KLM mala nihanja

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1. Ponovi izpeljavo matričnega opisa enačb za mala nihanja, zapiši njihovo rešitev v obliki lastnih nihajnih načinov! Pokaži, da velja posplošena ortogonalnostna relacija med vektorji amplitud odmikov! Zapiši Lagrangeovo funkcijo z uporabo normalnih koordinat!
2. Na vodoravno nameščeno ravno vodilo namestimo zaporedno 3 uteži z masami m, m', m in jih spnemo z vzmetema z enakima koeficientoma k, dolžina neraztegnjene vzmeti l. Uteži po vodilu gladko drsijo. Izračunaj lastne nihajne načine, lastne frekvence! Zapiši Lagrangeovo funkcijo v normalni obliki! Zapiši položaj uteži ob kasnejših časih, če ob t=0 prvo maso sunemo, tako da ima hitrost v0! Koliko energije je shranjeno v posamičnih nihajnih načinih? Preveri, da je vsota energij enaka začetni energiji!

1. Sistem opisan s $q_1, \dots, q_N$. potencial $V(\underline{q})$.
Razvijemo okoli ekstrema (za mala nihanja minimuma potenciala) pri $\underline{q}^0$

$q_i = q_i^0 + \eta_i$

$$V = V_0 + \frac{1}{2}\sum_{ij} \underbrace{\frac{\partial^2 V}{\partial q_i \partial q_j}}_{V_{ij}}\Bigg|_{\underline{q}=\underline{q}^0} \eta_i \eta_j + \dots$$

$$V = \frac{1}{2}\underline{\eta}^T \underline{\underline{V}}\,\underline{\eta}$$

kin. energija za male η

$$T = \frac{1}{2}\dot{\underline{\eta}}\,\underline{\underline{T}}\,\dot{\underline{\eta}}$$

$$\begin{pmatrix} m_1 & \\ & m_2 \end{pmatrix}$$

lahko pa tudi bolj kompl. a vedno

$\underline{\underline{T}}, \underline{\underline{V}}$ simetrični, poz. definitni.

$$L = T - V$$

$$\underline{\underline{T}}\,\ddot{\underline{\eta}} + \underline{\underline{V}}\,\underline{\eta} = 0$$

nastavek $\underline{\eta} = \underline{\eta}^0 e^{-i\omega t}$

$(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}})\,\underline{\eta}^0 = 0$; posplošen problem l.v.

$\det(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}}) = 0$ … ω^2 … lastne frekv l.v.

$$\det\left(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}}\right) = 0 \quad \dots \quad \omega^2 \dots \text{lastna frekv.}$$

$$\left(-\omega_n^2 \underline{\underline{T}} + \underline{\underline{V}}\right) \underline{a}_n = 0 \qquad n\text{-ti l. vektor}$$

Splošna rešitev:

$$\boxed{\underline{\eta}(t) = \mathrm{Re} \sum_i \underline{a}_i \begin{cases} d_i \, e^{-i\omega_i t} & ; \omega_i > 0 \\ c_i + \tilde{c}_i t & ; \omega_i = 0 \end{cases}}$$

Zač. pogoji $\underline{\eta}(t=0) = \dots \qquad \dot{\underline{\eta}}(t=0) = \dots$

Velja posp. ortog.:

$$\underline{a}_i^T \underline{\underline{T}} \underline{a}_j = \delta_{ij}$$

(l.v., ki pripadajo različnim ω_n so ort.) lahko izberemo normo tako, da $\underline{a}_i \underline{\underline{T}} \underline{a}_i = 1$

D.: $a_i^T / \quad \omega_j^2 \underline{\underline{T}} \underline{a}_j = \underline{\underline{V}} \underline{a}_j \quad (1)$

$$\omega_i^2 \underline{a}_i^T \underline{\underline{T}} = a_i^T \underline{\underline{V}} \qquad / \underline{a}_j \ (2) \ ; \ (1)-(2)$$

$$\left(\omega_i^2 - \omega_j^2\right) a_i^T \underline{\underline{T}} a_j = 0$$

normalna oblika:

normalne koord.

$$\underline{\eta} = \sum_i d_i(t) \, \underline{a}_i$$

$$L = \frac{1}{2} \sum_{ij} \dot{d}_i \dot{d}_j \underbrace{\underline{a}_i^T \underline{\underline{T}} \underline{a}_j}_{\delta_{ij}} - \frac{1}{2} \sum_{ij} d_i d_j \underbrace{\underline{a}_i \underline{\underline{V}} \underline{a}_j}_{\omega_i^2 \delta_{ij}}$$

$$\boxed{L = \frac{1}{2} \sum_i \left(\dot{d}_i^2 - \omega_i^2 d_i^2\right)}$$

L izražen v norm. obliki

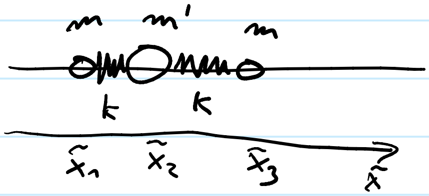
energija je $E = \frac{1}{2} \sum_i \left(\dot{d}_i^2 + \omega_i^2 d_i^2\right) = \sum_i E_i$

$$d_i = d_i^0 \cos \omega t \qquad E_i = \begin{cases} \frac{d_{i0}^2}{2} \omega_i^2 & ; \omega_i \neq 0 \\ \dots \end{cases}$$

$a_i = a_i^0 \cos \omega t$ $\qquad E_i = \begin{cases} \frac{a_{i0}^2}{2}\,\omega_i^2 \,; & \omega_i \neq 0 \\ \dot a_i^2/2 \,; & \omega_i = 0 \end{cases}$

energija v i-tem nih. načinu.

2.



$\tilde x_1 = x_{10} + x_1 \qquad \tilde x_2 = x_{20} + x_2 \qquad \tilde x_3 = x_3^0 + x_3$

$\qquad\qquad = x_{10} + \ell + x_2$

$\tilde x_1 < \tilde x_2 < \tilde x_3$

raztezek ~~/~~skrček vzmeti glede na neraztegnjeno

$$V = \frac{k(\tilde x_2 - \tilde x_1 - \ell)^2}{2} + \frac{k(\tilde x_3 - \tilde x_2 - \ell)^2}{2}$$

$$V = \frac{k(x_2 - x_1)^2}{2} + \frac{k}{2}(x_3 - x_2)^2$$

potencial že v kvad. obliki

$$T = \sum_i \frac{m_i \dot x_i^2}{2} \qquad V = \frac{1}{2}k(x_1^2 + x_2^2 - 2x_1x_2 + x_3^2 + x_2^2 - 2x_2x_3)$$

$$\underline{\underline{T}} = \begin{bmatrix} m & & \\ & m' & \\ & & m \end{bmatrix} \qquad \underline{\underline{V}} = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$\left(-\omega^2 \underline{\underline{T}} + \underline{\underline{V}}\right)\underline{a}_i = 0$$

2. načina: direktno

$$\det\begin{pmatrix} -\omega^2 m + k & -k & 0 \\ -k & -\omega^2 m' + 2k & -k \\ 0 & -k & -\omega^2 m + k \end{pmatrix} = 0$$

$$(-\omega^2 m + k)\left[(-\omega^2 m + k)(-\omega^2 m' + 2k) - k^2\right] + k\left(-k(-\omega^2 m + k)\right) = 0$$

$$(-\omega^2 m + k)\left[(-\omega^2 m + k)(-\omega^2 m' + 2k) - 2k^2\right] = 0$$

$$(-\omega^2 m + k)\left[\omega^2 m m' \omega^2 - \omega^2(m\,2k + m'k)\right] = 0$$

$$(-\omega^2 m + k)\,\omega^2\left[\omega^2 m m' - (m2k + m'k)\right] = 0$$

$\boxed{\omega_1^2 = k/m} \qquad \boxed{\omega_2^2 = 0} \qquad \boxed{\omega_3^2 = \frac{k}{m}\left(2\frac{m}{m'} + 1\right)} = \frac{k}{m'}\left(2 + \frac{m'}{m}\right)$

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l. v. npr. 3:

$$\begin{pmatrix} -k\frac{2m}{m'} & -k & 0 \\ -k & -\frac{m'}{m}k & -k \\ 0 & -k & -k\frac{2m}{m'} \end{pmatrix} =$$

$$\underline{a}_3 = \left(1, -2\frac{m}{m'}, 1\right)c$$

$$\underline{a}_3^T \underline{\underline{T}}\, \underline{a}_3 = c^2\left(m + 4\frac{m^2}{m'} + m\right) = 1 \qquad c = \frac{1}{\sqrt{2m + 4m^2/m'}}$$

$$\underline{a}_3^T = \left(1, -2m/m', 1\right)\frac{1}{\sqrt{2m + 4m^2/m'}}$$

$$\underline{a}_1^T = \frac{(1,1,1)}{\sqrt{2m+m'}} \qquad \underline{a}_2^T = \frac{(1,0,-1)}{\sqrt{2m}}$$

2. o kombinacija ugomjemnja in antagonalnostne relacije

1. translacija (gibanje težišča)

$$\underline{a}_1^T = (1,1,1)c_1$$

$$\underline{\underline{V}}(1,1,1) = 0 \Rightarrow \omega_1^2 = 0$$

2. m_2 miruje, težišče miruje

$$\underline{a}_2^T = (1,0,-1)c_2 \qquad (\tilde{a}, 0, b)\cdot(1,1,1) = 0 \Rightarrow a = -b$$

3. $\underline{a}_3 = (\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$

$$\underline{a}_3^T \underline{\underline{T}}\, \underline{a}_2 = 0 \Rightarrow \tilde{a}_1 = \tilde{a}_3$$

$$(\tilde{a}_1, \tilde{a}_2, \tilde{a}_1)\,\underline{\underline{T}}\begin{pmatrix}1\\1\\1\end{pmatrix} = 0$$

$$m\tilde{a}_1 + m'\tilde{a}_2 + m\tilde{a}_1 = 0$$

$$m\tilde{a}_1 + m'\tilde{a}_2 + m\tilde{a}_1 = 0$$

$$\tilde{a}_2 = -\frac{2m}{m'}\tilde{a}_1$$

$$\underline{a}_3^T = \left(1, -\frac{2m}{m'}, 1\right)$$

Lagrang. v norm. obliki

$$L = \frac{1}{2}\left[\dot{\alpha}_1^2 + \left(\dot{\alpha}_2^2 - \frac{k}{m}\alpha_2^2\right) + \dot{\alpha}_3^2 - \frac{k}{m}(2m/m' + 1)\alpha_3^2\right]$$

Splošna rešitev

$$(x_1, x_2, x_3)^T(t) = \underline{a}_1\,(c_1 + \tilde{c}_1 t)$$
$$+ \underline{a}_2\,(c_2 \cos\omega_2 t + \tilde{c}_2 \sin\omega_2 t)$$
$$+ a_3\,(c_3 \cos\omega_3 t + \tilde{c}_3 \sin\omega_3 t)$$

Zač. pogoj

$$\underline{x}(t=0) = (0,0,0)^T \;; \quad c_1 = c_2 = c_3 = 0$$
$$\dot{\underline{x}}(t=0) = (v_0,0,0)^T \;; \quad (v_0,0,0)^T = \underline{a}_1\tilde{c}_1 + \underline{a}_2\tilde{c}_2\omega_2 + \underline{a}_3\tilde{c}_3\omega_3$$

$$\underline{a}_1^T \underline{\underline{T}} \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix} = \tilde{c}_1 = \frac{m}{\sqrt{2m+m'}}v_0$$

$$\underline{a}_2^T \underline{\underline{T}} \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix} = \tilde{c}_2\omega_2 = \frac{m v_0}{\sqrt{2m}} \;; \quad \tilde{c}_2$$

$$a_3^T \underline{\underline{T}} \begin{pmatrix} v_0 \\ 0 \\ 0 \end{pmatrix} = \tilde{c}_3\omega_3 = \frac{m v_0}{\sqrt{2m + 4m^2/m'}}$$

$$(x_1, x_2, x_3)(t) = \frac{1}{\sqrt{2m+m'}}\,\frac{m}{\sqrt{2m+m'}}\,v_0 t\,(1,1,1)$$
$$+ \frac{1}{\sqrt{2m}}\,\frac{m v_0}{\sqrt{2m}}\,(1,0,-1)\left(\frac{\sqrt{m}}{\sqrt{k}}\right)\sin(\omega_2 t)$$

$$+ \frac{1}{\sqrt{2m+4m^2/m'^2}} \frac{m v_0}{\sqrt{2m+4m^2/m'}} \frac{1}{\omega_3} \left(1, \frac{-2m}{m'}, 1\right) \sin(\omega_3 t)$$

$$\underline{x} = \hat{c}_1 t \, \underline{a_1} + \hat{c}_2 \sin \omega_2 t \, \underline{a_2} + \tilde{c}_3 \sin \omega_3 t \, \underline{a_3}$$

energija v lastnih načinih.

$$E_1 = \frac{\tilde{c}_1^2}{2} = \qquad E_2 = \frac{\omega_2^2 \tilde{c}_2^2}{2} = \frac{m v_0^2}{4}$$

$$= \frac{m^2}{2(2m+m')} v_0^2 \qquad E_3 = \frac{m^2 v_0^2}{2(2m + 4m^2/m')}$$

$$E_1 + E_2 + E_3 = \frac{m^2 v_0^2}{2} \left[\frac{1}{2m+m'} + \frac{1}{2m} + \frac{1}{2m+4m^2/m'} \right]$$

$$= \frac{m v_0^2}{2} \left[\frac{1}{2+\mu} + \frac{1}{2} + \frac{1}{2+4/\mu} \right] =$$

$$= \frac{m v_0^2}{2} \left[\frac{4 + 8/\mu + (2+\mu)(2+4/\mu) + 4 + 2\mu}{(4+2\mu)(2+4/\mu)} \right]$$

$$= \frac{m v_0^2}{2} \left[\frac{1/\mu \, \cancel{(4+2\mu)} [2 + (2+\mu) + \mu]}{(2+4/\mu)\cancel{(4+2\mu)}} \right]$$

$$= \underline{\frac{m v_0^2}{2}}$$