ORBITE V SFERIČNEM HARMONSKEM POTENCIALU

Poišči vse možne orbite delca v potencialu $V = \frac{1}{2} m \omega_0^2 r^2$, kjer je m masa delca in r oddaljenost od izhodišča.

V KARTEZIČNIH KOORDINATAH

Rešitev je zelo enostavna, saj so koordinate x, y, z nesklopljene:

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$V = \frac{1}{2} m \omega_0^2 (x^2 + y^2 + z^2)$$

$$L = T - V$$

$$x: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0 \Rightarrow m\ddot{x} + m\omega_0^2 x = 0$$

$$y: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0 \Rightarrow$$

$$z: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0 \Rightarrow$$

$$\boxed{\begin{aligned} \ddot{x} + \omega_0^2 x &= 0 \\ \ddot{y} + \omega_0^2 y &= 0 \\ \ddot{z} + \omega_0^2 z &= 0 \end{aligned}}$$

Enačbe gibanja so kot za tri neodvisne harmonske oscilatorje = harmonska nihala.

Rešitev enačb je:

$$x(t) = x_0 \sin(\omega t + \delta_x)$$

$$y(t) = y_0 \sin(\omega t + \delta_y)$$

$$z(t) = z_0 \sin(\omega t + \delta_z)$$

V ~~SFERIČNIH~~ POLARNIH KOORDINATAH

$$T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\varphi}^2)$$

$$V = \frac{1}{2} m \omega_0^2 r^2$$

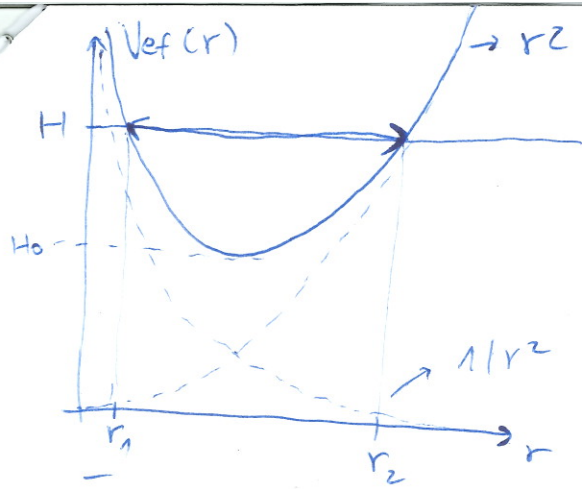
$L = T - V \rightarrow$ ne vsebuje φ, torej $\frac{\partial L}{\partial \dot{\varphi}} = p_\varphi = \text{konst.}$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \dot{\varphi} \Rightarrow \dot{\varphi} = \frac{p_\varphi}{m r^2}$$

Energija sistema, ki konstanta gibanja:

$$H = \frac{1}{2} m \dot{r}^2 + \frac{p_\varphi^2}{2 m r^2} + \frac{1}{2} m \omega_0^2 r^2 = \frac{1}{2} m \dot{r}^2 + V_{ef}(r)$$

$$V_{ef}(r) = \frac{p_\varphi^2}{2 m r^2} + \frac{1}{2} m \omega_0^2 r^2$$



Iz oblike Vef(r) vidimo, da ni nevezanih orbit, delec je ujet v potencialu in njegovo gibanje je omejeno na $r \in [r_1, r_2]$.

Izračun orbit:

1. Ker nas zanima $r(\varphi)$, zamenjamo $\frac{dr}{dt} \to \frac{dr}{d\varphi}$:

$$\dot r = \frac{dr}{dt} = \frac{d\varphi}{dt}\frac{dr}{d\varphi} = \dot\varphi\frac{dr}{d\varphi} = \frac{p_\varphi}{mr^2}\frac{dr}{d\varphi}$$

Enačbo gibanja dobimo iz izraza za energijo H, ki je konstanta gibanja. Da to diferencialno enačbo lažje rešimo, vpeljemo novo spremenljivko.

$$u = \frac{1}{r^2} \to r^2 = \frac{1}{u}$$

$$\to \frac{du}{d\varphi} = -2\frac{1}{r^3}\frac{dr}{d\varphi} \to \cancel{\frac{d\varphi}{d\varphi}}\frac{dr}{d\varphi} = -\frac{1}{2}\frac{1}{u^{3/2}}\frac{du}{d\varphi}$$

Vstavimo v izraz za H in dobimo:

$$H = \frac{1}{2}m\left(\frac{p_\varphi}{mr^2}\right)^2\left(-\frac{1}{2}\frac{1}{u^{3/2}}\frac{du}{d\varphi}\right)^2 + \frac{1}{2}m\omega_0^2 r^2 + \frac{p_\varphi^2}{2mr^2} =$$

$$H = \frac{1}{8}\frac{p_\varphi^2}{m}\frac{1}{u}u'^2 + \frac{1}{2}m\omega_0^2\frac{1}{u} + \frac{p_\varphi^2}{2m}u \quad / \cdot \frac{8m}{p_\varphi^2}u$$

$$\frac{8m}{p_\varphi^2}Hu = u'^2 + \frac{4m^2\omega_0^2}{p_\varphi^2} + 4u^2$$

$$u'^2 = -4u^2 + \frac{8m}{p_\varphi^2}Hu - \frac{4m^2\omega_0^2}{p_\varphi^2} \underbrace{- \left(2\frac{Hm}{p_\varphi^2}\right)^2 + \left(2\frac{Hm}{p_\varphi^2}\right)^2}_{\text{Želimo spraviti izraz na popoln kvadrat.}}$$

$$u'^2 = -4\left(u - \frac{Hm}{p_\varphi^2}\right)^2 + \left(2\frac{Hm}{p_\varphi^2}\right)^2 - \left(\frac{2m\omega_0}{p_\varphi}\right)^2$$

Vpeljemo novo spremenljivko:

$$x = u - \frac{Hm}{p_\varphi^2}$$

$$x' = u'$$

$$\boxed{x'^2 + 4x^2 = \frac{4m^2}{p_\varphi^2}\left(\frac{H^2}{p_\varphi^2} - \omega_0^2\right)}$$

Za reševanje DE uporabimo nastavek:

$$x(\varphi) = A\cos(2\varphi+\delta)$$
$$x'(\varphi) = -2A\sin(2\varphi+\delta)$$

Vstavimo v enačbo:

$$4A^2\sin^2(2\varphi+\delta) + 4A^2\cos^2(2\varphi+\delta) = \frac{4m^2}{p_\varphi^2}\left(\frac{H^2}{p_\varphi^2} - \omega_0^2\right)$$
$$4A^2 = \frac{4m^2}{p_\varphi^2}\left(\frac{H^2}{p_\varphi^2} - \omega_0^2\right)$$
$$\Rightarrow \quad \underline{A = \frac{m}{p_\varphi}\sqrt{\frac{H^2}{p_\varphi^2} - \omega_0^2}}$$

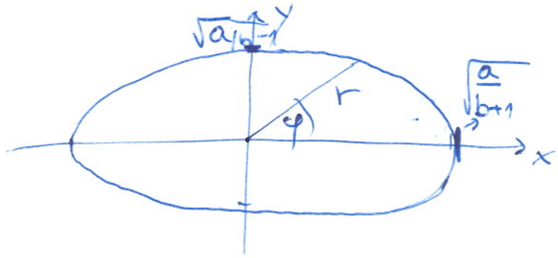
$$\frac{1}{r^2} = u = x + \frac{Hm}{p_\varphi^2} = \frac{m}{p_\varphi}\sqrt{\frac{H^2}{p_\varphi^2} - \omega_0^2}\cos(2\varphi+\delta) + \frac{Hm}{p_\varphi^2}$$

$$= \frac{Hm}{p_\varphi^2}\left(1 + \frac{p_\varphi}{H}\sqrt{\frac{H^2}{p_\varphi^2} - \omega_0^2}\cos\right) = \frac{Hm}{p_\varphi^2}\left(1 + \sqrt{1 - \left(\frac{p_\varphi}{H}\omega_0\right)^2}\right)$$

$$\boxed{r^2 = \frac{p_\varphi^2}{Hm}\frac{1}{1 + \sqrt{1 - \frac{p_\varphi^2\omega_0^2}{H^2}}\cos(2\varphi+\delta)}}$$

To je rezultat in predstavlja elipso s centrom v izhodišču.

$$r^2(\varphi) = \frac{a}{b\cos(2\varphi) + 1}$$



Dokaz:

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi = \frac{x^2}{r^2} - \frac{y^2}{r^2}$$
$$r^2 = \frac{a}{b\frac{x^2}{r^2} - b\frac{y^2}{r^2} + 1} \quad \Big/ \frac{1}{r^2}$$
$$1 = \frac{a}{bx^2 - by^2 + r^2} = \frac{a}{bx^2 - by^2 + x^2 + y^2}$$
$$1 = \frac{a}{x^2(b+1) + y^2(-b+1)} \quad \Big/^{-1}$$

$$\boxed{1 = \frac{x^2}{\frac{a}{b+1}} + \frac{y^2}{\frac{a}{-b+1}}}$$

$H \to p_\varphi\omega_0 = H_0$ sicer mi orbite! (glej graf V_{ef})

Enačba elipse za $b < 1$.

, vsi relevantni primeri.

Pri nas je $b = \sqrt{1 - \frac{p_\varphi^2\omega_0^2}{H^2}} < 1$ za $H \neq 0$ in $p_\varphi \neq 0$ in $\omega_0 \neq 0$.