DELEC V POTENCIALU $V(r) = -\frac{k}{r^2}$

Poišči vse možne orbite delca v $V(r) = -\frac{k}{r^2}$ potencialu.

Centralni potencial $\Rightarrow$ Ohranitev vrtilne količine $\Rightarrow$ ravninsko gibanje : (r, φ). Iščemo $r(\varphi)$!

Konstante gibanja:

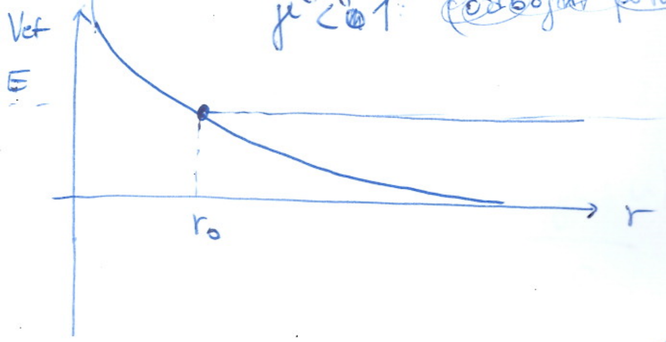
- celotna energija: $H = \frac{m}{2}\dot{r}^2 + V_{ef}(r)$; $V_{ef}(r) = V(r) + \frac{p_\varphi^2}{2mr^2}$
- $p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi} = $ konst saj $L \neq L(\varphi)$

Najprej poglejmo, kakšne so možne orbite:

$\varkappa < 1$: ~~(odbojni potencial)~~

$$V_{ef}(r) = -\frac{k}{r^2} + \frac{p_\varphi^2}{2mr^2} = \frac{1}{r^2}\left(\frac{p_\varphi^2}{2m} - k\right)$$

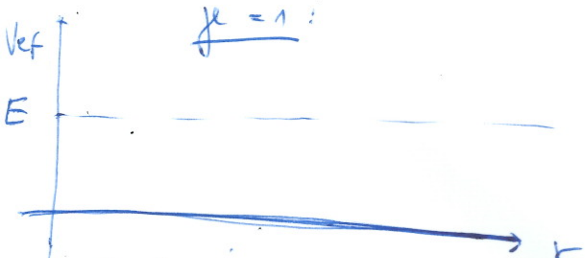
$$V_{ef}(r) = \frac{p_\varphi^2}{2m}(1-\varkappa)\frac{1}{r^2} \; ; \; \varkappa = \frac{2mk}{p_\varphi^2}$$



Neomejeno gibanje:

$\rightarrow$ simbolična slika, parabola / hiperbola le pri $1/r$ potencialu.

$\varkappa = 1$:



$E > 0$: Padanje v središče:

Če $E = 0$, potem

$\frac{m\dot{r}^2}{2} = 0 \Rightarrow$ kroženje pri katerem koli r, če izpolnjeno $p_\varphi^2 = 2mk$.

$\varkappa > 1$

$E_1 > 0$: padanje v središče:

$E_2 < 0$: padanje v središče:

Poiščimo enačbo gibanja za $r(\varphi)$:

$$H = \frac{m}{2}\dot{r}^2 + \frac{p_\varphi^2}{2m}(1-\varkappa)\frac{1}{r^2}$$

$$\dot{r}^2 = \frac{2}{m}H - \frac{p_\varphi^2}{m^2}(1-\varkappa)\frac{1}{r^2}$$

Uvedemo $u = \frac{1}{r}$, lažje rešimo to diferencialno enačbo s tem nastavkom:

$r = \frac{1}{u}$; $\dot{r} = \frac{d}{dt}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\varphi}\dot{\varphi} =$

$$= -\frac{1}{u^2}\frac{du}{d\varphi}\frac{p_\varphi}{m}u^2 = -\frac{p_\varphi}{m}\frac{du}{d\varphi} = -\frac{p_\varphi}{m}u'$$

$$\dot{\varphi} = \frac{p_\varphi}{mr^2} = \frac{p_\varphi}{m}u^2$$

Vstavimo v DE:

$$\left(\frac{p_\varphi}{m}\right)^2 (u')^2 = \frac{2}{m}H - \left(\frac{p_\varphi}{m}\right)^2(1-\varkappa)u^2$$

$$\boxed{(u')^2 = \frac{2m}{p_\varphi^2}H - (1-\varkappa)u^2}$$

Poiščimo rešitve te DE za različne vrednosti $\varkappa$ je še malo preuredimo enačbo: $/\frac{d}{d\varphi}$ → diferenciramo

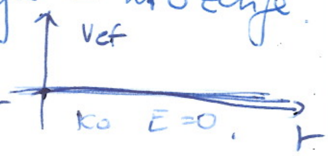
Dobimo: $2u'u'' = -(1-\varkappa)2uu'$

$$u'(u'' + (1-\varkappa)u) = 0$$

1. rešitev: $\boxed{u' = 0}$ ⇒ $u(\varphi) = u_0$ in $r(\varphi) = \frac{1}{u_0}$

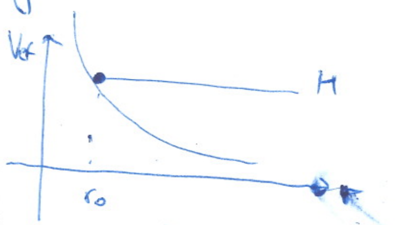
Gibanje pri konstantnem radiju = kroženje.

za primer



2. rešitev: $\boxed{u'' + (1-\varkappa)u = 0}$

$\varkappa < 1$:



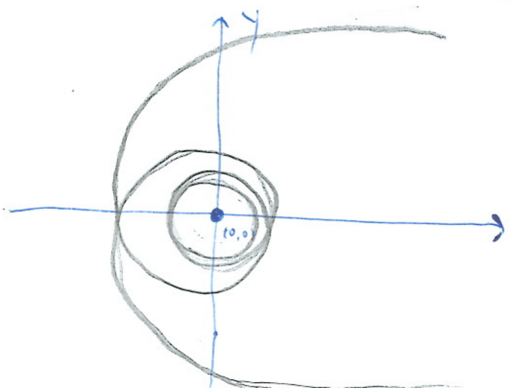
Označimo $1-\varkappa = \alpha^2$; $\alpha \in \mathbb{R}$

$$u'' + \alpha^2 u = 0$$

Rešitev: $u(\varphi) = A\cos[\alpha(\varphi-\varphi_0)]$

$$r(\varphi) = \frac{1}{A\cos[\alpha(\varphi-\varphi_0)]} \quad ; \quad \varphi \in \left(-\frac{\pi}{2\alpha}, \frac{\pi}{2\alpha}\right)$$

To so take orbite:



Niso parabole, hiperbole

→ $\alpha = 0,12$.

$\varkappa = 1$:

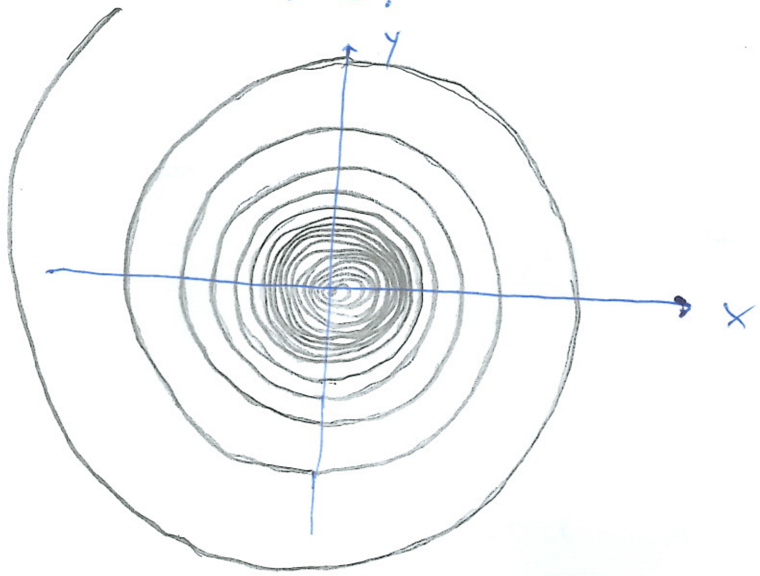
$$u'' = 0$$

Rešitev: $u(\varphi) = A\varphi + B$

$$r(\varphi) = \frac{1}{A\varphi + B} \quad ; \quad \varphi = -\frac{B}{A} \quad r \to \infty$$

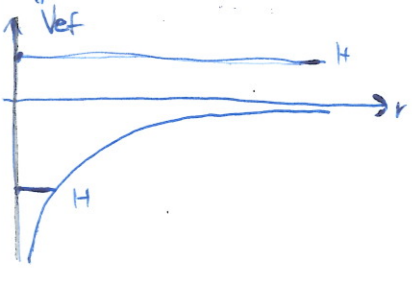
To je spirala, po njej pada masa v maso 2. (proti izhodišču).

Orbite so take:



Ko $\varphi \to \infty$ pade v izhodišče.

$\varkappa > 1$:



$$u'' - \alpha^2 u = 0 \quad ; \quad \alpha^2 = \varkappa - 1, \quad \alpha \in \mathbb{R}$$

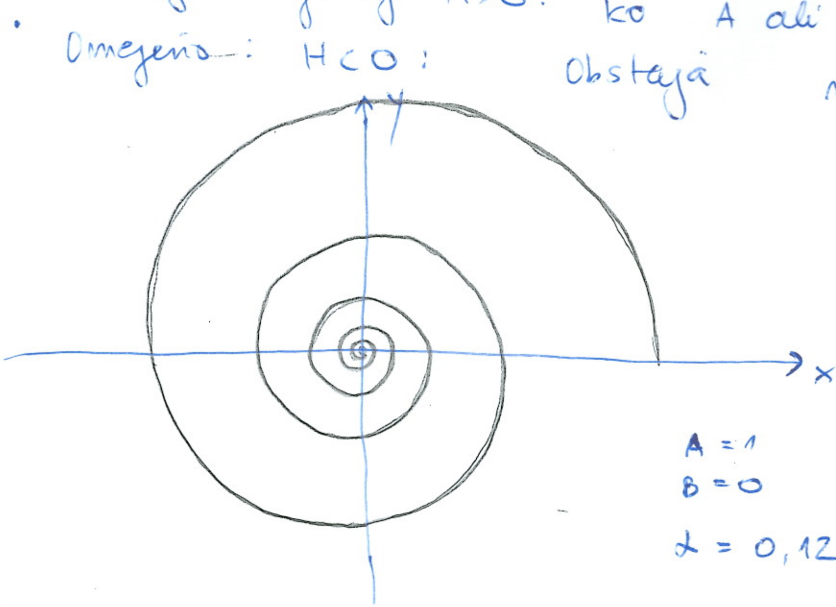
Rešitev: $u(\varphi) = Ae^{-\alpha\varphi} + Be^{\alpha\varphi}$

$$r(\varphi) = \frac{1}{Ae^{-\alpha\varphi} + Be^{\alpha\varphi}}$$

To je spirala za $A, B\ \cancel{\alpha} \geq 0$

- Neomejeno gibanje: $H > 0$: ko A ali $B = 0$
- Omejeno: $H < 0$: Obstaja meh $r_{max} = r(\varphi_{max})$, $\varphi_{max} = \frac{1}{2\alpha} \ln \frac{A}{B}$

$$r_{max} = \frac{1}{2\sqrt{AB}}$$



$A = 1$
$B = 0$
$\alpha = 0{,}12$

Hitro padanje v izhodišče

Računam r_{max}:

$$\left.\frac{dr}{d\varphi}\right|_{\varphi=\varphi_{max}} = 0 = \frac{-1}{(\quad)^2}\left(-\alpha A e^{-\alpha\varphi_{max}} + \alpha B e^{\alpha\varphi_{max}}\right) = 0$$

$$\Rightarrow \quad A e^{-\alpha\varphi_{max}} = B e^{\alpha\varphi_{max}}$$

$$e^{2\alpha\varphi_{max}} = \frac{A}{B} \Rightarrow \quad \varphi_{max} = \frac{1}{2\alpha}\ln\frac{A}{B}$$

$$r_{max} = r_{max}(\varphi_{max}) = \frac{1}{A e^{-\frac{1}{2}\ln A/B} + B e^{\frac{1}{2}\ln A/B}} = \frac{1}{A\sqrt{\frac{B}{A}} + B\sqrt{\frac{A}{B}}} = \frac{1}{2\sqrt{AB}}$$

Hkrati pa je pri r_{max} $H = V_{ef}$ (glej graf $V_{ef}(r)$):

$$H = \left(\frac{p_\varphi^2}{2m} - K\right)\frac{1}{r_{max}^2} = \left(\frac{p_\varphi^2}{2m} - K\right) 2AB$$

~~začetna ener~~

Velikost vrtilne količine, celotna energija in K določajo parametra A in B.

Če $A = B$:

$$r_{max} = \frac{1}{2A}$$

$$A = \frac{1}{2}\sqrt{\frac{H}{\frac{p_\varphi^2}{2m} - K}}$$