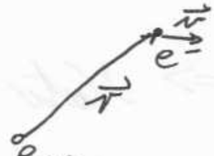
GIBANJE ELEKTRONA V POLJU MAGNETNEGA MONOPOLA

Gostoto magnetnega polja $\vec B$ okoli magnetnega monopola opišemo z enačbo

$$\vec B = \frac{\mu_0 e_m}{4\pi r^2}\,\frac{\vec r}{|\vec r|}.$$

e_m = „magnetni naboj"



Upoštevamo še, da je za naš sistem $\vec E = 0$ in uporabimo Lorentzovo silo:

$$\vec F = e(\vec E + \vec v \times \vec B).$$

~~Upoštevamo~~ 2. Newtnov zakon in dobimo

$$m\ddot{\vec r} = e\,\dot{\vec r}\times\frac{g}{r^2}\frac{\vec r}{r}$$, kjer je $g = \frac{\mu_0 e_m}{4\pi}$,

kar nas pripelje do enačbe gibanja:

$$\boxed{\ddot{\vec r} = \frac{eg}{m}\,\dot{\vec r}\times\frac{\vec r}{r^3}.}$$

Nato si ogledamo nekaj ~~lastnosti~~ zanimivosti naše enačbe gibanja, do katerih pridemo z lastnostmi skal. in vekt. produkta.

i) $\cdot\vec r$: $\quad \ddot{\vec r}\cdot\vec r = 0 \to \ddot{\vec r}\perp\vec r$

ii) $\cdot\dot{\vec r}$: $\quad \ddot{\vec r}\cdot\dot{\vec r} = \frac{1}{2}(\dot r^2)^\cdot = \frac{1}{2}(v^2)^\cdot = 0$

$$\frac{d}{dt}(\dot{\vec r}\cdot\dot{\vec r}) = \ddot{\vec r}\cdot\dot{\vec r} + \dot{\vec r}\cdot\ddot{\vec r}$$

$$\left(\frac{1}{2}mv^2\right)^\cdot = 0 \quad\to\quad \underline{W_k = \text{konst.}\,!}$$

iii) $\times\vec r$ $\quad \ddot{\vec r}\times\vec r = \frac{eg}{m}\left(\dot{\vec r}\times\frac{\vec r}{r^3}\right)\times\vec r$

Pomislimo: $\vec\Gamma = \vec r\times\vec p = \vec r\times m\dot{\vec r}$ (1)

in $\frac{d}{dt}(\vec r\times\dot{\vec r}) = \underbrace{\dot{\vec r}\times\dot{\vec r}}_{0} + \vec r\times\ddot{\vec r}$ (2) !!

(2) $\Rightarrow \frac{d}{dt}(\vec r\times\dot{\vec r}) = \frac{eg}{m}\left(\frac{\vec r}{r^3}\times\dot{\vec r}\right)\times\vec r \quad /m$

~~skalarni produkt!!!~~

$\cancel{(\vec r\times\dot{\vec r})\times\vec r\times(\vec r\times\dot{\vec r})}$

$\cancel{(\dot{\vec r}\times\vec r)\times(\vec r\times\dot{\vec r})}$

$$\dot{\vec\Gamma} = \frac{eg}{m}\frac{1}{r^3}\,\vec\Gamma\times\vec r \quad /\cdot\vec\Gamma$$

$$\dot{\vec\Gamma}\cdot\vec\Gamma = 0 = \frac{1}{2}(\Gamma^2)^\cdot \quad\to\quad \underline{\Gamma = \text{konst.}\,!}$$

$\Rightarrow$ 1) $\dfrac{\dot{\vec{\Gamma}}}{eg} = \dfrac{\vec{\Gamma} \times \vec{r}}{m r^3}$ 3) $|\vec{\Gamma}| = konst.$

2) $\dot{\vec{r}}^{\,2} = konst$ 4) $\dot{\vec{r}} \cdot \vec{r} = 0$

$\Rightarrow$ Nato iščemo $\vec{r}(t)$.

Iz 1):

$$\frac{\dot{\vec{\Gamma}}}{em} = \frac{(\vec{r} \times \dot{\vec{r}}) \times \vec{r}}{r^3} = \frac{\overset{r}{(\vec{r})^2}\, \dot{\vec{r}} - (\vec{r} \cdot \dot{\vec{r}})\vec{r}}{r^3} =$$

$$\frac{d}{dt}\left(\frac{\vec{r}}{\sqrt{\vec{r}\cdot\vec{r}}}\right) = \frac{\sqrt{\vec{r}\cdot\vec{r}}\,\dot{\vec{r}} - \frac{1}{\cancel{2}}\frac{1}{\sqrt{\vec{r}\cdot\vec{r}}}\cancel{2}(\dot{\vec{r}}\cdot\vec{r})\cdot\vec{r}}{(\vec{r}\cdot\vec{r})}$$

$$= \frac{(\sqrt{\vec{r}\cdot\vec{r}})^2\dot{\vec{r}} - (\dot{\vec{r}}\cdot\vec{r})\cdot\vec{r}}{(\vec{r}\cdot\vec{r})^{3/2}} = \frac{r^2\dot{\vec{r}} - (\dot{\vec{r}}\cdot\vec{r})\vec{r}}{r^3}$$

$$= \frac{d}{dt}\left(\frac{\vec{r}}{\sqrt{\vec{r}\cdot\vec{r}}}\right) = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

$$\frac{\dot{\vec{\Gamma}}}{eg} = \frac{d}{dt}\left(\frac{\vec{r}}{r}\right) \quad \Big/ \int dt$$

$$\frac{\vec{\Gamma}}{eg} + \frac{\vec{\Gamma}_0}{eg} = \frac{\vec{r}}{r}$$

$$\frac{\vec{\Gamma}}{eg} = \frac{\vec{r}}{r} - \frac{\vec{\Gamma}_0}{eg} \quad \Big/^2$$

$$\frac{\Gamma^2}{e^2g^2} = 1 - 2\frac{\vec{r}\vec{\Gamma}_0}{r\,eg} + \frac{\Gamma_0^2}{eg^2}$$

$\rightarrow$ konst!

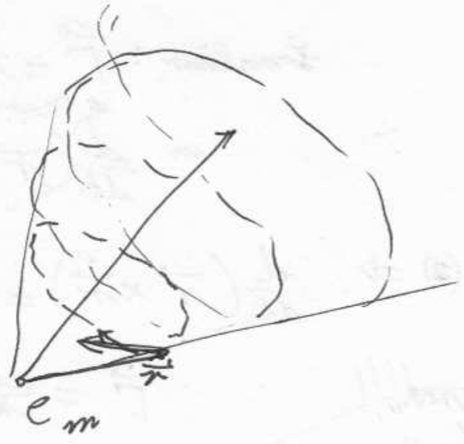
$$2\frac{\vec{r}\vec{\Gamma}_0}{r\,eg} = konst$$

$\Rightarrow$ $\dfrac{\vec{r}}{r}\cdot\vec{\Gamma}_0 = konst$

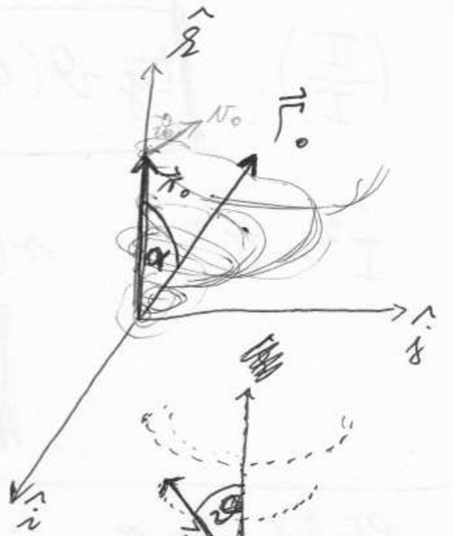
$\Downarrow$

$\Rightarrow$ <u>r oklepa konstanten kot</u>
<u>z vektorjem $\vec{\Gamma}_0$</u>

$\boxed{\rightarrow \text{STOŽEC!}}$



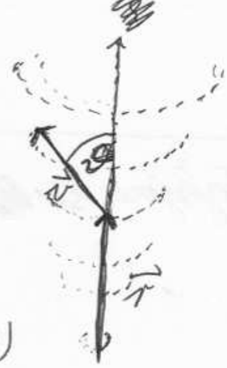
Nato nas zanima kako ugotovimo $\vec{\Gamma}_0$, torej os stožca.



Začetni pogoji:

$$\vec{r}_0 = z_0 \hat{k}$$

$$\vec{v}_0 = \dot{\vec{r}}_0 = v_0 \sin\vartheta_0 \hat{j} + v_0 \cos\vartheta_0 \hat{k}$$



⇝ ker se delec giblje po stožcu, bo hitrost $\dot{\vec{r}}$ vedno v tangentni ravnini stožca!

⇝ ϑ torej kot med hitrostjo in položajem ($\vec{v}$ in $\vec{r}$)

⇝ $\vec{v}_0$ si izberemo tako, da leži v ravnini $\hat{k}, \hat{j}$

$$\vec{\Gamma}_{zač.} = m\vec{r}_0 \times \vec{v}_0 = -\underbrace{m z_0 v_0 \sin\vartheta_0}_{\Gamma_z} \hat{i}$$

$$\frac{\vec{r}}{r} = \frac{\vec{\Gamma}}{eg} + \frac{\vec{\Gamma}_0}{eg} \rightarrow \vec{\Gamma}_0 = eg\hat{k} + \Gamma_z \hat{i}$$

$$\boxed{|\vec{\Gamma}_0| = \sqrt{(eg)^2 + \Gamma_z^2}}$$

$$\vec{\Gamma}_0 \cdot \frac{\vec{r}}{r} = eg = \cos\alpha\, |\Gamma_0|$$

$$\boxed{\cos\alpha = \frac{eg}{\sqrt{(eg)^2 + \Gamma_z^2}}}, \quad tg\,\alpha = \frac{\Gamma_z}{eg}$$

⇝ Γ_0 torej določen z sklopitvijo in vrtilno količino

Nato nas seveda zanima $r(t)$:

uporabimo 2) $\dot{\vec{r}}^2 = konst$
4) $\ddot{\vec{r}} \cdot \vec{r} = 0$ →

$$(\dot{\vec{v}} \cdot \vec{r})' = \ddot{\vec{r}} \cdot \vec{r} + \dot{\vec{r}}^2 = v^2 \;/\!\int$$

$$(\dot{\vec{r}} \cdot \vec{r})^2 = C \;;\; C = v_0^2$$

I) $\dot{\vec{r}} \cdot \vec{r} = r v_0 \cos\vartheta = v_0^2 t + D$

$t = 0 : D = z_0 v_0 \cos\vartheta_0$

II) $|\vec{r} \times m\dot{\vec{r}}| = r v_0 \sin\vartheta \cdot m = \Gamma_z$

$\left(\frac{II}{I}\right)$: $\boxed{\mathrm{tg}\,\vartheta(t) = \frac{\Gamma_z}{m(v_0^2 t + D)}} = \boxed{\frac{\Gamma_z}{m(v_0^2 t + z_0 v_0 \cos\vartheta_0)} = \mathrm{tg}\,\vartheta(t)}$

$I^2 + II^2$: $r(t) = \frac{1}{v_0}\sqrt{\left(\frac{\Gamma_z}{m}\right)^2 + (v_0^2 t + D)^2}$

$$\boxed{r(t) = \frac{1}{v_0}\sqrt{\left(\frac{\Gamma_z}{m}\right)^2 + (v_0^2 t + v_0 z_0 \cos\vartheta_0)^2}}$$

Oglejmo si posebni primer, ko je ob $t = 0$:

$\dot{\vec{r}} \cdot \vec{r} = 0 \rightarrow \vartheta_0 = 90°$ → elektron torej poženemo pravokotno na zveznico z magnetnim monopolom

$\cos 90° = 0 \Rightarrow D = 0$

$\Gamma_z = m z_0 v_0$

Dobimo: $\boxed{\mathrm{tg}\,\vartheta = \frac{z_0 v_0}{v_0^2 t} = \frac{z_0}{v_0 t}}$

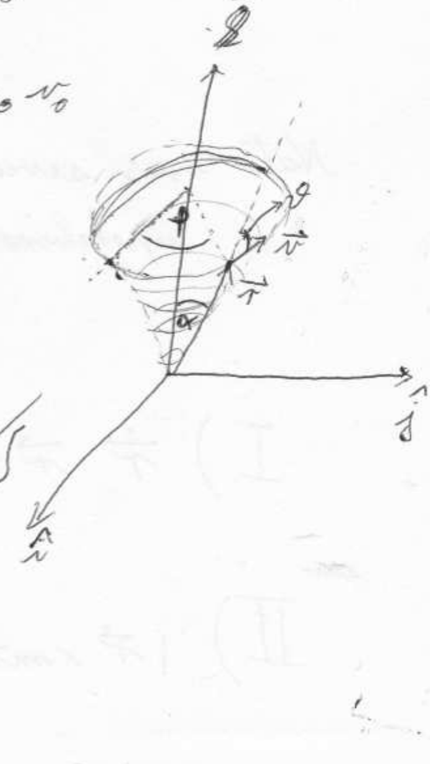
$$\boxed{r(t) = \frac{1}{v_0}\sqrt{(z_0 v_0)^2 + (v_0^2 t^2)^2} = \sqrt{z_0^2 + v_0^2 t^2}}$$

Relevanten za boljše predstavljanje problema je še sistem, kjer $\hat{k} \parallel \vec{\Gamma}_0$

$r(t), \vartheta(t) \Rightarrow r(t), \varphi(t)$ → klasičen φ cilindričnih koordinat

→ sistem $(\hat{i}, \hat{j}, \hat{k})$ najprej zavrtimo za α okrog $\hat{j}$

$$\alpha: \mathrm{tg}\,\alpha = \frac{\Gamma_z}{eg} = \frac{m z_0 v_0}{eg} = \frac{m}{eg} \cdot z_0 v_0$$



→ lok, ki ga elektron opiše v dt: $dl = v_0 \sin\vartheta\, dt$

→ kot, — // — : $d\varphi = \frac{dl}{\rho} = \frac{v_0 \sin\vartheta\, dt}{r \sin\alpha} = \frac{r v_0 \sin\vartheta}{r^2 \sin\alpha} dt = \frac{z_0 v_0}{\sin\alpha} \frac{dt}{(z_0^2 + v_0^2 t^2)}$ $\left(\frac{\Gamma_z}{m}\right)$

$$\varphi = \frac{z_0 v_0}{\sin\alpha} \int_0^t \frac{dt'}{z_0^2 + v_0^2 t'^2}$$

$$= \frac{\mathrm{arctg}\left(\frac{v_0 t}{z_0}\right)}{\sin\alpha}$$

Torej:

$$\boxed{\begin{aligned} r(t) &= \sqrt{r_0^2 + v_0 t^2} \\ \varphi(t) &= \frac{1}{\sin\alpha} \operatorname{arctg}\left(\frac{v_0 t}{r_0}\right) \end{aligned}}$$

Rešitev za $r(t)$ nam pokaže, da se bo elektron po dolgem času oddaljeval od monopola s konstantno hitrostjo. Še bolj zanimiva je rešitev za $\varphi(t)$, ki pove, da se po dolgem času $\varphi(t)$ skoraj ne bo več spreminjal! arctg se namreč pri $t \to \infty$ približuje $\frac{\pi}{2}$. To je ~~ne~~ nekako smiselno, saj bi pri ogromnih r ~~se~~ spreminjanje φ (torej $\dot{\varphi} \neq 0$) hitro povzročilo ~~spremembo~~ ~~zelo~~ veliko Γ. Ker pa ~~vemo~~ je ta, kot smo pokazali na začetku, konstantna, velja

$$\dot{\varphi}(t) \xrightarrow{t\to\infty} 0$$