 u + \left(2 - \frac{1}{4u} - \frac{1}{2u^2} - \frac{1}{4(2 - \bar{u})}\right) \text{Li}_2(\bar{u})$$

$$+ \left(-\frac{5}{2u} + \frac{3}{u^2} - \frac{1}{2(2 - \bar{u})}\right) g(\bar{u}) + \left(-\frac{1}{2u} + \frac{1}{2(2 - \bar{u})}\right) h(\bar{u})$$

$h_F = \ldots$

with

$$g(u) = \int_0^1 dy \frac{\ln [1 - uy(1 - y)]}{y} = \ldots$$

$$h(u) = \int_0^1 dy \frac{\ln [1 - uy(1 - y)]}{1 - uy} = \ldots$$
Outline

- Motivation
- QCD Factorization and SCET as applied to $B \to V\gamma$
- Matching calculations and hard-scattering kernels
- Numerical results and comparison with experiment
The vertex and hard spectator amplitudes

The branching fraction for $B \rightarrow K^* \gamma$ decays is

$$\mathcal{B}(B \rightarrow K^* \gamma) = \frac{\tau_B m_B}{4\pi} \left( 1 - \frac{m_{K^*}^2}{m_B^2} \right) |A_v + A_{hs}|^2$$

The vertex ($v$) and hard-spectator ($hs$) amplitudes are

$$A_v = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cb} \sum_i C_i(\mu_{QCD}) \Delta_i C^A(m_b, \mu_{QCD}, \mu) \zeta_{K^*}(\mu)$$

$$A_{hs} = \frac{G_F}{\sqrt{2}} V_{cs}^* V_{cb} \sum_i C_i(\mu_{QCD}) t^I_i(\mu_{QCD}, \mu) \star (\phi_B \star \phi_{K^*})(\mu)$$

- $A_v$ and $A_{hs}$ are separately RG invariant
- Can study their contribution to the amplitudes separately
Vertex corrections to NNLO

The ratio of NNLO to LO is:

\[
\frac{A_v^{\text{NNLO}}}{A_v^{\text{LO}}} = 1 + (0.096 + 0.057i) [\alpha_s] + (-0.007 + 0.030i) [\alpha_s^2]
\]

In terms of individual contributions

\[
1 + \left( (0.264 + 0.034i) [Q_1] - (0.184) [Q_7] + (0.016 + 0.023i) [Q_8] \right) [\alpha_s] \\
+ \left( (0.073 + 0.022i) [Q_1] - (0.081) [Q_7] + (0.002 + 0.008i) [Q_8] \right) [\alpha_s^2]
\]

- NNLO correction is small because of large cancellation between $Q_1$ and $Q_7$
- That $Q_1$ is only large-$\beta_0$ limit result can be significant (See branching fractions)
Hard spectator corrections to NNLO

Total corrections:

\[ \frac{A_{\text{NNLO}}}{A_{\text{LO}}} = (0.11 + 0.05i) [\alpha_s] + (0.03 + 0.01i) [\alpha_s^2] \]

In terms of individual operators:

\[ = \left( (0.023 + 0.046i) [Q_1] + 0.074 [Q_7] + 0.010 [Q_8] \right) [\alpha_s] \]
\[ + \left( (0.004 + 0.003i) [Q_1] + 0.025 [Q_7] + 0.003 + 0.005i) [Q_8] \right) [\alpha_s^2] \]

([Q_1] = \Delta_1 C^{B1(0)} \ast f^{(1)}_{\perp})

- The NNLO corrections are individually small
  - Errors associated this term are small
  - Resummation effects \( \sim 10\% \) (but stabilize \( \mu \)-dependence)
Determining $\zeta_{V\perp}$ from the QCD form factor

Results depend strongly on $\zeta_{V\perp}$. To determine it:

- Require that $\langle V\gamma|Q_7|\bar{B}\rangle \propto F_{B\rightarrow V\perp}$
- Use the QCD factorization formula for $F_{B\rightarrow V\perp}$
- Use recent sum rule results $F_{B\rightarrow V\perp} = 0.31 \pm 0.4$ (Ball, Jones, Zwicky ’05)

At NNLO:

$$\zeta_{V\perp}(\mu = m_b) \simeq 0.35 \pm 0.05$$

The vertex corrections dominate the spectator ones:

$$\frac{F_{B\rightarrow V\perp}}{\zeta_{V\perp}} = (1 - 0.15[\alpha_s] - 0.06[\alpha_s^2])[v] + (0.07[\alpha_s] + 0.03[\alpha_s^2])[hs]$$
Branching Fractions at NNLO

Results at NNLO in units of $10^{-5}$

\begin{align*}
\mathcal{B}(B^+ \to K^{*+}\gamma) &= 4.6 \pm 1.2 \, [\zeta_{K^*}] \pm 0.4 \, [m_c] \pm 0.2 \, [\lambda_B] \pm 0.1 \, [\mu] \\
\mathcal{B}(B^0 \to K^{*0}\gamma) &= 4.3 \pm 1.1 \, [\zeta_{K^*}] \pm 0.4 \, [m_c] \pm 0.2 \, [\lambda_B] \pm 0.1 \, [\mu] \\
\mathcal{B}(B_s \to \phi\gamma) &= 4.3 \pm 1.1 \, [\zeta_{\phi}] \pm 0.3 \, [m_c] \pm 0.3 \, [\lambda_B] \pm 0.1 \, [\mu]
\end{align*}

Matching not complete because of $Q_1$:

- 100% uncertainty to NNLO vertex correction in large-$\beta_0$ limit: 
  \[ \Rightarrow \Delta B \approx \pm 0.5 \]

- 100% uncertainty to NLO hard-spectator correction: 
  \[ \Rightarrow \Delta B \approx \pm 0.1 \]

- Results beyond large-$\beta_0$ limit would reduce errors (but 3 loops)
Comparison with experiment

Compared to current experimental numbers (HFAG, LP 2007):

\[
\frac{\mathcal{B}(B^+ \to K^{*+}\gamma)}{\mathcal{B}(B^+ \to K^{*+}\gamma)_{\text{expt}}} = 1.1 \pm 0.35 \text{ [theory]} \pm 0.07 \text{ [expt.]} \\
\frac{\mathcal{B}(B^0 \to K^{*0}\gamma)}{\mathcal{B}(B^0 \to K^{*0}\gamma)_{\text{expt}}} = 1.1 \pm 0.35 \text{ [theory]} \pm 0.06 \text{ [expt.]} \\
\frac{\mathcal{B}(B_s \to \phi\gamma)}{\mathcal{B}(B_s \to \phi\gamma)_{\text{expt}}} = 0.8 \pm 0.2 \text{ [theory]} \pm 0.3 \text{ [expt.]} 
\]

- Theory errors about 30%
- Dominant error is in $\zeta V_\perp$
Limitations of QCD factorization approach to $B \to V\gamma$

1) Power corrections in $\Lambda_{QCD}/m_b$:
   - Some power corrections factorize, some don’t
   - While SCET has potential to deal with power corrections, no serious attempt so far
   - Treatment of these may rely on QCD sum rules (Ball, Zwicky, Jones 2007)

2) Hadronic uncertainties:
   - The branching fractions are very sensitive to $\zeta_{V\perp}$
   - The soft function $\zeta_{V\perp}$ has large uncertainties
     \[
     \zeta_{V\perp}(\mu = m_b) \approx 0.35 \pm 0.05
     \]

Theory errors in branching fractions stuck at 20-30% until these improve.
Conclusions

$B \rightarrow V\gamma$ decays are interesting and will probe $b \rightarrow d\gamma$.

The branching fractions obey a QCD factorization formula.

Improving it requires:

- **NNLO perturbative corrections**
- More precise knowledge of $\zeta V_\perp$
- Treatment of power corrections

We obtained most of the NNLO perturbative corrections.

The other points must be addressed for precise branching fractions.

**But:** can look for ratios where hadronic uncertainties drop out