A variational estimate of the Higgs mass

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Outline

- Motivation
- Reminder about the SM Higgs
- Brief intro to variational methods in QFT
- Application of variational approx to SM
- Stevenson et al’ results and problems
- Solution to the gauge invariance problem
- Summary and outstanding questions

Motivation:

- One (last?) open question in the Standard Model (*modulo* neutrino masses & CP violation), is the Higgs particle: it should (must?) be found @ LHC (2010-13?)
- All of the precision studies of the Standard Model have been based on perturbative calculations. What if the Higgs dynamics are non-perturbative?

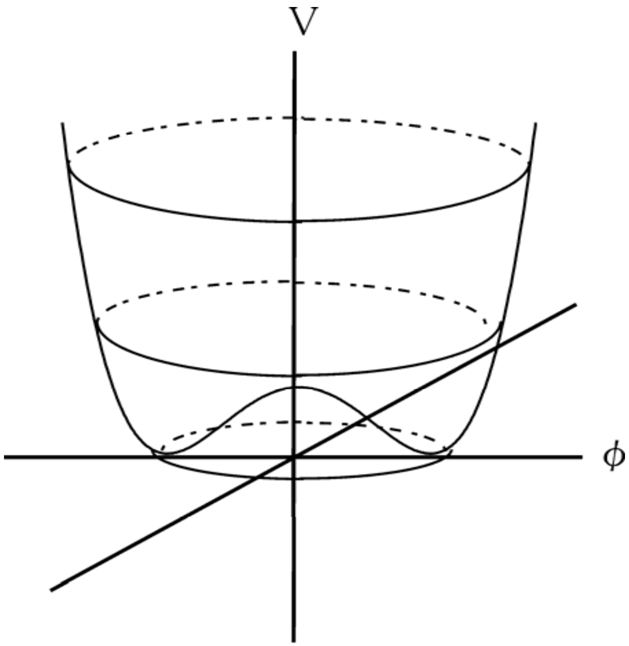
The Higgs sector of the Standard Model (=linear sigma model)

$$V_{Higgs,\sigma\,\mathrm{model}}(\phi_i) = -\frac{m_0^2}{2}(\phi_i\phi_i) + \frac{\lambda}{4}(\phi_i\phi_i)^2$$

• *The "Mexican hat" potential.*

$$\langle \phi_0 \rangle = \varphi_0 = f =$$

$$= (\sqrt{2}G_F)^{-1/2} = 246 GeV$$



Higgs boson in the Standard Model

- The Higgs meson mass is determined by the v.e.v. and the square root of the four-meson coupling λ
- All masses in the Standard Model are proportional v.e.v. (in the Born approximation)
- Higher order perturbative corrections introduce non-linear dependences.

$$m_{Higgs}{}^2 \cong 2\lambda \langle \phi_0 \rangle^2$$

$$m_{W,Z} \approx g \langle \phi_0 \rangle$$

$$m_{fermions} \approx g' \langle \phi_0 \rangle$$

Higgs boson vs. sigma meson

- The Higgs meson is the analog of the sigma meson in the sigma model (QCD). Its mass is determined by the v.e.v. and the square root of the four-meson coupling is larger than one, i.e. it is in the strong-coupling regime
- In the Standard Model the v.e.v. is larger than the calculated Higgs mass, so the coupling is weak. This result is consistent with the assumption that the perturbation theory is applicable, but does not prove its necessity.

$$\sigma(450)$$

$$\langle \phi_0 \rangle_{\sigma-\mathrm{model}} = f_\pi = 93 MeV$$

$$\lambda_{\sigma-\mathrm{model}} > 12$$

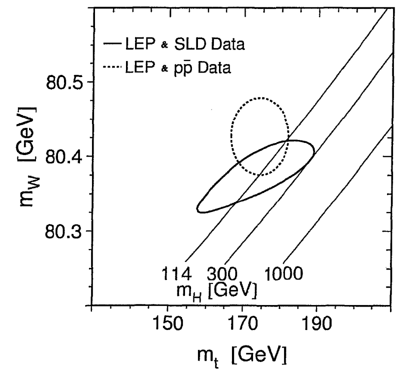
$$\langle \phi_0 \rangle_{SM} = 246 GeV$$

$$m_{Higgs} \approx 115 - 170 GeV$$

$$\lambda_{SM} \approx 0.125$$

Expectations from LHC: find the Higgs!

- Higgs mass is (severely) constrained by other measurements: What if LHC fails to find Higgs here?
- There are many scenarios for heavier Higgs(es). But the most mundane loophole is that these constraints were derived at the one-loop approximation level!
- What if the SM is OK, but higher loops are just as important? Need a non-perturbative approximation! e.g. variational !?!



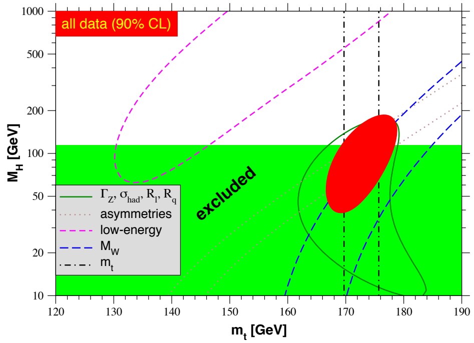


Figure 10.2: One-standard-deviation (39.35%) uncertainties in M_H as a function of m_t for various inputs, and the 90% CL region ($\Delta\chi^2 = 4.605$) allowed by all data. $\alpha_s(M_Z) = 0.120$ is assumed except for the fits including the Z-lineshape data. The 95% direct lower limit from LEP 2 is also shown. See full-color version on color pages at end of book.

Basics of the variational approximation to Quantum Field Theory

- Ground state energy evaluated variationally.
- Work in the Schroedinger picture L.I. Schiff, PR130, 458 (1963).
- Lorentz symmetry is not obvious, but the end result is Lorentz invariant (see Brian Hatfield's textbook.

$$\left|\Psi_0\right\rangle = \left|\Psi_0(\alpha_i)\right\rangle$$

$$E_0(\alpha_i) = \frac{\left\langle \Psi_0 \middle| H \middle| \Psi_0 \right\rangle}{\left\langle \Psi_0 \middle| \Psi_0 \right\rangle}$$

$$\left.\frac{\partial}{\partial \alpha_j} E_0(\alpha_i)\right|_{\min} = 0$$

Trial wave function(al): the Gaussian

- Need a trial wave function(al) for the ground state: one Gaussian in momentum space for each harmonic oscillator -- eigen-mode of the field
- The Gaussian wave functional is just a product of infinitely many Gaussian wave functions, i.e. HO wave functions. This is the "free" scalar quantum field theory ground state.

$$\left|\Psi_0[\phi_k]\right\rangle =$$

$$\left(\prod_{\vec{k}}^{\infty}\left(\frac{\omega_k}{\pi}\right)^{1/4}\right)\exp\left[-\frac{1}{2}\int\frac{d\vec{k}}{(2\pi)^3}\omega_k\,\tilde{\phi}(\vec{k})\tilde{\phi}(-\vec{k})\right]$$

Gaussian wave functional

- Evaluate the ground state energy with a Gaussian wave functional Ansatz (QFT) in configuration space with allowance made for non-vanishing vacuum expectation values (v.e.v.) of fields.

$$\left|\Psi_0(m_i,\langle\varphi_i\rangle)\right\rangle = N\exp(-\frac{1}{4}\int d\vec{x}\int d\vec{y}[\varphi_i(\vec{x})-\langle\varphi_i\rangle]G_{ij}^{-1}(\vec{x},\vec{y})[\varphi_j(\vec{y})-\langle\varphi_j\rangle])$$

Gaussian wave functional

- Must do functional integrals to find the necessary normalization and ground state energy matrix elements: the integrals are Gaussians, so they can be done exactly.
- Theory with N scalar fields [O(N) sigma model], function of 2N variational parameters (N masses and N v.e.v.s):

$$\langle \Psi_0 | \Psi_0 \rangle = 1$$

$$|\Psi_0[\phi_i]\rangle = |\Psi_0[\phi_i(m_i, \varphi_i)]\rangle$$

FGA in the linear sigma model

- Evaluate the energy expectation value in the ground state and minimize
- The resulting ground state ("vacuum") energy is related to the "effective potential"
- It modifies the old (up to 4th order) and induces new higher-order vertices.

$$E_0(m_i, \varphi_i) = \langle \Psi_0 | H | \Psi_0 \rangle \langle \Psi_0 | \Psi_0 \rangle^{-1}$$

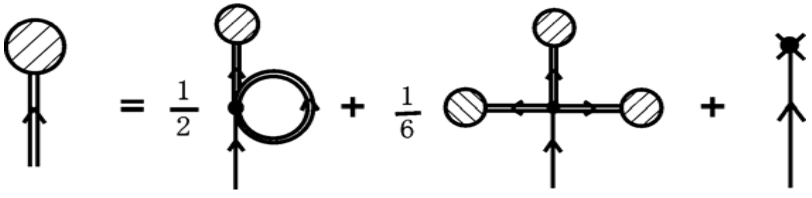
$$\left. \frac{\partial}{\partial m_j} E_0(m_i, \varphi_i) \right|_{\min} = 0$$

$$\left. \frac{\partial}{\partial \varphi_j} E_0(m_i, \varphi_i) \right|_{\min} = 0$$

$$V_{eff}(\varphi_i) = E_0(m_i, \varphi_i)_{\min} - E_0(m_i, \varphi_i = 0)$$

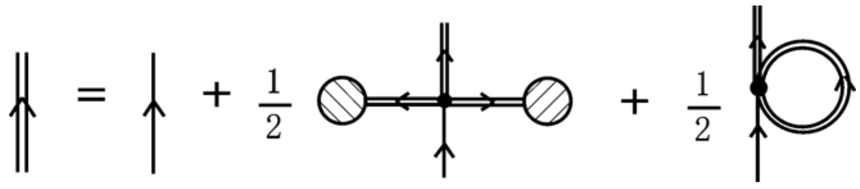
"Miracle #1": diagrammatic interpretation

- Minimization w.r.t. v.e.v. yields the zero-particle ("vacuum") Schwinger-Dyson equation.



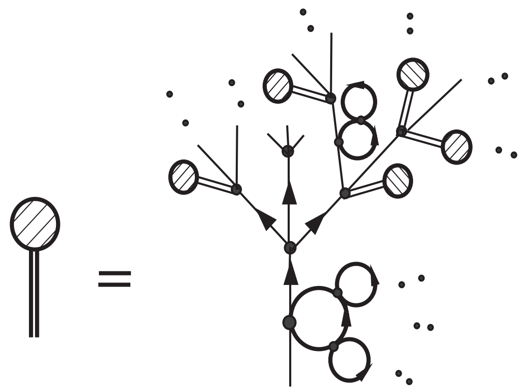
"Miracle #1": diagrammatic interpretation

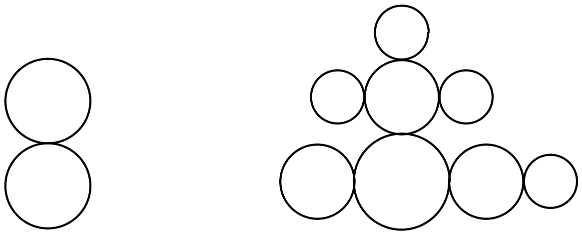
- Minimization w.r.t. masses yields the one-particle ("gap") Schwinger-Dyson equations.
- This is a truncation of the exact SD eqs: (only) one diagram is "missing". Contains many features of the exact eq.!
- This is the leading-N result in the 1/N expansion. Is the "missing" diagram negligible in the large-N limit?



(Physical) meaning of FGA eqs.?

- These SD eqs are integral eqs.: solve by iteration.
- Iteration produces (sums of) infinitely many ordinary Feynman diagrams
- FGA sums all "cactus", or "daisy" diagrams. Manifestly non-perturbative approximation.
- Each loop by itself contains a numerical infinity, so one must first regularize and then renormalize the FGA effective potential/ground state energy!



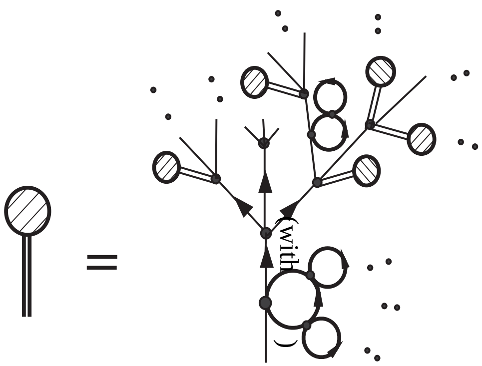


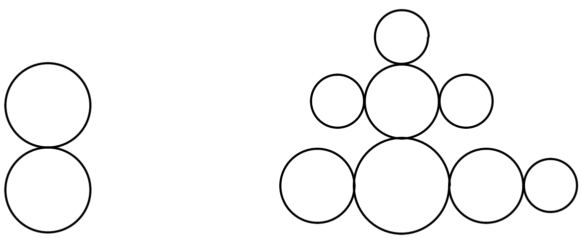
(1.a) (1.b)

Renormalization of FGA

- Must renormalize! Two non-perturbative renormalization prescriptions
- “autonomous” $(\lambda_B \geq 0)$

  Stevenson, Alles and Tarrach PRD35, 2407 (1987)
- “precarious” $(\lambda_B \leq 0)$ Stevenson PRD32, 1389 (‘85) (Symanzik’s ’70 “asymptotically free” ϕ^4 ?)
- corresponding to two possible approaches of space-time dimensionality d to 4: $(d \uparrow 4)$ and $(d \downarrow 4)$.





(1.a) (1.b)

Gaussian effective potential for the U(1) Higgs model

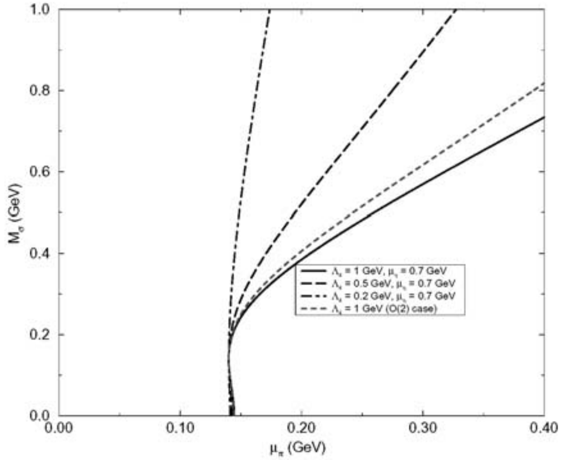
- Ibanez-Meier, Stancu and Stevenson. Z.Phys. C70: 307 - 320, (1996) [hep-ph/9207276] evaluated the U(1) Abelian Higgs model using "autonomous" renormalization and found a huge Higgs mass (2 TeV).

$$m_{Higgs} \approx 2\sqrt{2}\pi f = 2\sqrt{2}\pi \times 246 GeV \approx 2 TeV$$

- The calculation was challenged on the grounds that "it is not gauge invariant" What does this mean?
- The Higgs mechanism turns the would-be Nambu-Goldstone bosons into longitudinal components of gauge bosons; the Higgs mechanism works only if the NG bosons are exactly massless.
- Otherwise, violation of the underlying U(1) symmetry, hence "not gauge invariant".

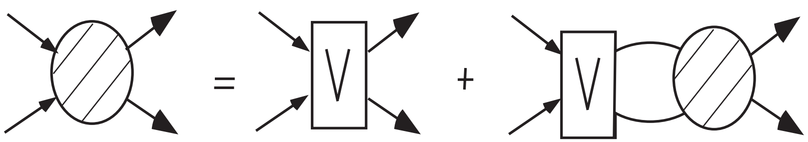
Properties of Nambu-Goldstone fields in FGA

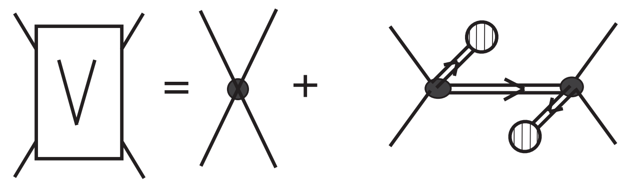
- Two coupled nonlinear equations in two unknowns, M, μ, and a new loop parameter, e.g. cutoff Λ, necessary to regularize the infinite integral.
- One free parameter can be fixed by demanding that $f = 246 GeV$
- The solutions do not have the (naively) expected properties:
- The (Nambu-Goldstone) pion field is not massless ($\mu \neq 0$), even in the chiral limit !?! first noted by Kamefuchi and Umezawa, Nuov. Cimento 31, 429 (1964). Problem rediscovered several times thereafter, finally solved in 1994!



"Miracle #2": Solution of the NG boson problem in FGA

- The problem was solved in [hep-th/9406151], published two years later (V.D., McNeil, Shepard, Z. Phys.C69, 359 (1996)):
- The "gap" mass μ is not the (physical) Nambu-Goldstone mass.
- Physical NG boson mass is given by the pole in the 4-point function, which satisfies the Bethe-Salpeter equation.





- The "kernel" of the BS eq. is defined by the Gaussian effective action: solve the BS eq.!
- The solution to the Bethe-Salpeter equation has a pole at: $m_{NG} = 0$
  q.e.d.

Two-body FGA Schroedinger eqn

- Set up a two-body wave functional
- Its e.o.m. is uniquely determined by the F. Gaussian Approx.
- Solve the two-meson eq., defined by: the solution to the Bethe-Salpeter equation has a pole at: $m_{NG} = 0$ *q.e.d.*

$$\left|\Psi_{2-body}\right\rangle = b_p^+ a_{-p}^+ \left|\Psi_0\right\rangle$$

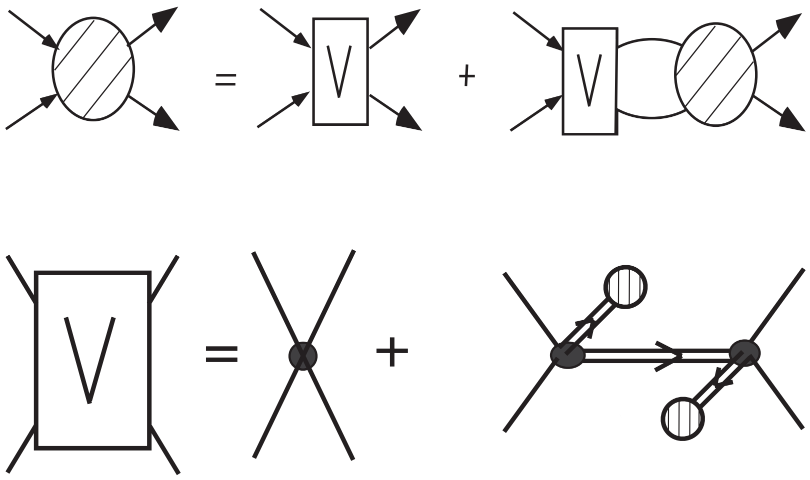
$$\left|\Psi_{NG}\right\rangle = \left|\Psi_{2-body}\right\rangle + \varepsilon\left|\Psi_{1-body}\right\rangle$$

$$E_{NG}(m_i,\varphi_i) = \frac{\left\langle\Psi_{NG}\right|H\left|\Psi_{NG}\right\rangle}{\left\langle\Psi_{NG}\middle|\Psi_{NG}\right\rangle} - E_0(m_i,\varphi_i)$$

$$\frac{\partial}{\partial m_j} E_{NG}(m_i,\varphi_i)\Big|_{\min} = 0 = \frac{\partial}{\partial \varphi_j} E_{NG}(m_i,\varphi_i)\Big|_{\min}$$

Two-body FGA Schroedinger eqn is equivalent to the Bethe-Salpeter eqn

- Set up a two-body BS equation
- Its kernel V (potential) is uniquely determined by the Gaussian app.
- Solve the two-body eq.: same solution as in variational approx.!
- The Nambu-Goldstone boson is “composite”!



Two-body variational equation for the Higgs/sigma meson

- Set up a two-body wave functional

$$\left|\Psi_{2-body}\right\rangle = \frac{1}{2}(b_p^+ b_{-p}^+ + a_p^+ a_{-p}^+)\left|\Psi_0\right\rangle$$

- Two coupled channels

$$\left|\Psi_\sigma\right\rangle = \left|\Psi_{2-body}\right\rangle + \varepsilon\left|\Psi_{1-body}\right\rangle$$

- Two different two-meson thresholds: 2M and 2μ.

$$E_\sigma(m_i,\varphi_i) = \frac{\left\langle\Psi_\sigma\right|H\left|\Psi_\sigma\right\rangle}{\left\langle\Psi_\sigma\middle|\Psi_\sigma\right\rangle} - E_0(m_i,\varphi_i)$$

- Solve the eqs..

$$\frac{\partial}{\partial m_j}E_\sigma(m_i,\varphi_i)\Big|_{\min} = 0 = \frac{\partial}{\partial \varphi_j}E_\sigma(m_i,\varphi_i)\Big|_{\min}$$

"Miracle#3": Connection with the Gaussian effective potential

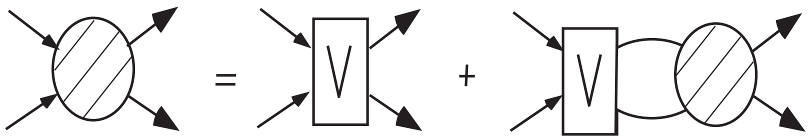
- NG boson mass via differentiating the Gaussian effective potential same as the FGA result. A. Okopinska (1995)
- Also yields the Higgs mass as propagator^{-1} value at s=0.

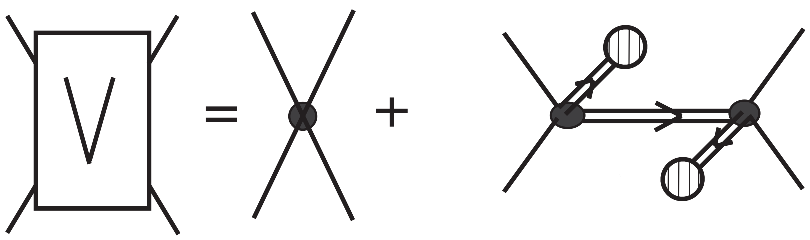
$$m_{NG}^2 = \frac{d^2}{d\left\langle \varphi_i \right\rangle^2} V_{eff}(\varphi_i) \Big|_{i=1,2,3} = 0$$

$$m_{Higgs}^2 = \frac{d^2}{d\left\langle \varphi_i \right\rangle^2} V_{eff}(\varphi_i) \Big|_{i=0} = M^2 \left(\frac{1 + 2\lambda_0 \left(3I_{MM} - I_{\mu\mu}\right) - 24\lambda_0{}^2 I_{MM} I_{\mu\mu}}{1 - \lambda_0 \left(3I_{MM} + 5I_{\mu\mu}\right) + 12\lambda_0{}^2 I_{MM} I_{\mu\mu}} \right)$$

Two-body Bethe-Salpeter is equivalent to the “Bootstrap” N/D equation

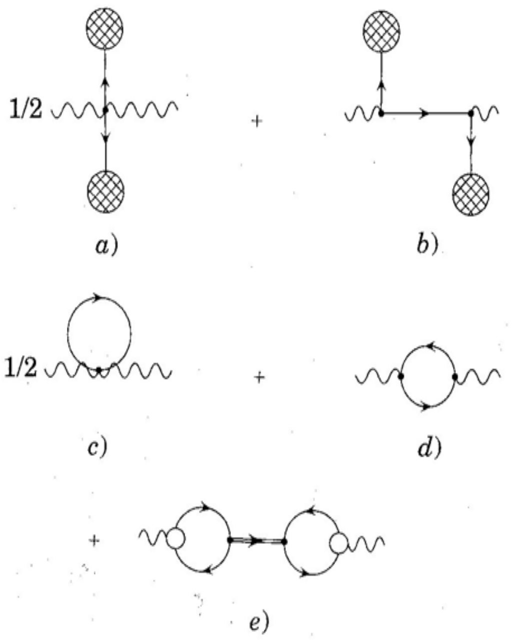
- Nagata, V.D. NPA713, 133 (‘03) have shown that the solution to the BS eq. is precisely equivalent to the N/D eq. (S-matrix theory) solution!
- All subtraction constants, CDD pole masses & couplings are entirely determined by the FGA
- Causality and unitarity are automatically ensured!
- Basic properties of QFT fulfilled by this non-perturbative approximation.





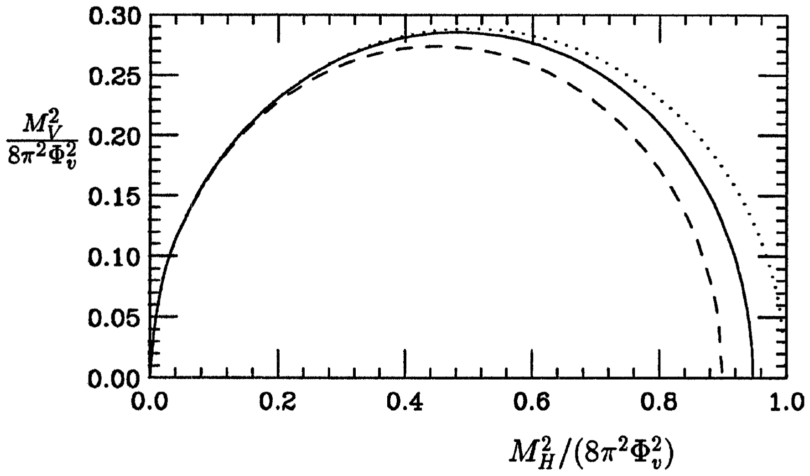
Gauge invariance of Higgs mechanism in FGA

- V.D. Nuov. Cim. 109A, 1187 (1996) constructed the vacuum polarization tensor in FGA and explicitly showed its gauge invariance and confirmed the Higgs mass
- All gauge problems resolved - may use in the Standard Model!



Gaussian approximation to the SU(2) Higgs model

- Consoli and Stevenson [hep-ph/9303256] extended the U(1) Abelian Higgs model to non-Abelian SU(2). Leads to only a small shift of the Higgs mass: 2 ->2.2 TeV.
- They found the following relation between the Higgs and the vector boson (W,Z) masses:
- The left-hand corner corresponds to the perturbative regime, but new non-perturbative solutions found!



Gaussian approximation to the SU(2) Higgs model II

- Interesting analytic result for the (renormalized) lambda coupling:
- What does this mean?
- Is it a renormalization group fixed point?
- How do these results reflect on the "triviality" of the lambda phi**4 theory.

$$\lambda_{SM} \cong 2\pi$$

Gaussian approximation and "triviality"

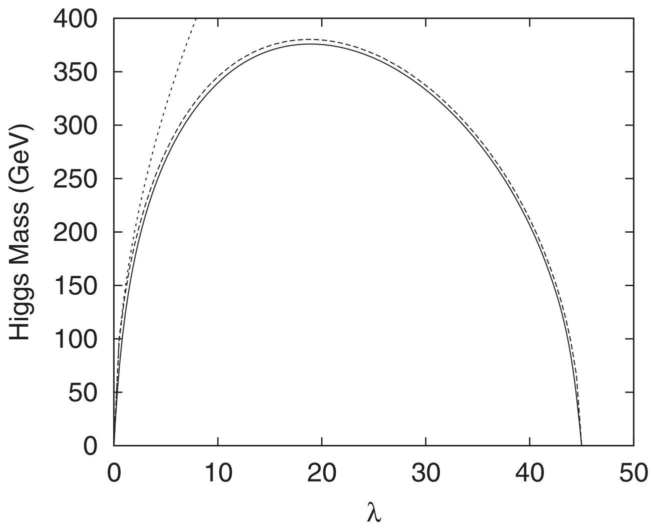
- In 4+epsilon dimensions this theory is non-interacting ("trivial"). In 4-epsilon dimensions it is perturbatively renormalizable. What about 4 dimensions?
- This led to a series of papers on "triviality": "'Triviality' made easy: The real Lambda phi**4 in four-dimensions story".
- "The Nontrivial effective potential of the 'trivial' (lambda phi**4) theory: A Lattice test." Z.Phys.C63:427-436,1994.
- "Landau poles, ultraviolet fixed points, and 'triviality' in the perturbative expansion of (Lambda phi**4) in four-dimensions." Mod.Phys.Lett.A11:2511-2524,1996.
- "Mode-dependent field renormalization and triviality in lambda phi**4 theory." Phys.Lett.B391:144-149,1997.
- At which stage I left the field to experts on triviality …

Summary of the FGA in the Higgs/sigma model

- Functional Gaussian Approximation is: (a) self-consistent and non-perturbative; (b) Lorentz and gauge invariant; (c) causal and unitary.
- May be used to calculate infinitely-many-loop corrections in QFT.
- Challenges: (a) Renormalization is difficult/ambiguous; (b) physical interpretation is not straight-forward.
- Interesting if perturbative predictions of Higgs mass do not pan out at LHC.

The latest (2008) from Catania

- **Siringa, Marotta PRD78: 016003 (2008) "Gaussian effective potential for the standard model SU(2) x U(1) electroweak theory"**
- **Higgs mass as a function of self-coupling constant λ w/o renormalization, i.e. with a cut-off $\Lambda = 9TeV$**



Back-up

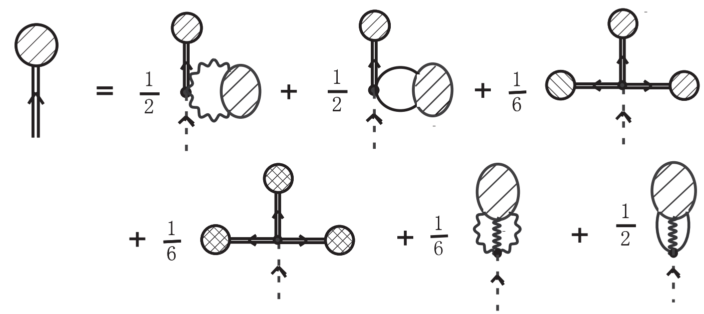


Figure 3.1: The one-point Green function Schwinger-Dyson equation in the linear Σ model with $\varepsilon = 0$: The dotted line denotes the bare meson multiplet. The shaded blob (the double-shaded blob) called "tadpole" denotes the vacuum expectation value of the $\phi_0(x)$ field ($\phi_{i=1,2,3}(x)$ fields), i.e. the one-point Green function. The solid dot in the intersection of the four lines denotes the bare four-point coupling. The diagrams are explicitly multiplied by their symmetry numbers.

$V^{ijkl}(s)$ denotes the potential.[‡] The symmetry factor, $\frac{1}{2}$ comes from the loop diagram. The diagrammatic interpretations are shown in Fig 3.5 and Fig 3.6 and those equations should be solved self-consistently with the one and two-point Green functions.[§] These self-consistent diagrams can be also seen in Ref. [11], while they applied GFA to the ϕ^4 theory.[¶] The diagrams include the effects of infinite many one-loops, for instance, as shown in Fig. 3.7. Of course, at this stage, we do not know whether or not our choice(or amputation) of the diagrams is good approximation. However, we see that the choice can be justified in sec. 3.8 because it maintains chiral symmetry.

Thus, in this section, we saw the explicit non-perturbative calculations based on path integral formalism in the linear Σ model, where the obtained self-consistent diagrams are identified with the diagrams in GFA presented later. As seen here, the path integral formalism is straightforward, which gives us a picture of particle-scattering, while it

[‡]By the careful replacements between the diagrams (one, two, three and four-point Green functions), we can obtain Eq. (3.14).

[§]We assumed $W_2^{ij}(k) = \delta_{ij}\frac{1}{-k^2 + m_i^2}$ in the momentum space, where m_i is the dressed mass of the mesons, and neglected the three and four-point Green functions in Fig. 3.1 and Fig. 3.2. Such approximated two-point Green function, represented by the double line, appears in Fig. 3.5 and Fig. 3.6.

[¶]The Ref. [11] showed the case of non-SSB with a single field. The SSB case is shown in Ref. [35].

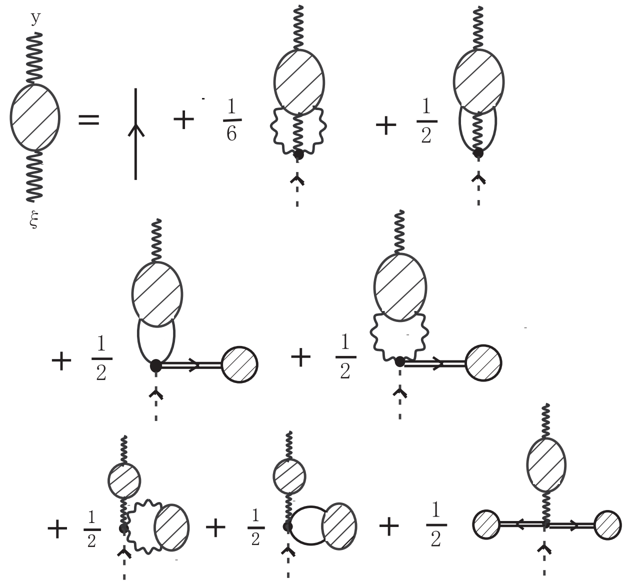


Figure 3.2: The two-point Green function Schwinger-Dyson equation in the linear Σ model: The symbols have the same meaning as in Fig 3.1.

has very complicated structure to solve. Therefore, in general, one suffers from how to approximate it and where to stop the iteration, choosing the diagrams which can maintain chiral symmetry. In addition, the physical meaning of the dressed masses in the two-point Green function are not clear at this stage. The three states of the dressed particles, i.e, $\phi_{i=1,2,3}$ never become Nambu-Goldstone bosons because the masses do not go to zero even in the chiral limit. Even if we can justify the masses in the theory, how can we recognize such dressed particles experimentally ? It looks as if there were extra particles, which do not apper to be Nambu-Goldstone boson in the theory. It is emphasized again that in terms of GFA, the masses of these dressed particles appear as the variational parameters, which never appear in the spectrum as a single particle excitation. More detailed discussions are presented in sec. 3.5 and sec. 3.9.3.