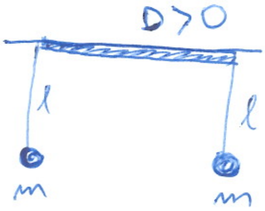
TORZIJSKO SKLOPLJENI TEŽNI NIHALI

Imamo dve nihali, ki nihata pravokotno okrog skupne osi, ki deluje navrati kot torzijska sklopitev z velikostjo D. Poišči stacionarne stabilne lege in lastna nihanja okrog njih!



Sistem opišemo z odmiki

$$\Phi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

- Kinetična energija:

$$T = \frac{1}{2} m l^2 (\dot{\varphi}_1^2 + \dot{\varphi}_2^2)$$

- Potencialna energija:

$$V = -mgl(\cos\varphi_1 + \cos\varphi_2) + \frac{D}{2}(\varphi_1 - \varphi_2)^2 =$$

$$= -mgl\left(\cos\varphi_1 + \cos\varphi_2 \mp \frac{\alpha}{2}(\varphi_1 - \varphi_2)^2\right) \quad ; \quad \alpha = \frac{D}{mgl}$$

- Ravnovesne lege:

$$\left.\frac{\partial V}{\partial \varphi_1}\right|_{\varphi_0} = +mgl(\sin\varphi_{10} + \alpha(\varphi_{10} - \varphi_{20})) = 0 \quad (1)$$

in hkrati:

$$\left.\frac{\partial V}{\partial \varphi_2}\right|_{\varphi_0} = mgl(\sin\varphi_{20} + \alpha(\varphi_{20} - \varphi_{10})) = 0 \quad (2)$$

(1) + (2) : $mgl(\sin\varphi_{10} + \sin\varphi_{20}) = 0 \Rightarrow \sin\varphi_{10} = -\sin\varphi_{20} \Rightarrow \varphi_{10} = -\varphi_{20}$

$$\Rightarrow \begin{cases} \varphi_{10} = -\varphi_{20} + 2\pi n \\ \varphi_{10} = +\varphi_{20} + \pi + 2\pi n \end{cases}$$

mnogo možnih rešitev (* druga stran),

(1) - (2) : $mgl(\sin\varphi_{10} \underbrace{-\sin\varphi_{20}}_{\sin\varphi_{10}} + 2\alpha(\varphi_{10} - \underset{\varphi_{10}}{\varphi_{20}}))$

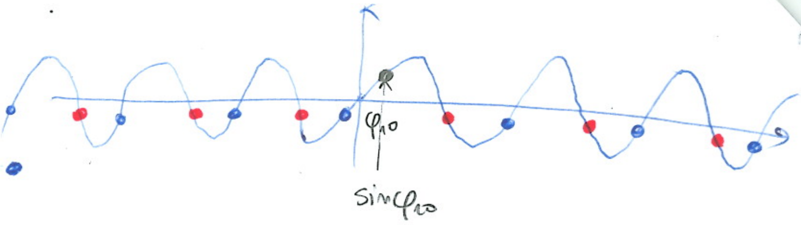
Mi si izberemo le rešitve za $n=0$ in $\varphi_{10} = -\varphi_{20}$ $\Rightarrow$

$$\sin\varphi_{10} = -2\alpha\varphi_{10}$$

To enačbo moramo grafično rešiti:

$\Rightarrow$ Tri možne rešitve (razen če nekako 2α prevelik, potem le 1, nezanimivo - ko prešibka D)

φ_{1r} – $\sin\varphi_{10} = -\sin\varphi_{20}$:



Možne rešitve: $\varphi_{10} = \begin{cases} -\varphi_{20} + 2\pi m \\ +\varphi_{20} + \pi + 2\pi m \end{cases}$

To ustavimo v ① – ②: $mgl\left(\sin\varphi_{10} - \sin\varphi_{20} + 2\lambda(\varphi_{10} - \varphi_{20})\right) = 0$

a) $2\sin\varphi_{10} + 4\lambda\varphi_{10} - \lambda 2\pi m = 0$

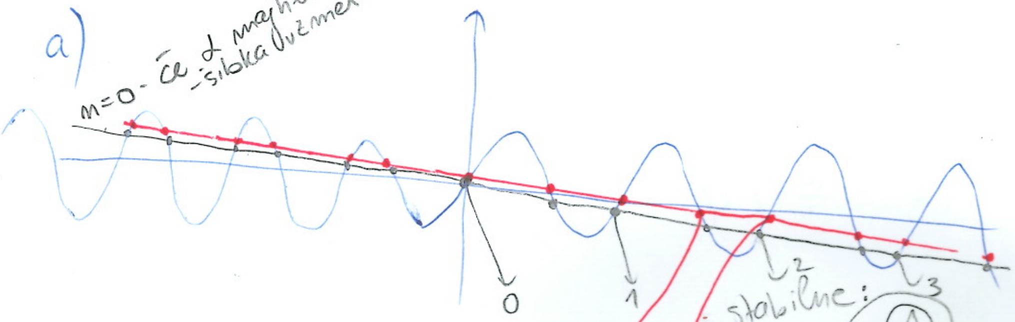
b) $2\sin\varphi_{10} + \cancel{4\lambda\varphi_{10}} - \lambda 2\pi m - \pi\lambda = 0$

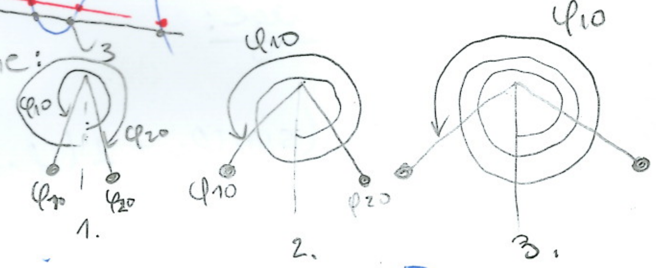
$\sin\varphi_{10} = \lambda 2\pi m + \frac{\pi}{2}\lambda$

~~ni rešitev~~ !!

Mnogo rešitev za φ_{10}!

a)

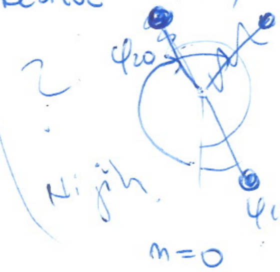




$m = 1, 2, 3, \ldots$ Dajo iste rešitve kot $m=0$, le oba φ_1 in φ_{20} sta pač za par skupnih zasukov zasukana. kar je pa itak vseeno.

Čeprav se vleži ogromno krat ovijeta okrog izhodišča, je to vseeno stacionarna rešitev, saj teža kaže se vedno dol, vzmet gor.

Rešitve b) $\varphi_{10} = +\varphi_{20} + \pi + 2\pi m$ so zanimive, saj so take:



Njih so labilne...

Preveriti moramo, katere so stabilne. Pogoj za stabilnost je $V(\underline{\varphi_0} + \delta\varphi) > V(\varphi_0)$ za v bližnji okolici $\underline{\varphi_0}$ - stacionarne lege. Če razvijemo v Taylorjevo vrsto, se to prevede na pogoj:

$$V(\underline{\varphi_0} + \underline{\delta\varphi}) = V(\underline{\varphi_0}) + \underbrace{\sum_i \frac{\partial V}{\partial \varphi_i}\Big|_{\varphi_0} \delta\varphi_i}_{0} + \frac{1}{2}\sum_{ij} \frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j}\Big|_{\varphi_0} \delta\varphi_i \delta\varphi_j + \dots$$

$$= V(\underline{\varphi_0}) + \frac{1}{2}\underline{\delta\varphi}^T \underline{\underline{\tilde{V}}}\, \underline{\delta\varphi} \quad ; \quad \tilde{V}_{ij} = \frac{\partial^2 V}{\partial\varphi_i \partial\varphi_j}$$

$\Rightarrow$ φ_0 je stabilna če je $\underline{\delta\varphi}^T \underline{\underline{\tilde{V}}}\, \underline{\delta\varphi} > 0$. - če je $\tilde{V}$ pozitivno definitna.

Kako to preveriti?

$\rightarrow$ Če ~~zap~~ diagonaliziramo $\underline{\underline{\tilde{V}}}$ (kar se da, ker je realna in simetrična, torej hermitska), je $\underline{\underline{\tilde{V}}} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. λ_1 in $\lambda_2 > 0$, da je pozitivno definitna. Prav tako ima $\det\underline{\underline{\tilde{V}}} > 0$ in $\mathrm{Tr}\,\underline{\underline{\tilde{V}}} > 0$. To preverimo:

③ $\mathrm{Tr}\,\underline{\underline{\tilde{V}}} = \frac{\partial^2 V}{\partial\varphi_1^2}\Big|_{\varphi_0} + \frac{\partial^2 V}{\partial\varphi_2^2}\Big|_{\varphi_0} = mgl\left(\cos\varphi_{10} + \alpha + \cos\varphi_{20} + \alpha\right) \underset{\varphi_{20} = -\varphi_{10}}{=} mgl\, 2(\cos\varphi_{10} + \alpha) > 0$

④ $\mathrm{Det}\,\underline{\underline{\tilde{V}}} = \frac{\partial^2 V}{\partial\varphi_1^2}\Big|_{\varphi_0} \cdot \frac{\partial^2 V}{\partial\varphi_2^2}\Big|_{\varphi_0} - \left(\frac{\partial^2 V}{\partial\varphi_1 \partial\varphi_2}\right)^2\Big|_{\varphi_0} = (mgl)^2\left((\cos\varphi_{10} + \alpha)(\cos\varphi_{20} + \alpha) - (-\alpha)^2\right) > 0$

$$\boxed{\underline{\underline{\tilde{V}}} = mgl\begin{pmatrix} \cos\varphi_{10} + \alpha & -\alpha \\ -\alpha & \cos\varphi_{10} + \alpha \end{pmatrix}}$$

stranski produkt, rabimo za uosmeje

$\Rightarrow$ ③ $\cos\varphi_{10} + \alpha > 0$

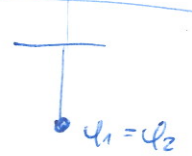
④ $\cos\varphi_{10}^2 + 2\alpha\cos\varphi_{10} > 0$

Preverimo tri stacionarne lege:

a) $\varphi_{10} = 0$: ③ $= 1 + \alpha > 0$ ✓

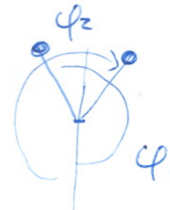
④ $= 1 + 2\alpha > 0$ ✓

stabilna



b) $\varphi_{10} \in [\pi, \frac{3\pi}{2}]$: ③ $= -|\cos\varphi_{10}| + \alpha$

④ $= |\cos\varphi_{10}|^2 - 2\alpha|\cos\varphi_{10}|$



če ③ > 0 : $|\cos\varphi_{10}| < \alpha$, potem ④ $= |\cos\varphi_{10}|\left(|\cos\varphi_{10}| - 2\alpha\right) < 0$

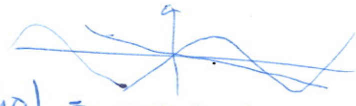
Prav tako obratno.

$\Rightarrow$ labilna!

c) $\varphi_{10} \in [\frac{3\pi}{2}, 2\pi]$: ③ $= |\cos\varphi_{10}| + \alpha > 0$ ✓

④ $= |\cos\varphi_{10}|\left(|\cos\varphi_{10}| + 2\alpha\right) > 0$ ✓

$\Rightarrow$ stabilna

Iz slike rešitev:  vidimo, da rešitev c) ostaja, ko je $f(\varphi_{10}) = -2\alpha\varphi_{10}$ v $\varphi_{10} = \frac{3\pi}{2}$ večja od $\sin\varphi_{10}$!

$$-2\alpha\frac{3\pi}{2} > \sin\frac{3\pi}{2} = -1 \quad \Rightarrow \quad \alpha < \frac{1}{3\pi}.$$

• <u>Razvoj okrog stabilnih leg:</u>

$$\varphi_1 = \varphi_{10} + \delta\varphi_1 \qquad \underline{\delta\varphi} = \begin{pmatrix} \delta\varphi_1 \\ \delta\varphi_2 \end{pmatrix}$$
$$\varphi_2 = \varphi_{20} + \delta\varphi_2$$

Razviti moramo do 2. reda.

-<u>Potencial</u> smo že:

$$V = V(\underline{\varphi_0}) + \frac{1}{2}\underline{\delta\varphi}^T \underline{\underline{\tilde{V}}}\,\underline{\delta\varphi} \quad ; \quad \underline{\underline{\tilde{V}}} = mgl\begin{pmatrix} \cos\varphi_{10}+\alpha & -\alpha \\ -\alpha & \cos\varphi_{10}+\alpha \end{pmatrix}$$

- <u>Kinetična energija</u>

$$T = \frac{ml^2}{2}\left(\delta\dot{\varphi}_1^2 + \delta\dot{\varphi}_2^2\right) = \frac{1}{2}\underline{\delta\dot{\varphi}}^T \underline{\underline{\tilde{T}}}\,\underline{\delta\dot{\varphi}} \quad ; \quad \underline{\underline{\tilde{T}}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ml^2$$

• <u>Iskanje lastnih nihanj:</u>

$\underline{\delta\varphi} = \varphi_0\,\underline{a}\,e^{-i\omega t}$, vstavimo v E-L enačbe in dobimo $\sum_j \tilde{T}_{ij}\delta\ddot{\varphi}_j + \sum_j \tilde{V}_{ij}\varphi_j = 0$

$$\underline{\underline{\tilde{V}}}\,\underline{a} = \omega^2\underline{\underline{\tilde{T}}}\,\underline{a} \Rightarrow \det\left(\underline{\underline{\tilde{V}}} - \omega^2\underline{\underline{\tilde{T}}}\right) = 0$$

$$mgl\left(\begin{pmatrix} \cos\varphi_{10}+\alpha & -\alpha \\ -\alpha & \cos\varphi_{10}+\alpha \end{pmatrix} - \underbrace{\omega^2\frac{ml^2}{1}\frac{1}{mgl}}_{\lambda = \omega^2/\omega_0^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\underline{a} = 0$$

$$\lambda = \omega^2/\omega_0^2 \quad ; \quad \omega_0^2 = g/l$$

$$\det\left(\begin{pmatrix} \cos\varphi_{10}+\alpha-\lambda & -\alpha \\ -\alpha & \cos\varphi_{10}+\alpha-\lambda \end{pmatrix}\right) = 0$$

$$\Rightarrow (\cos\varphi_{10}+\alpha-\lambda)^2 - \alpha^2 = 0$$

$$\Rightarrow \cos\varphi_{10}+\alpha - \lambda_\pm = \pm\alpha$$

$$\Rightarrow \lambda_{1,2} = +\cos\varphi_{10} + \alpha \mp \alpha \quad \Rightarrow \quad \begin{aligned}\lambda_1 &= +\cos\varphi_{10} \\ \lambda_2 &= +\cos\varphi_{10} + 2\alpha\end{aligned}$$

Lastni frekvenci nihanj sta:

$$\omega_1 = \sqrt{\lambda_1\omega_0^2} = \sqrt{\cos\varphi_{10}}\,\omega_0 \quad ; \quad \omega_2 = \sqrt{\cos\varphi_{10}+2\alpha}\,\omega_0$$

• <u>Lastni vektorji:</u>

$$\lambda_1: \begin{pmatrix} +\alpha & -\alpha \\ -\alpha & +\alpha \end{pmatrix}\begin{pmatrix} a_{11} \\ a_{12} \end{pmatrix} = \vec{0} \Rightarrow \begin{aligned} a_{11} &= +a_{12} \\ \underline{a}_1 &= \begin{pmatrix} 1 \\ +1 \end{pmatrix}\end{aligned}$$

$$\lambda_2: \begin{pmatrix} -\alpha & -\alpha \\ -\alpha & -\alpha \end{pmatrix}\begin{pmatrix} a_{21} \\ a_{22} \end{pmatrix} = \vec{0} \Rightarrow \begin{aligned} a_{21} &= -a_{22} \\ \underline{a}_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\end{aligned}$$