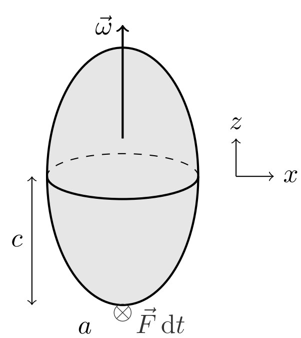
3. izpit iz Klasične mehanike, 30.8.2023

1. Consider a weight with mass m that slides on a surface of an icy planet of perfect spherical shape. Neglect friction.
a) Let first the planet be at rest with respect to an inertial frame. Write down the Lagrangian and derive the equation of motion! Which quantity is conserved?
b) From equations of motion derive the trajectory $\theta(\phi)$ of the weight!
c) Let now the planet turn around an axis that goes through the center of the planet with angular velocity ω. Write down the velocity vector *in the rotating frame* (that is at rest with respect to the planet) using the spherical coordinates and corresponding spherical base vectors.
d) At time $t = 0$ the weight is released with vanishing velocity with respect to the surface of the planet at a latitude that corresponds to $\theta = \pi/6$ (measured from the North pole). Describe the motion followed by the weight at later times (from the point of view of both inertial and rotating reference frame).

2. Consider the motion of a rigid and homogeneous spheroid that is axially symmetric along the axis z. Let us denote the semi-axes of the spheroid as a and c (see the sketch).
a) Calculate the volume and the tensor of inertia! Express the results in terms of a, c, and the mass of the spheroid m. Hint: The surface of the spheroid is described by the equation $\frac{x^2+y^2}{a^2} + \frac{z^2}{c^2} = 1$.

Let the spheroid rotate around the axis z with an angular velocity ω_z. At a certain moment, we apply a sudden force $\vec{F}\,\mathrm{d}t$ to the bottom of the spheroid, as shown in the sketch (imagine a small bump on the bottom of the spheroid). Let J_x and J_z denote the eigenvalues of the tensor of inertia. Consider cases where $J_x < J_z$ and $J_x > J_z$.

b) After the impact, in which of the two cases can the vector of angular velocity $\vec{\omega}$ never dip below the horizontal plane, regardless of the strength of the impact? Justify the result with the help of sketches for both cases.
c) Determine the critical value of the J_x/J_z ratio for which $\vec{\omega}$ does not dip below the horizontal plane. Additionally, find the ratio of the semi-axes c/a for this critical case.



3.
a) Let the motion in 1d be described by a Hamiltonian $H = p^2/2 - (1/2)q^{-2}$, where q and p are the canonical position and canonical momentum, respectively, $\{q, p\}$ =1. Using the Poisson brackets calculate the total time dependence of a quantity $D = pq/2 - Ht$, where t is time!
b) Now consider the generalization of the problem to arbitrary dimensionality as described by the Hamiltonian $H = |\mathbf{p}|^n - a|\mathbf{q}|^{-n}$, where a and n are constants. What is the time dependence of $D = \mathbf{p} \cdot \mathbf{q}/n - Ht$?