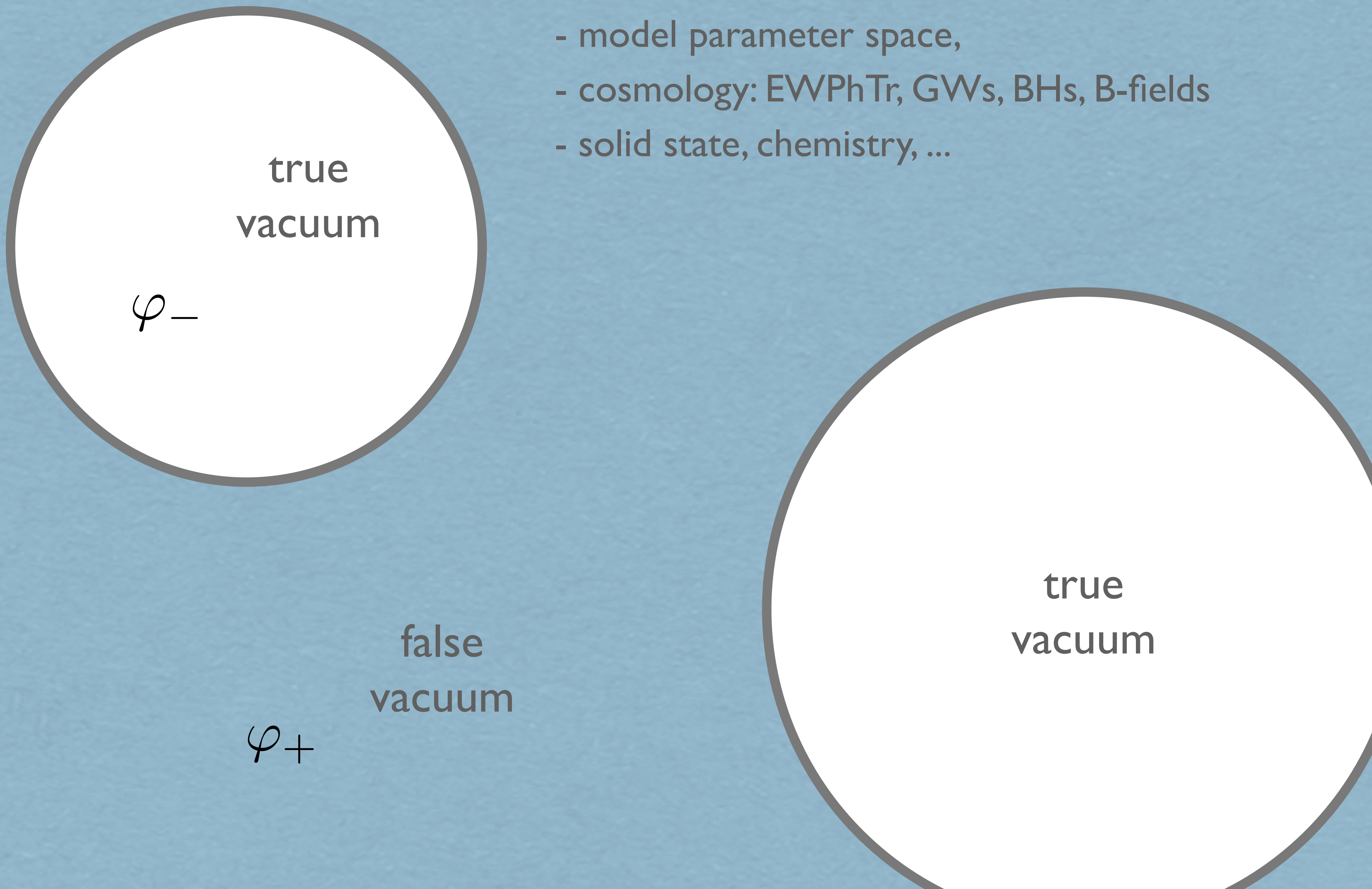
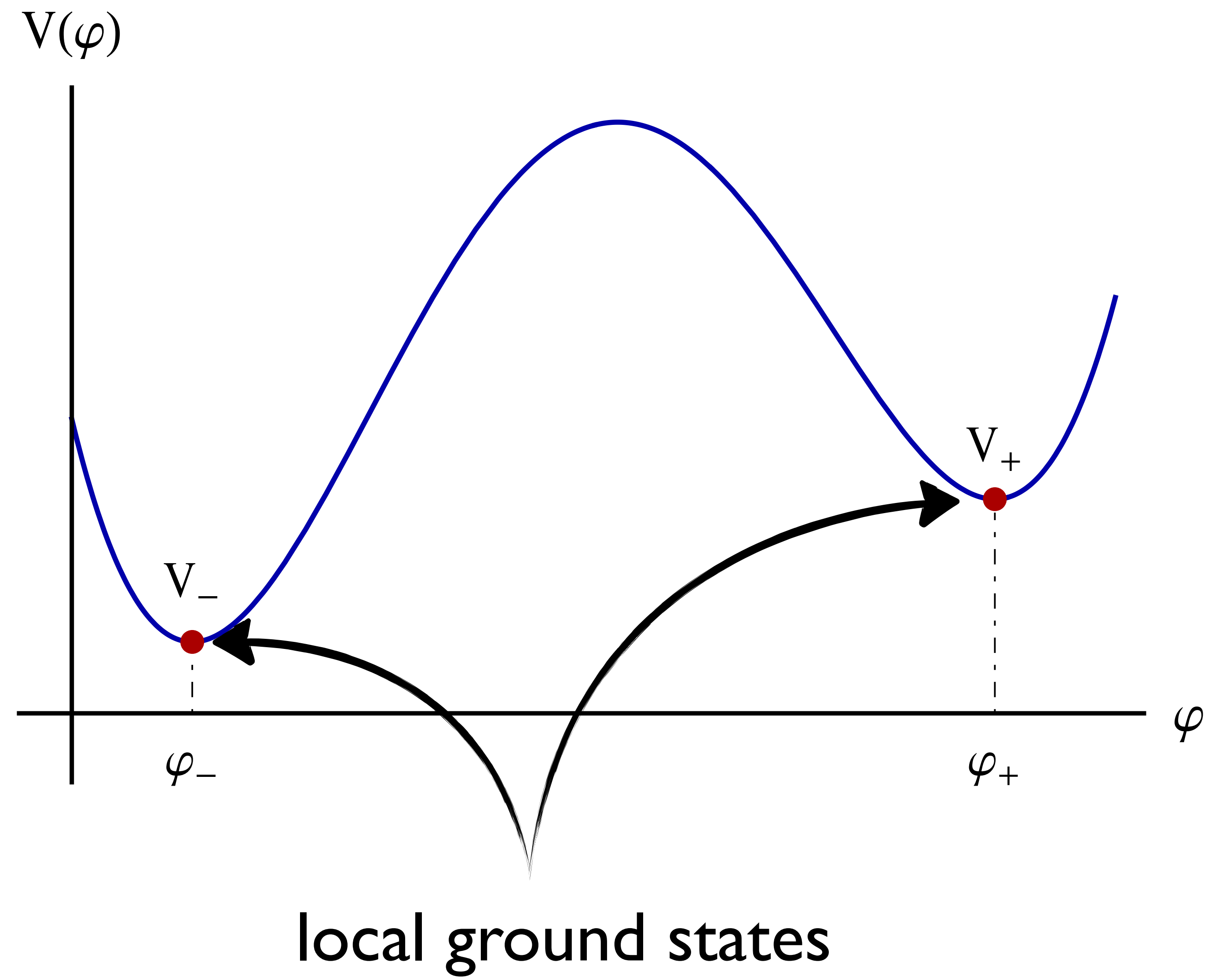
Phase transitions
and
Gravitational Waves

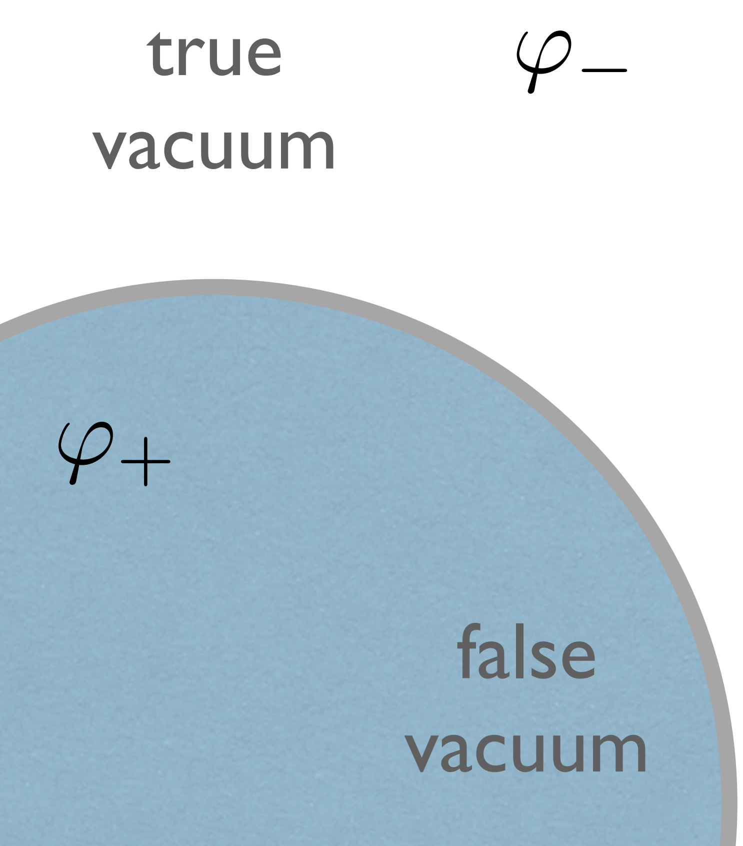
Phase transitions

Phase Transitions

- model parameter space,
- cosmology: EWPhTr, GWs, BHs, B-fields
- solid state, chemistry, ...




V(φ)
V₋
V₊
φ
φ₋
φ₊
local ground states


true
vacuum
φ₋
φ₊
false
vacuum

Effective potential

A barrier can come from tree-level, quantum or thermal corrections

$$V = V_{\text{tree}} + V_{\text{quantum}} + V_{\text{thermal}}$$

SM complex doublet

$$\phi = \begin{pmatrix} G^+ \\ \frac{h+v+iG^0}{\sqrt{2}} \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ \frac{h}{\sqrt{2}} \end{pmatrix}$$

fields

Effective potential

A barrier can come from tree-level, quantum or thermal corrections

$$V = V_{\text{tree}} + V_{\text{quantum}} + V_{\text{thermal}}$$

all renormalizable terms

$$\phi = \begin{pmatrix} G^+ \\ \frac{h+v+iG^0}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} 0 \\ \frac{h}{\sqrt{2}} \end{pmatrix}$$

fields

$$V_{\text{tree}} = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{4!}\left(\phi^\dagger\phi\right)^2$$

tree level

Effective potential

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fields

$$V_{\rm tree} = -\frac{\mu^2}{2}\phi^\dagger\phi + \frac{\lambda}{4!}\left(\phi^\dagger\phi\right)^2$$

tree level

$$\frac{dV_{\rm tree}}{dh} = 0, \quad \langle h\rangle^2 = v^2 = 12\frac{\mu^2}{\lambda}$$

minimize

$$m_h^2 = \frac{d^2V_{\rm tree}}{dh^2}\bigg|_{h=v} = \mu^2 = \frac{\lambda v^2}{12}$$

mass

minimize
get masses

Effective potential

$$V = V_{\text{tree}} + V_{\text{quantum}} + V_{\text{thermal}}$$

Quantum fluctuations

given by vacuum fluctuations from off-shell particles

includes fields that couple to the scalar at a given loop level

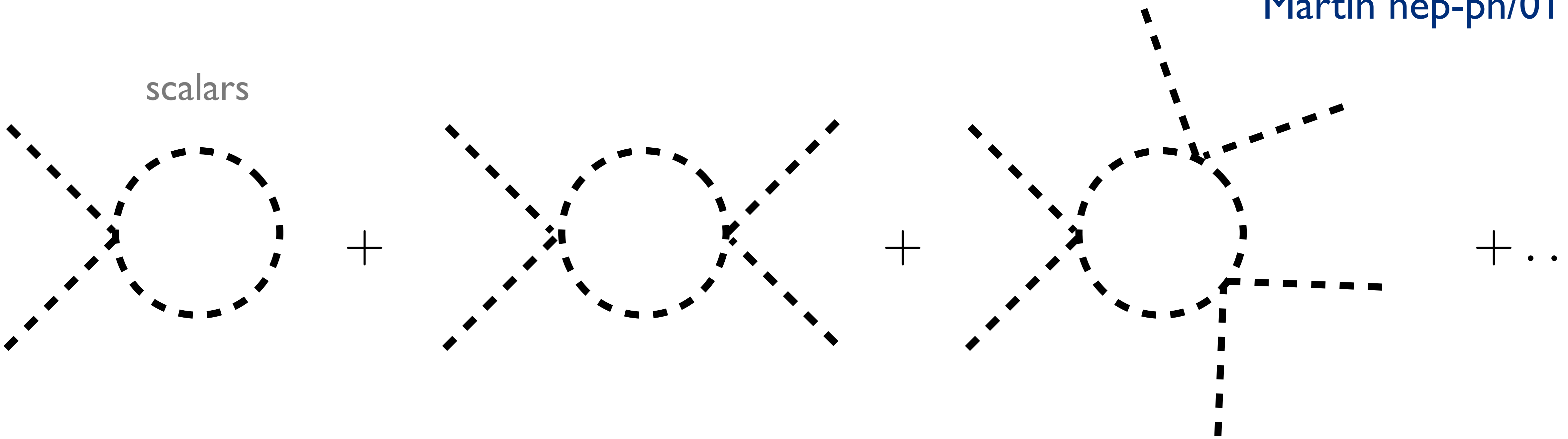
infinite result, as expected in *D=4* QFT

made finite by renormalization in dimensional regularization

Effective (quantum) potential @ 1 loop

Coleman, Weinberg '73, Jackiw '74,
Iliopoulous et al. '75

Martin hep-ph/0111209



Effective (quantum) potential: 1 loop but infinite number of quartic insertions

$$V_{\text{quantum}} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + m^2(\varphi)\right)$$

field dependent mass

Effective potential

scalars

$$V_{\text{quantum}} = \frac{1}{2}\int \frac{d^4k}{(2\pi)^4} \log\left(k^2 + m^2(\varphi)\right)$$

field dependent mass

UV infinite, need to regularize and renormalize

$$\frac{dV_{\text{quantum}}}{m^2(\varphi)} = \frac{\mu^{4-d}}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2 + m^2(\varphi)}$$

see any QFT book,
e.g. Peskin-Schröder

Effective potential

scalars

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see any QFT book, e.g. Peskin-Schröder

remove with c.t.s in MS-bar

$$V_{\text{quantum}} = \frac{1}{64\pi^2} m^4(\varphi)\left(\log\frac{m^2(\varphi)}{\mu^2} - \frac{3}{2} - \frac{1}{2-\frac{d}{2}} - \gamma_E + \log 4\pi\right)$$

Final result

Generic effective (quantum) 1 loop potential

MS-bar scheme   simple but less 'physical', vevs and masses move at one loop

Landau gauge   also simple, would-be-goldstones massless in the minima

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$$V_{\text{quantum}} = \frac{1}{64\pi^2} \sum_i n_i m_i(\varphi)^4 \left(\frac{m_i^2}{\mu^2} - C_i \right)$$

Final result

Generic effective (quantum) 1 loop potential

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$$V_{\text{quantum}} = \frac{1}{64\pi^2} \sum_i n_i m_i(\varphi)^4 \left(\frac{m_i^2}{\mu^2} - C_i \right)$$

$m_i(\varphi)$ — field dependent masses

μ — renormalization scale

n_i — degrees of freedom

C_i — renormalization constants

vector bosons: $C_V = \frac{3}{2}$

fermions and scalars: $C_{f,s} = \frac{5}{6}$

Effective potential

$$V_{\text{quantum}} = \frac{1}{64\pi^2} \sum_i n_i m_i(h)^4 \left(\frac{m_i^2}{\mu^2} - C_i \right)$$

Effective potential in the SM

$$\phi = \begin{pmatrix} G^+ \\ \frac{h+v+iG^0}{\sqrt{2}} \end{pmatrix} \to \begin{pmatrix} 0 \\ \frac{h}{\sqrt{2}} \end{pmatrix}$$

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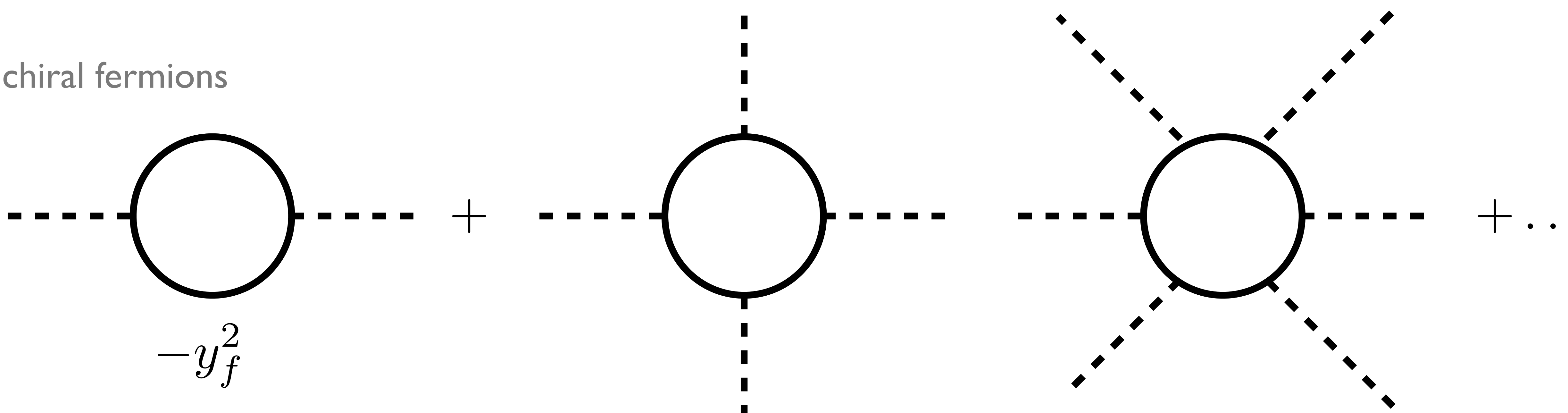
degrees of freedom

$$\{n_W, n_Z, n_t, n_{G^+}, n_{G^0}, n_h\} = \{6, 3, -12, 2, 1, 1\}$$

chiral fermions



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degrees of freedom

$$\{n_W, n_Z, n_t, n_{G^+}, n_{G^0}, n_h\} = \{6, 3, -12, 2, 1, 1\}$$

field dependent masses

$$m_h(h)^2 = \frac{M_W^2}{v^2} h^2, \qquad m_W(h)^2 = \frac{M_Z^2}{v^2} h^2, \qquad m_t = \frac{m_t}{v} h$$

Effective potential

$$V_{\text{quantum}} = \frac{1}{64\pi^2} \sum_i n_i m_i(h)^4 \left(\frac{m_i^2}{\mu^2} - C_i \right)$$

Effective potential beyond the SM

degrees of freedom $n_V = 3 \times d_V$ $n_b = 1 \times d_b$ $n_f = -d_f \times$ chiralities

Effective potential

$$V_{\text{quantum}} = \frac{1}{64\pi^2} \sum_i n_i m_i(h)^4 \left(\frac{m_i^2}{\mu^2} - C_i \right)$$

Effective potential beyond the SM

degrees of freedom $n_V = 3 \times d_V$ $n_b = 1 \times d_b$ $n_f = -d_f \times \text{ chiralities}$

counter-terms change e.g. to on-shell mass to relate to physical observables

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counter-terms change e.g. to on-shell mass to relate to physical observables

field dependent masses consider full mass matrices and mixings and diagonalize

$$M_{ij}^2(\varphi) \to \operatorname{diag} m_i^2(\varphi)$$

see Martin '01 for a general one and two loop V_{eff}

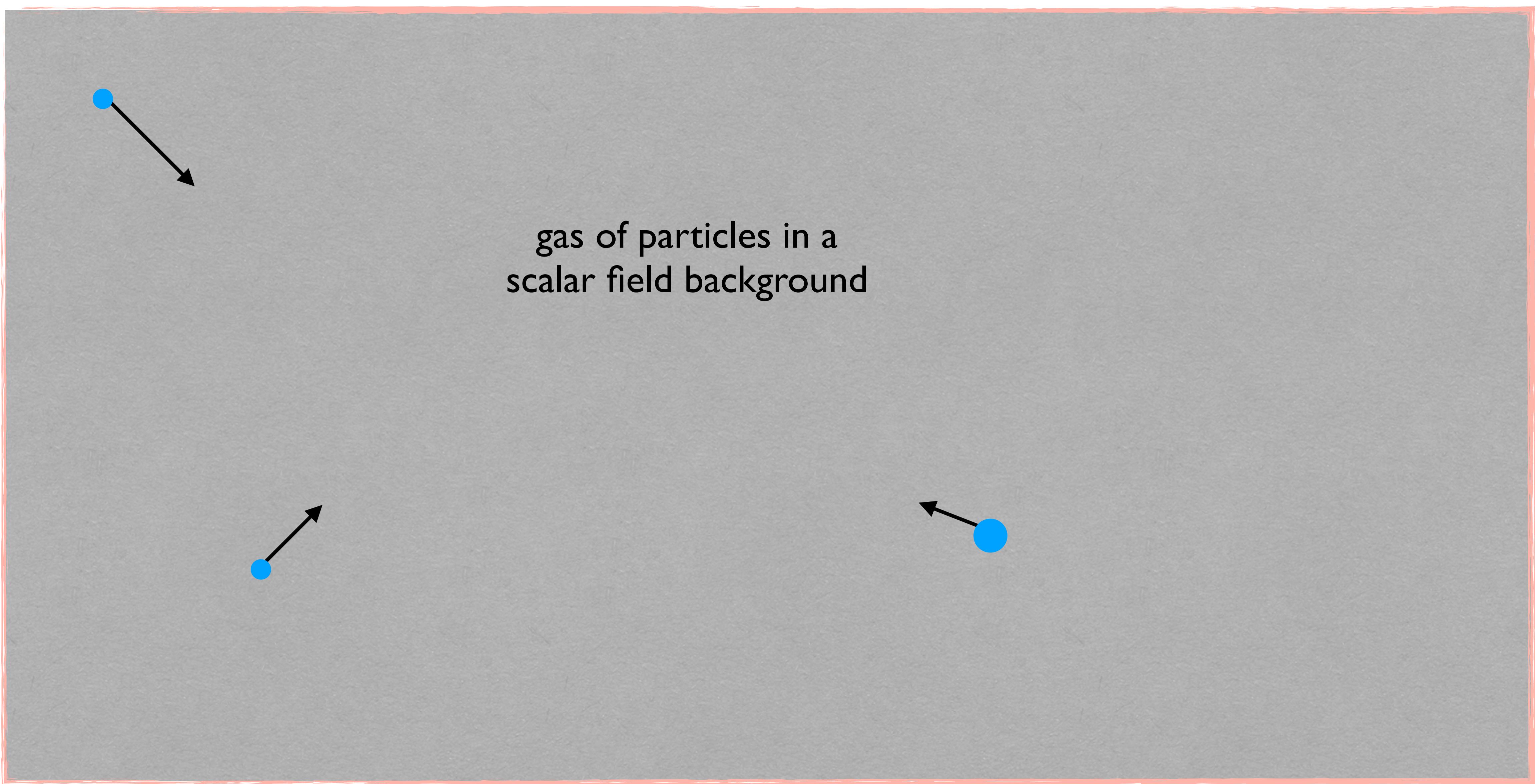
Thermal potential

Thermal fluctuations

Linde '83

Quiros review, books by Laine & Vuorinen 1701.01554

Kapusta & Gale, LeBellac



Thermal potential

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Kapusta & Gale, LeBellac

Compactify time to temperature *T* and go to *D=3*

$$V_{\text{thermal}} = \frac{T^4}{2\pi^2} \sum_i n_i J_{F/B}\left(\frac{m_i^2(\varphi)}{T^2}\right)$$

Thermal potential

Thermal fluctuations

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Compactify time to temperature *T* and go to *D=3*

$$V_{\rm thermal} = \frac{T^4}{2\pi^2} \sum_i n_i J_{F/B}\left(\frac{m_i^2(\varphi)}{T^2}\right)$$

thermal functions $J_{F/B}\left(y^2\right) = \int_0^\infty x^2 \log\left(1 \pm e^{-\sqrt{x^2+y^2}}\right)$

BSM may create a thermal barrier. The SM cannot, Higgs is too heavy.

Cosmological phase transitions

Order of the phase transitions characterized by the (dis)-continuity of the free energy (effective potential) at some temperature

Quiros hep-ph/9901312

$$V(\varphi,T) = D\left(T^2 - T_0^2\right)\varphi^2 - E\,T\,\varphi^3 + \frac{\lambda(T)}{4}\varphi^4$$

high-T expansion

(other expansions, tools to compute)

$$J_B \simeq -\frac{\pi^4}{45} + \frac{\pi^2}{12}\frac{m_2}{T^2} - \frac{\pi}{6}\left(\frac{m^2}{T^2}\right)^{3/2} + \ldots$$

$$J_F \simeq \frac{7\pi^4}{360} - \frac{\pi^2}{24}\frac{m^2}{T^2} + \ldots$$

Cosmological phase transitions

$$V(\varphi, T) = D\left(T^2 - T_0^2\right)\varphi^2 - E\,T\,\varphi^3 + \frac{\lambda(T)}{4}\varphi^4$$

SM calculation

$$D = \frac{2M_W^2 + M_Z^2 + 2m_t^2}{8v^2}$$

'mass term' important at the origin, drives symmetry restoration

$$T_0 = \frac{m_h^2 - 8Bv^2}{4D}$$

$$B = \frac{3}{64\pi^2 v^4}\left(2M_W^4 + M_Z^4 - 4m_t^4\right)$$

critical temperature, defines the point of SSB

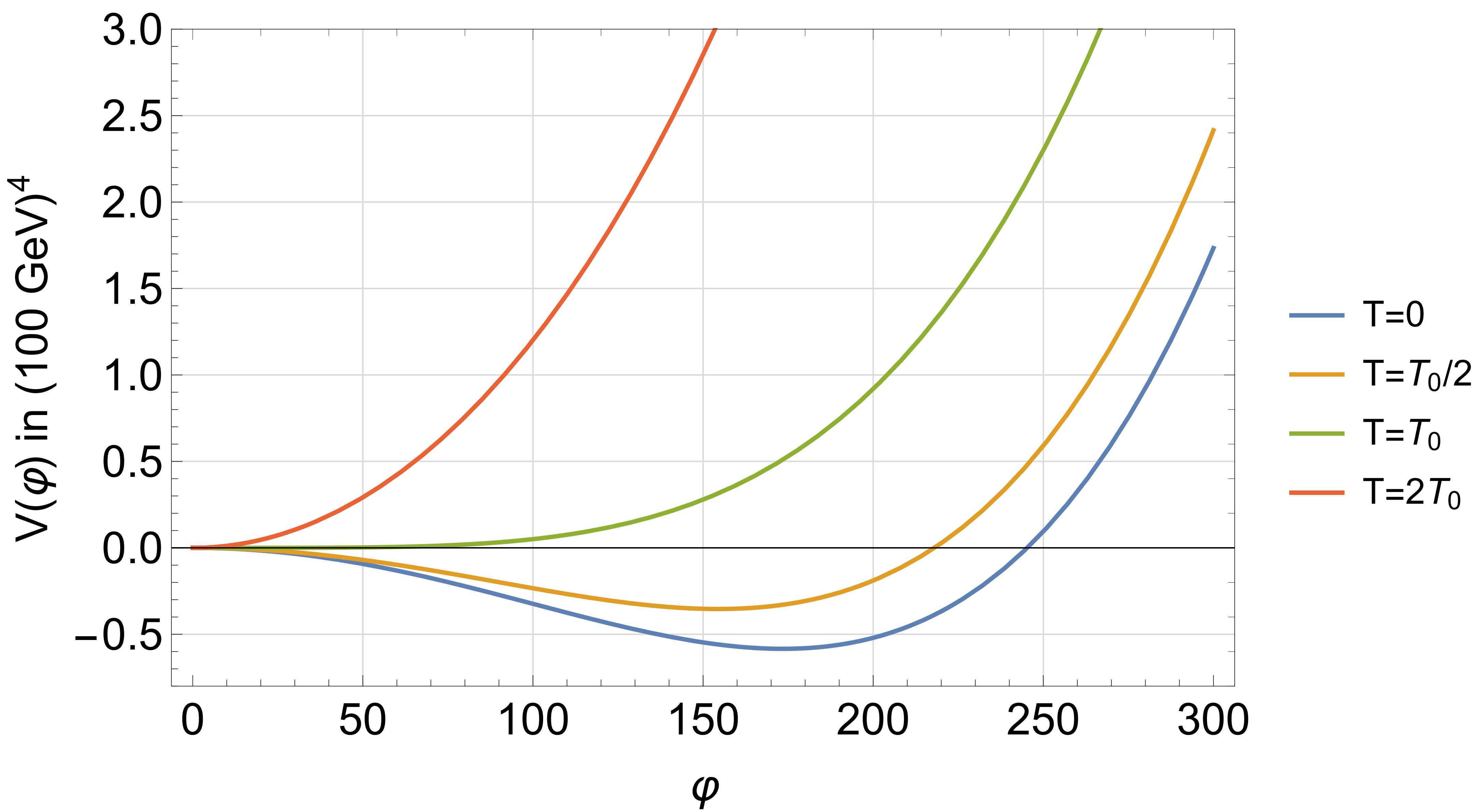
$$E = \frac{2M_W^3 + M_Z^3}{4\pi v^3}$$

cubic term creates a barrier, not from fermions

Cosmological phase transitions

$$V(\varphi, T) = D\left(T^2 - T_0^2\right)\varphi^2 - E\,T\,\varphi^3 + \frac{\lambda(T)}{4}\varphi^4$$

SM high T approximation



Second order PhTr

$$V(\varphi, T) = D\left(T^2 - T_0^2\right)\varphi^2 + \frac{\lambda}{4}\varphi^4$$

$$T = 0: \quad \varphi^2_{\rm min} = \frac{2D}{\lambda}T_0^2 \sim (174\ \mathrm{GeV})^2 > 0$$

broken phase

$$\frac{dV}{d\varphi} = 0: \quad \varphi_{\rm min} = \left(0,\ \sqrt{\frac{2D(T_0^2 - T^2)}{\lambda}}\right)$$

2nd min. exists only below T_0

Above T_0, only the origin is stable, symmetry gets restored

No barrier, the 2nd minimum goes to zero smoothly

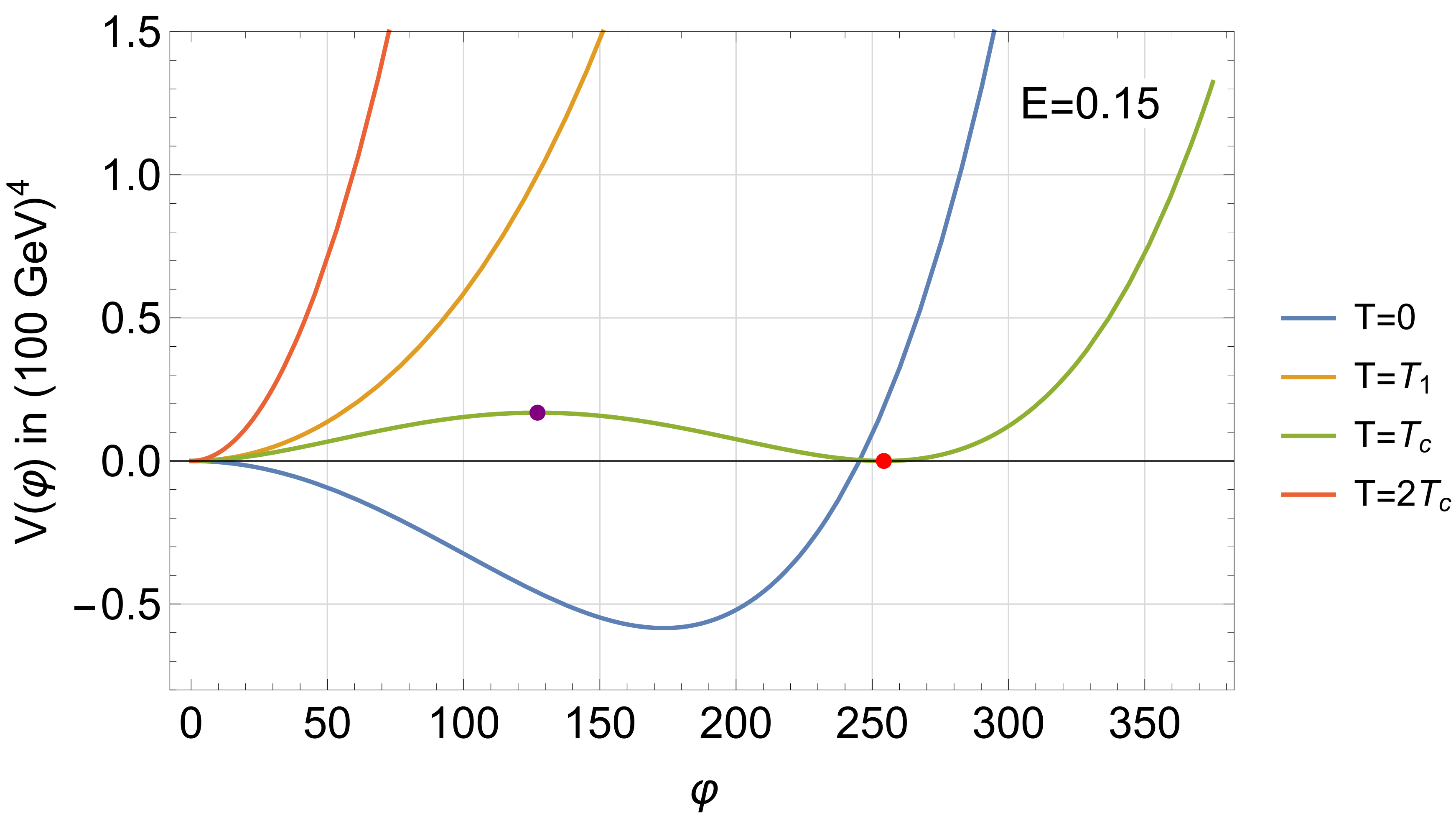
First order PhTr

$$V(\varphi, T) = D\left(T^2 - T_0^2\right)\varphi^2 - E\,T\,\varphi^3 + \frac{\lambda(T)}{4}\varphi^4$$

critical temperature

$$T_1 = \frac{T_0}{\sqrt{1 - \frac{9E^2}{8\lambda D}}} \xrightarrow{E \to 0} T_0 \qquad T_c = \frac{T_0}{\sqrt{1 - \frac{E^2}{\lambda D}}} \qquad v_c = \frac{2ET_c}{\lambda}$$

critical field value, vev



Electroweak Baryogenesis

Quiros hep-ph/9901312

v_c critical vev

T_c critical temperature

sphaleron rate at finite temp.

$$\Gamma_{\rm sph} = 2.8 \times 10^5 \, T^4 \kappa \, \left(\frac{\alpha_2}{4\pi} \right)^4 \left(\frac{E_{\rm sph}}{2\,T} \right)^7 e^{-E_{\rm sph}/T}$$

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$$\frac{E_{\rm sph}}{T_c} \gtrsim 37 - 45$$

requirement for sphaleron
rate to be active

Electroweak Baryogenesis

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$$\frac{E_{\rm sph}}{T_c} \gtrsim 37 - 45$$

requirement for sphaleron rate to be active

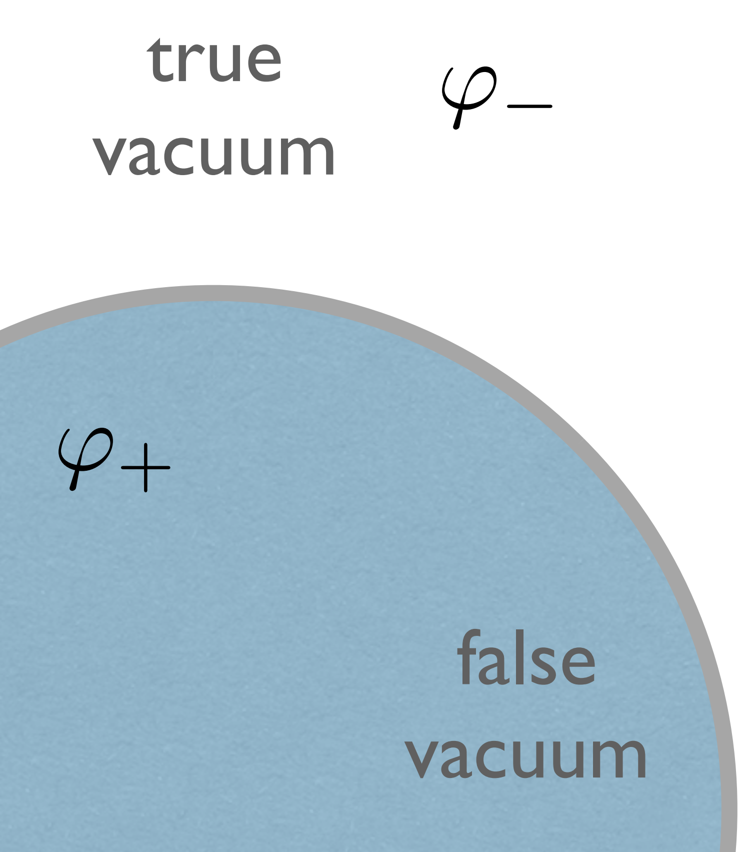
$$\frac{v_c}{T_c} \sim \frac{1}{36}\frac{E_{\rm sph}}{T_c} \gtrsim 1.3 - 1.0$$

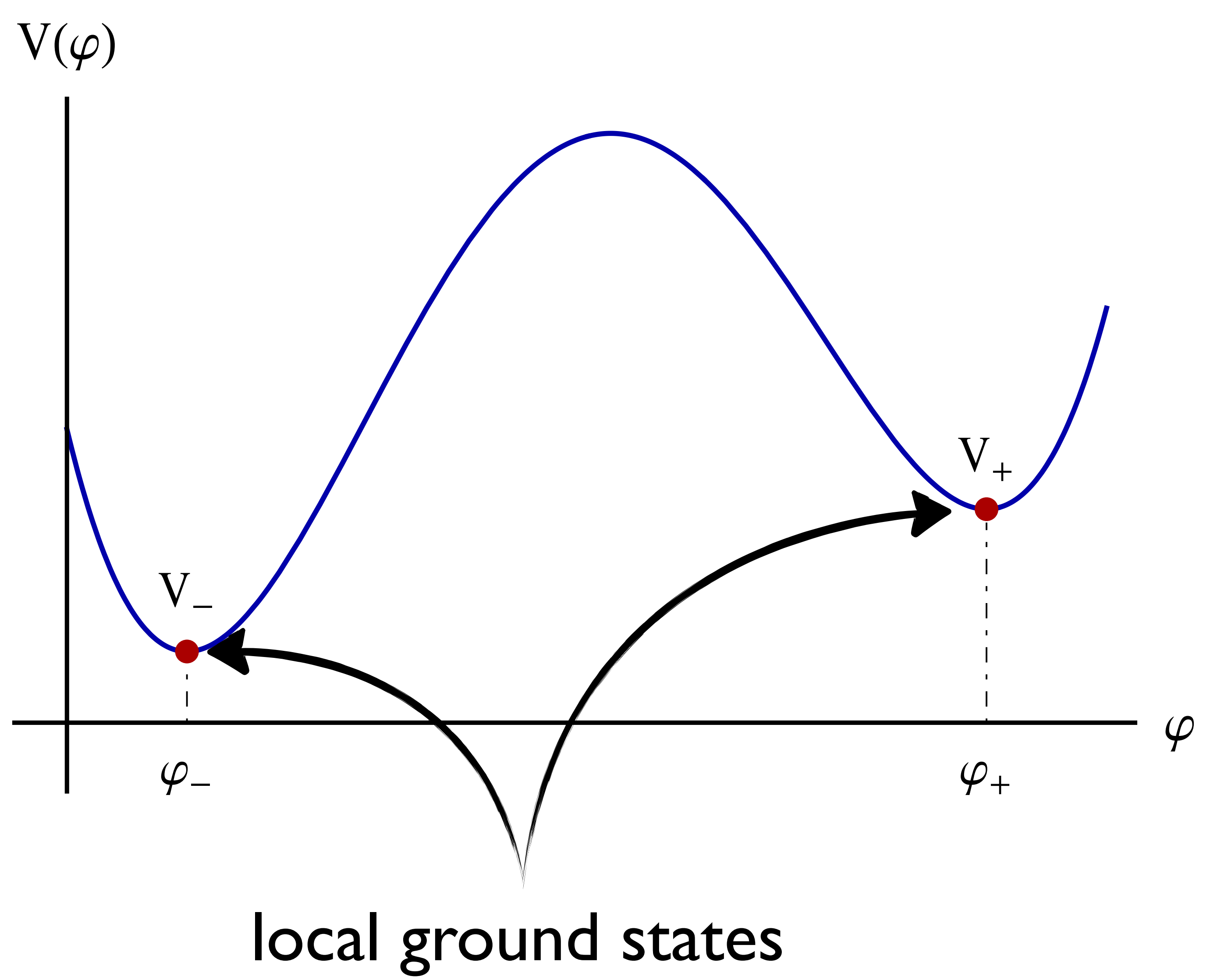
condition for a strong 1st order phase transition

False vacuum decay rate

Local minima may be meta-stable and long lived — Kobzarev, Okun, Voloshin '74

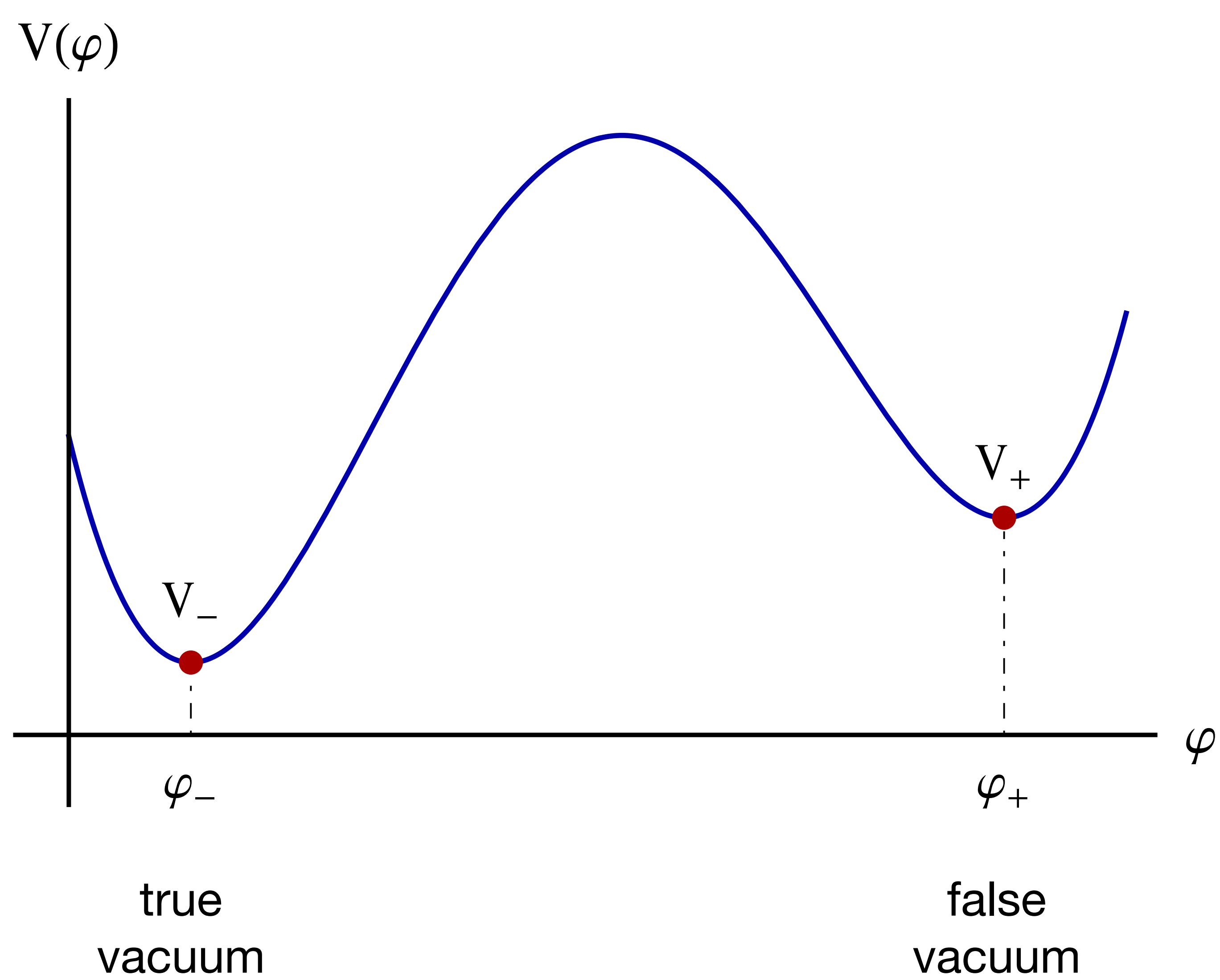
Theory of false vacuum decay — Coleman '77





False vacuum decay rate

Basics of false vacuum decay



Quantum mechanics

$$\hat{H}\psi = (E + i\Gamma)\psi$$

$$P = \int |\psi|^2 \propto e^{-\Gamma t}$$

Alpha decay

$$\Gamma = \Gamma_0 \, e^{-W}$$

$$\Gamma_0 \sim \frac{1}{t_0}$$

$$W = \int_a^b \sqrt{2V} \, \mathrm{d}x$$

The bounce

Computing the transition rate

$$\Gamma/V = A\, e^{-B/\hbar} + \mathcal{O}(\hbar)$$

Theory of B

Coleman '77

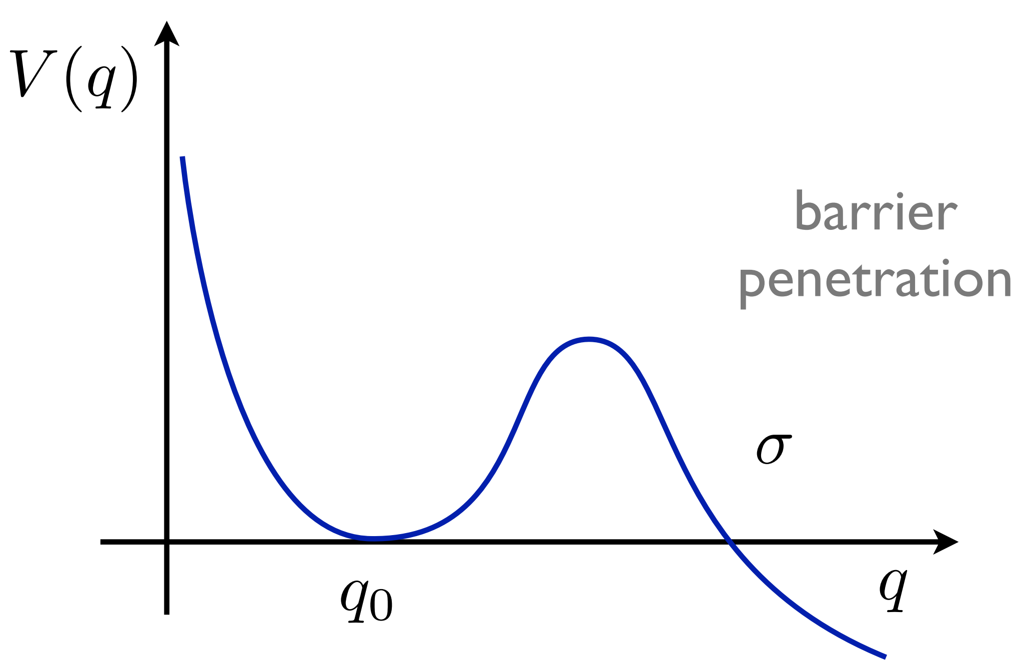
Theory of A

Semi-classical approximation

Callan, Coleman '77

1D QM $L = \frac{1}{2}\dot{q}^2 - V(q)$ $B = 2\int_{q_0}^{\sigma} \mathrm{d}q \sqrt{2V(q)}$

WKB '26



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multi-D $\quad L = \frac{1}{2}\dot{\vec{q}} \cdot \dot{\vec{q}} - V(\vec{q}) \qquad B = 2\int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2V(\vec{q})}$ Banks, Bender, Wu '73

The bounce

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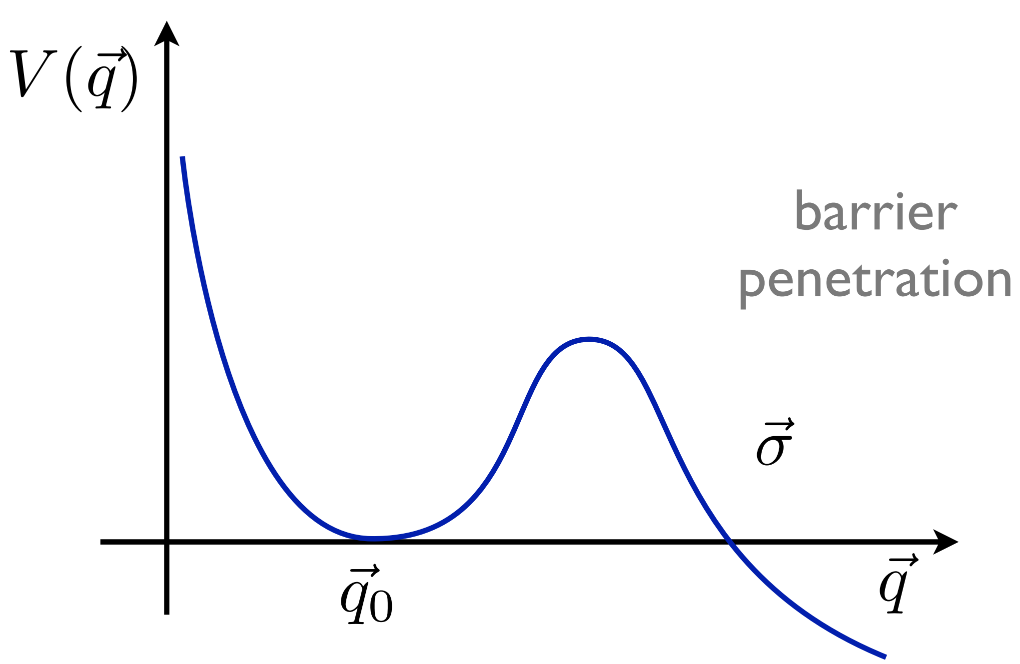

Recast to variational principle

$$\delta \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2V} = 0$$

$$\delta \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2(E-V)} = 0$$

equivalent when

$E = 0$

$V \to -V$

The bounce

Variational principle

$$\delta \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2(E-V)} = 0$$

from

$$\frac{d^2\vec{q}}{dt^2} = -\frac{\partial V}{\partial \vec{q}}$$

and

$$\frac{1}{2}\frac{d\vec{q}}{dt} \cdot \frac{d\vec{q}}{dt} + V = E$$

The bounce

Variational principle

$$\delta \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2(E-V)} = 0$$ from $$\frac{d^2\vec{q}}{dt^2} = -\frac{\partial V}{\partial \vec{q}}$$ and $$\frac{1}{2}\frac{d\vec{q}}{dt}\cdot\frac{d\vec{q}}{dt} + V = E$$

Transition to Euclidean time

$\tau = i\,t$ and $E = 0$ $\quad \frac{1}{2}\frac{d\vec{q}}{d\tau}\cdot\frac{d\vec{q}}{d\tau} - V = 0$

$$L_E = \frac{1}{2}\frac{d\vec{q}}{d\tau}\cdot\frac{d\vec{q}}{d\tau} + V \qquad \frac{B}{2} = \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s\sqrt{2V} = \int_{-\infty}^{0} d\tau L_E$$

bounce

$$q(\tau=-\infty) = q_0, \qquad V = 0, \qquad \dot{q} = 0$$
$$q(\tau=0) = \sigma, \qquad V = 0, \qquad \dot{q} = 0$$

$$B = \int_{-\infty}^{\infty} d\tau L_E = S_E$$ Euclidean action

The bounce

Variational principle

$$\delta \int_{\vec{q}_0}^{\vec{\sigma}} \mathrm{d}s \sqrt{2(E-V)} = 0$$ from $$\frac{d^2\vec{q}}{dt^2} = -\frac{\partial V}{\partial \vec{q}}$$ and $$\frac{1}{2}\frac{d\vec{q}}{dt}\cdot\frac{d\vec{q}}{dt} + V = E$$

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Generalize to single real scalar field theory in flat ST

Coleman '77

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \qquad B = S_E = \int d\tau d^3x \left(\frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial_i x}\right)^2\right) + V(\varphi)$$

The bounce

Generalize to single real scalar field theory

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi)$$

$$B = S_E = \int d\tau d^3x \left(\frac{1}{2}\left(\frac{\partial\varphi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\varphi}{\partial_i x}\right)^2 \right) + V(\varphi)$$

Bounce solution Euclidean $O(4)$ symmetric

Coleman, Glaser, Martin '78

$$\rho^2 = t^2 + \sum x_i^2$$

Euclidean time =
radius of the bubble

The bounce

Generalize to single real scalar field theory

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Bounce solution Euclidean $O(4)$ symmetric — Coleman, Glaser, Martin '78

$$\rho^2 = t^2 + \sum x_i^2$$

Euclidean time = radius of the bubble

> "...there always exists an *O(4)*-invariant bounce and it always has strictly lower action than any non-*O(4)* invariant bounce. The rigor of our proof is matched only by its tedium; I wouldn't lecture on it to my worst enemy."
> Coleman, Erice lectures '77

Extended to multi-fields — Blum, Honda, Sato, Takimoto, Tobioka '16

The bounce

D dimensional $O(D)$ spherically symmetric Euclidean action

$$S_D = \frac{2\pi^{\frac{D}{2}}}{\Gamma\left(\frac{D}{2}\right)} \int_0^\infty \rho^{D-1}\, \mathrm{d}\rho \left(\frac{1}{2}\dot{\varphi}^2 + V(\varphi)\right)$$

D = 4 : FV decay at *T = 0* — Coleman '77

D = 3 : FV nucleation at finite *T* — Affleck '81, Linde '83

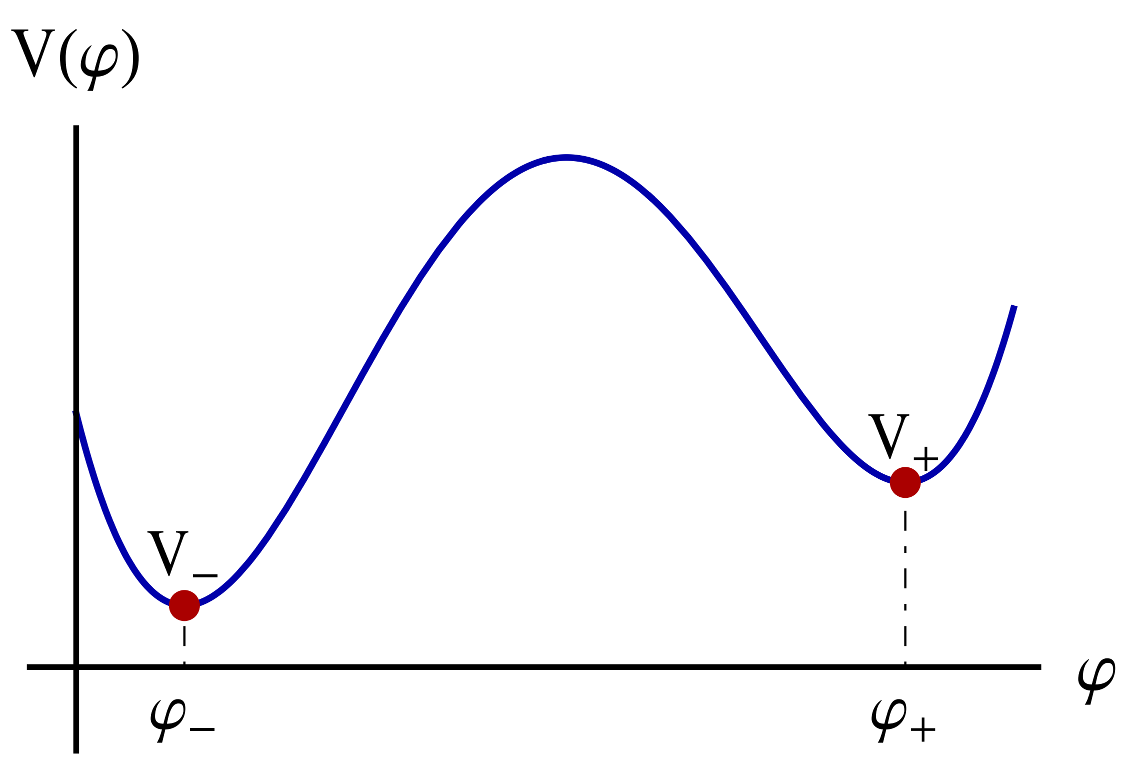
Bounce equation

$$\ddot{\varphi} + \frac{D-1}{\rho}\,\dot{\varphi} = dV$$

friction

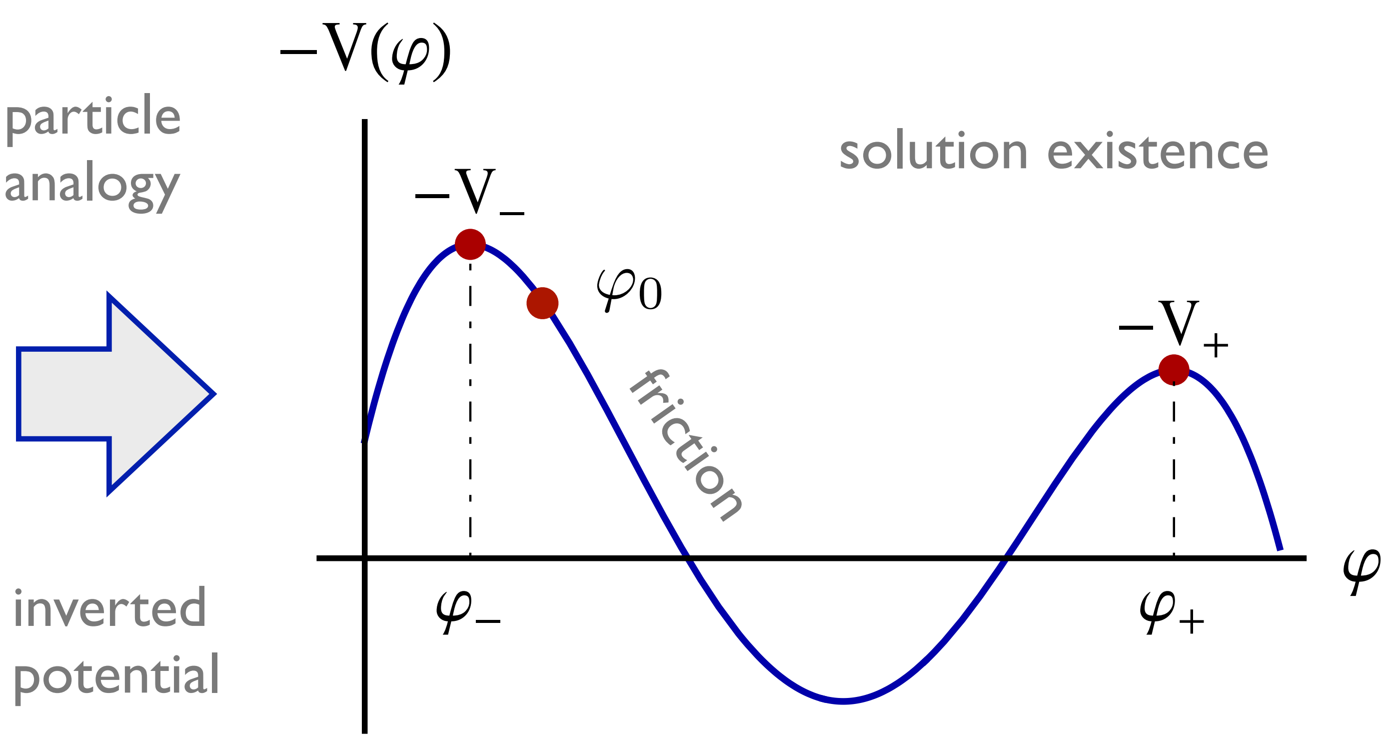
Boundary conditions

$$\varphi(0) = \varphi_0, \quad \varphi(\infty) = \varphi_+, \quad \dot{\varphi}(0,\infty) = 0$$

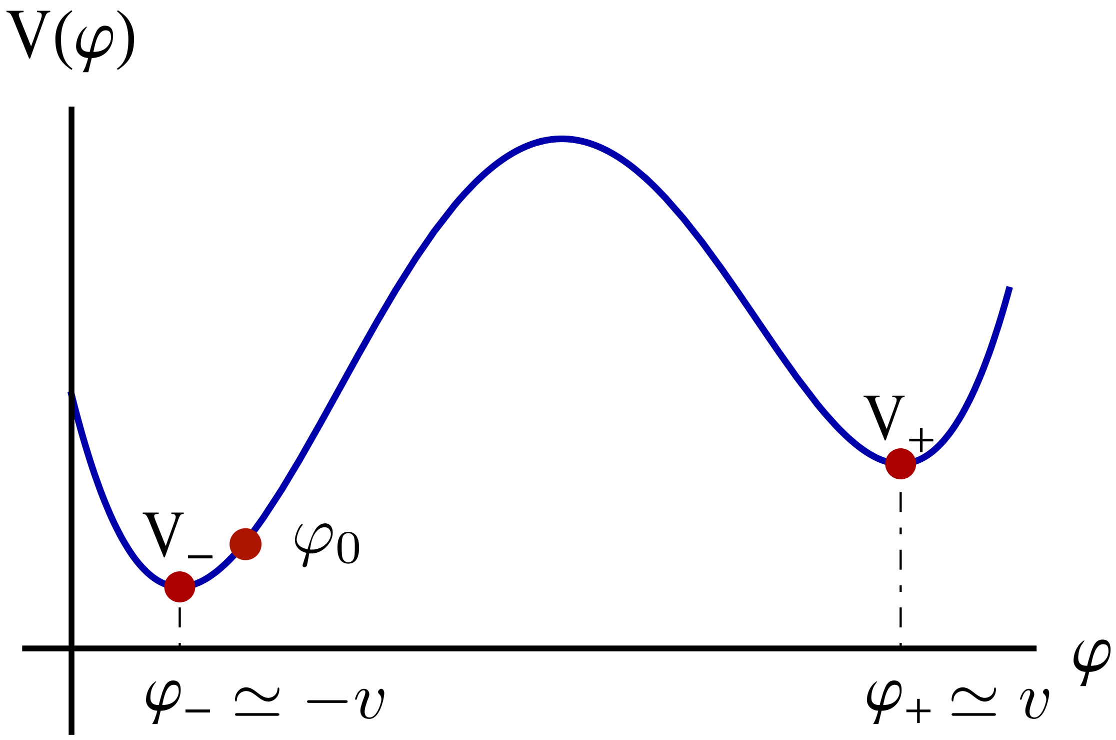


particle analogy

inverted potential



solution existence

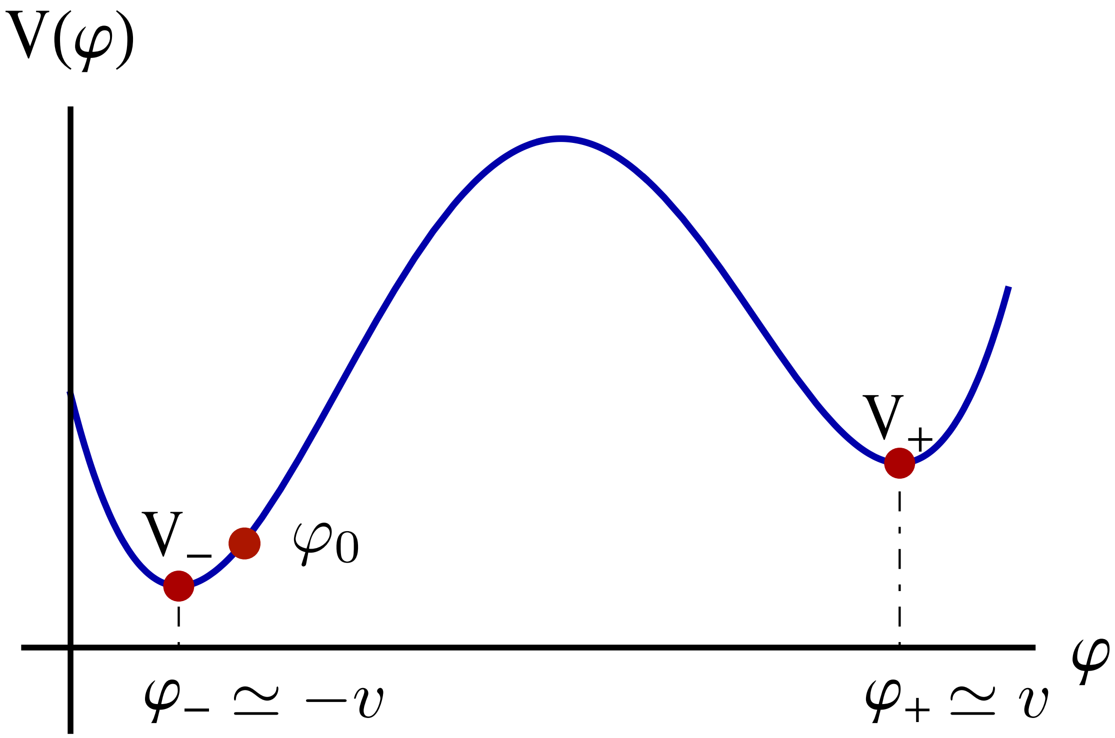


Thin wall approximation

Coleman '77

$$V(\varphi) = \frac{\lambda}{8}\left(\varphi^2 - v^2\right)^2 + \varepsilon\left(\frac{\varphi - v}{2v}\right), \qquad S_1 = \frac{v^3\sqrt{\lambda}}{3}$$

small ε limit $\quad \varphi_0 \simeq \varphi_-$ $\quad$ until $\quad \rho = R$



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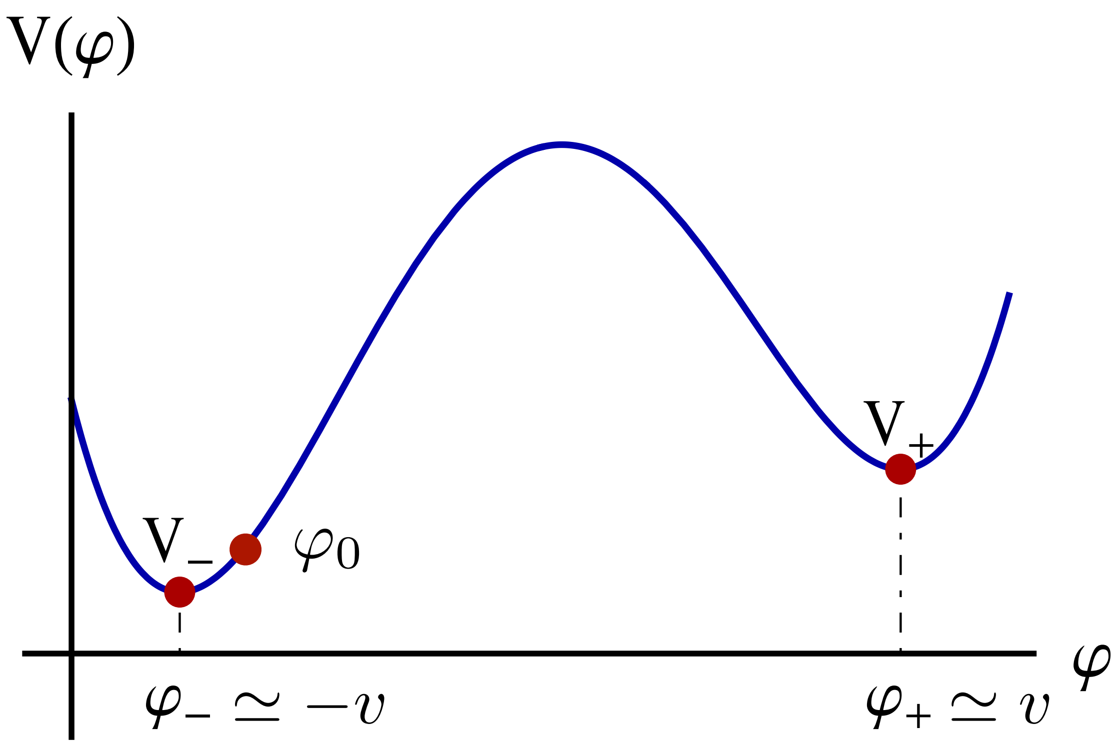
Field solution

$$\varphi(\rho) = \begin{cases} -v, & \rho \ll R \\ \varphi_1(\rho - R), & \rho \approx R \\ v, & \rho \gg R \end{cases}$$

$$\varphi_1(\rho) = v \tanh\left(\frac{\sqrt{\lambda}v}{2}\rho\right)$$

Extremize the action



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Extremize the action

Bounce action

$$S_E = 2\pi^2 \int_0^\infty \rho^3 d\rho \left(\frac{1}{2}\dot{\varphi}^2 + V\right)$$

$$= -\frac{1}{2}\pi^2 R^4 \varepsilon + \pi^2 R^3 S_1$$

volume surface

$$\frac{dS_E}{dR} = 0 \quad \Rightarrow \quad R = \frac{3S_1}{\varepsilon}$$

$$\boxed{S_E = \frac{27\pi^2}{2}\frac{S_1^4}{\varepsilon^3}}$$

Bounce actions

$$S_4 = 2\pi^2 \int_0^\infty \rho^3 d\rho \left(\frac{1}{2}\dot{\varphi}^2 + V\right)$$
$$= -\frac{1}{2}\pi^2 R^4 \varepsilon + \pi^2 R^3 S_1$$

4D = vacuum, *T=0*

$$\frac{\Gamma}{V} \sim A\, e^{-S_4}$$

quantum fluctuations

3D = finite temperature

$$S_3 = 4\pi \int_0^\infty \rho^2 d\rho \left(\frac{1}{2}\dot{\varphi}^2 + V\right)$$
$$\simeq -\frac{4\pi R^3}{3}\langle V\rangle + 2\pi R^2 \left(\frac{\delta\varphi}{\delta R}\right)^2 \delta R$$

$$\frac{\Gamma}{V} \sim T^4\, e^{-S_3/T}$$

thermal fluctuations

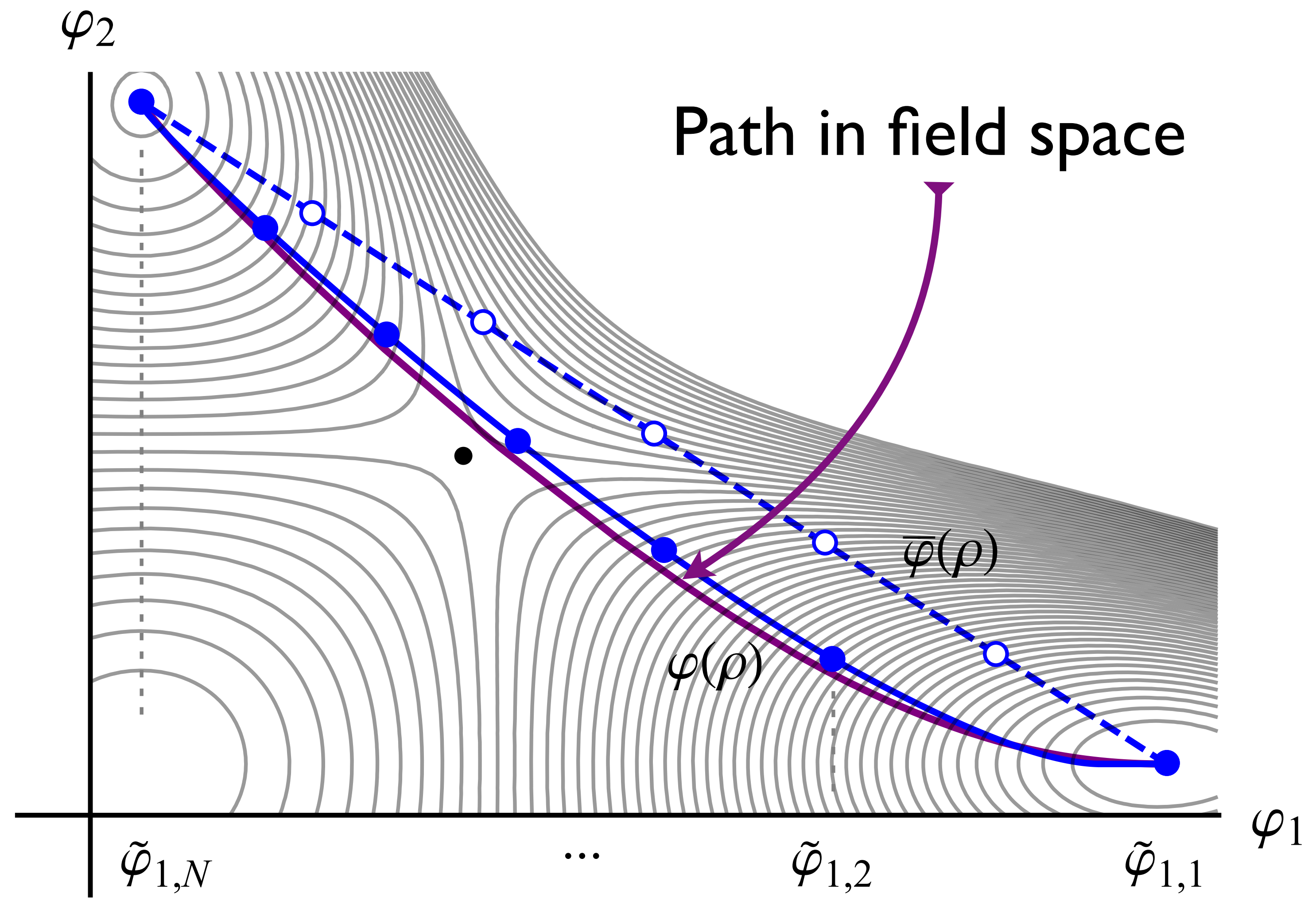
Multi-fields are harder, but possible to do semi-analytically.

Multi-fields

$$\ddot{\varphi}_i + \frac{D-1}{\rho}\dot{\varphi}_i = \frac{dV}{d\varphi_i} \qquad\qquad \varphi_i(0) = \varphi_{0i}$$

Shooting method impractical

- highly non-linear
- multi-dimensional field space



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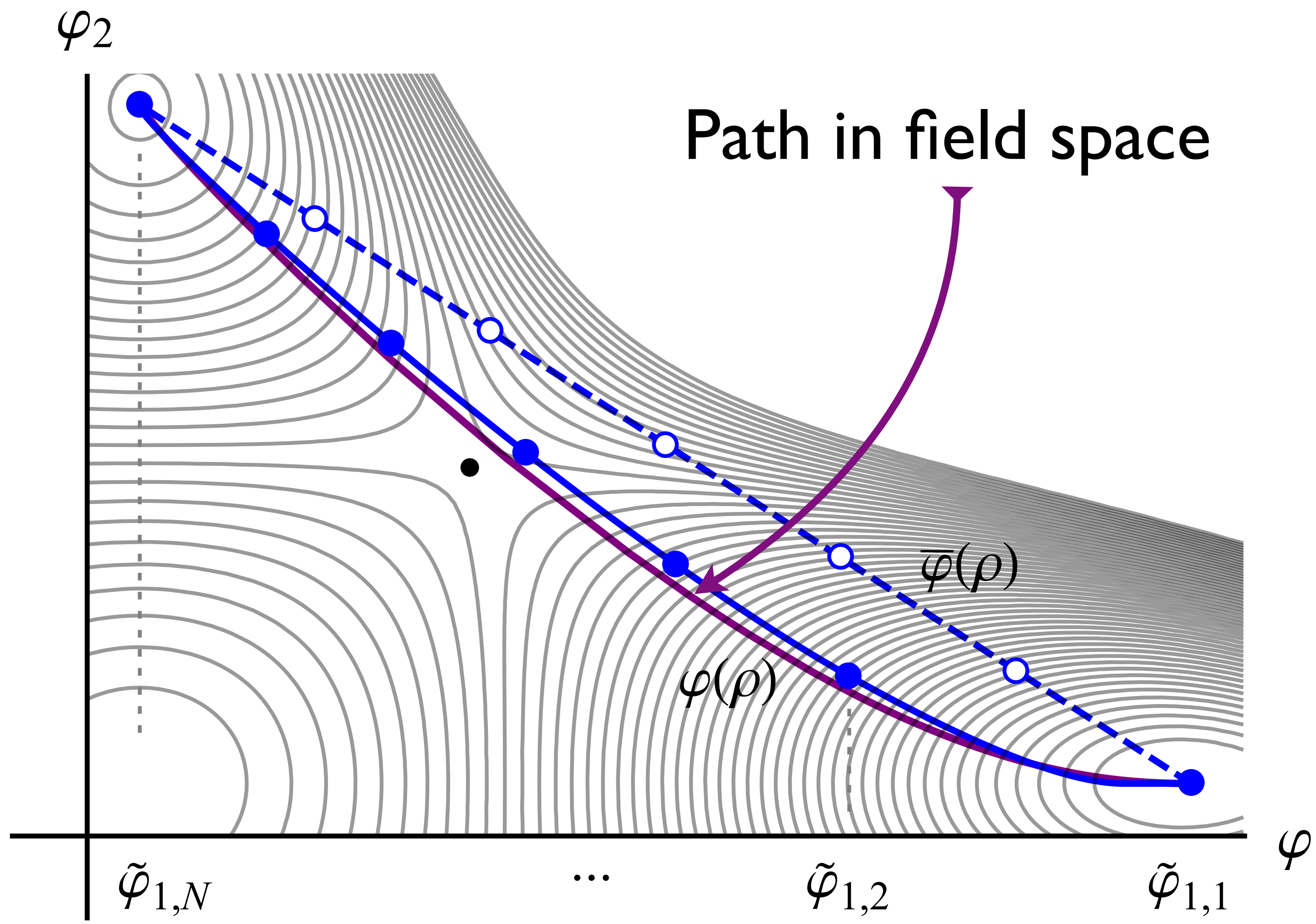


CosmoTransitions Wainwright '11

bounce and path deformation separate, oscillations, Runge-Kutta PDE solver

AnyBubble Masoumi, Olum, Shlaer '16

multiple shooting, damping approximations

Other recent approaches

tunneling potential

Espinosa, Konstandin '18

machine learning

Piscopo, Spannowsky, Waite '19

Multi-fields

$$V(\varphi_i) = \sum_{i=1}^{2} \left(-\mu_i^2 \varphi_i^2 + \lambda_i^2 \varphi_i^4\right) + \lambda_{12}\varphi_1^2\varphi_2^2 + \tilde{\mu}^3 \varphi_2$$

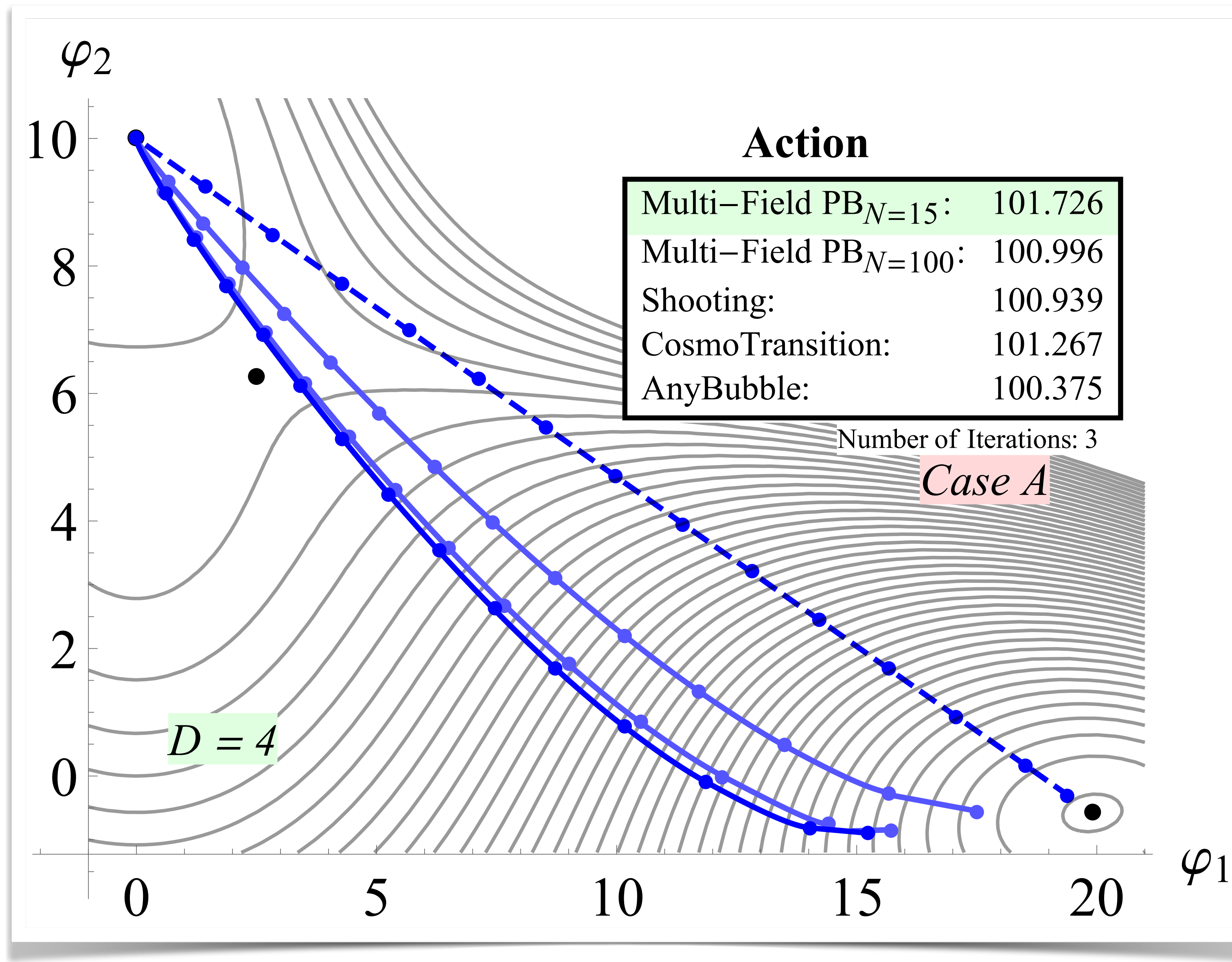
no oscillations

converges in a few iterations

works for thin wall

works for *D*=*3* and *4*

tested for up to 20 fields



FindBounce

Guada, Nemevšek, Pintar ’20

`FindBounce[V[x], x, {min1, min2}]`
computes the bounce for potential V[x] for field x.

`FindBounce[V[x, y, ...], {x, y, ...}, {min1, min2}]`
works with multiple scalar fields.

`FindBounce[{{x1, y1}, {x2, y2}, ...}]`
works with a single field potential given as a list of points.

▸ Details

▾ Examples (37)

▾ Basic Examples (4)

Load the package:

```
In[1]:= Needs["FindBounce`"]
```

Find bounce of a single field potential with known location on minima at 0 and 1.

```
In[2]:= bf = FindBounce[1/4 x^4 - 4/10 x^3 + 1/10 x^2 , x, {0, 1}]
```

Out[2]= BounceFunction[ Action: 139. Dimension: 4]

Resulting BounceFunction expression contains different properties of computed solution. In the notebook interface it is formatted with a nice expandable summary box. Value of Euclidean action can be extracted from the BounceFunction among other properties.

```
In[3]:= bf["Action"]
Out[3]= 139.25
```

Find field value for and radius. Result is always returned as a list of fields, even for single field case.

```
In[4]:= Through[bf["Bounce"][5.]]
Out[4]= {0.647905}
```

We can get the same result with a more general syntax which assumes that a single field potential is just a multi-field potential with length one.

```
In[5]:= FindBounce[1/4 x^4 - 4/10 x^3 + 1/10 x^2, {x}, {{0}, {1}}]
```

Out[5]= BounceFunction[ Action: 139. Dimension: 4]

Define a simple two field potential with known location of minima at {0,0} and {1,1}.

```
In[1]:= Clear[V]
V[x_, y_] := (x^2 + 5 y^2) (5 (x - 1)^2 + (y - 1)^2) + 2 (1/4 y^4 - 1/3 y^3)
minima = {{0, 0}, {1, 1}};
```

Calculate its bounce filed configuration in 3 dimensions.

```
In[2]:= bf2 = FindBounce[V[x, y], {x, y}, minima, "Dimension" → 3]
Out[2]= BounceFunction[ Action: 1790. Dimension: 3 ]
```

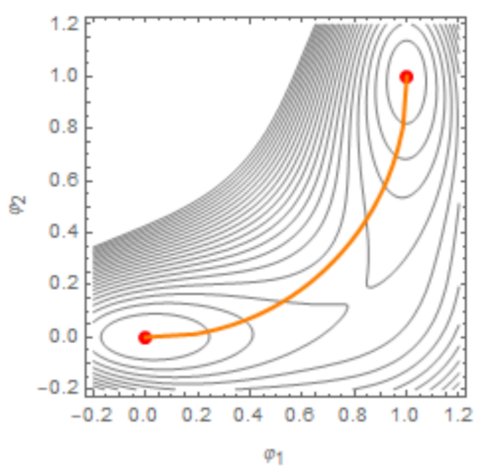
See what is the value of Euclidean action.

```
In[3]:= bf2["Action"]
Out[3]= 1788.76
```

Visualise contour plot of two field potential with bounce path between two minima marked with red points.

```
In[4]:= ContourPlot[
  V[x, y], {x, -0.2, 1.2}, {y, -0.2, 1.2},
  Contours → 25,
  ContourShading → LightGray,
  PlotRange → {-1, 5},
  FrameLabel → {"φ1", "φ2"},
  Epilog → {
    {Red, PointSize[Large], Point[minima]},
    {Orange, Thick, Line[bf2["Path"]]}
   }
 ]
```

Out[4]=



Ok, we got the rate. However, the universe is expanding and we have to take this into account.

Nucleation temperature

Quiros '99

Radiation dominated early universe

$$\rho_\gamma = \frac{\pi^2}{30}\, g_*(T)\, T^4$$

$$g_*(T > v) = 106.75$$

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$$g_*(T > v) = 106.75$$

time vs. temperature

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3M_{\rm Pl}^2}\rho_\gamma \quad \overset{a_1 T_1 = a_2 T_2}{\Rightarrow} \quad t = \zeta \frac{M_{\rm Pl}}{T^2} \qquad \zeta = \frac{1}{4\pi}\sqrt{\frac{45}{\pi g_*}}$$

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$$\rho_\gamma = \frac{\pi^2}{30}\, g_*(T)\, T^4 \qquad g_*(T > v) = 106.75$$

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$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3M_{\rm Pl}^2}\rho_\gamma \quad \overset{a_1 T_1 = a_2 T_2}{\Rightarrow} \quad t = \zeta \frac{M_{\rm Pl}}{T^2} \qquad \zeta = \frac{1}{4\pi}\sqrt{\frac{45}{\pi g_*}}$$

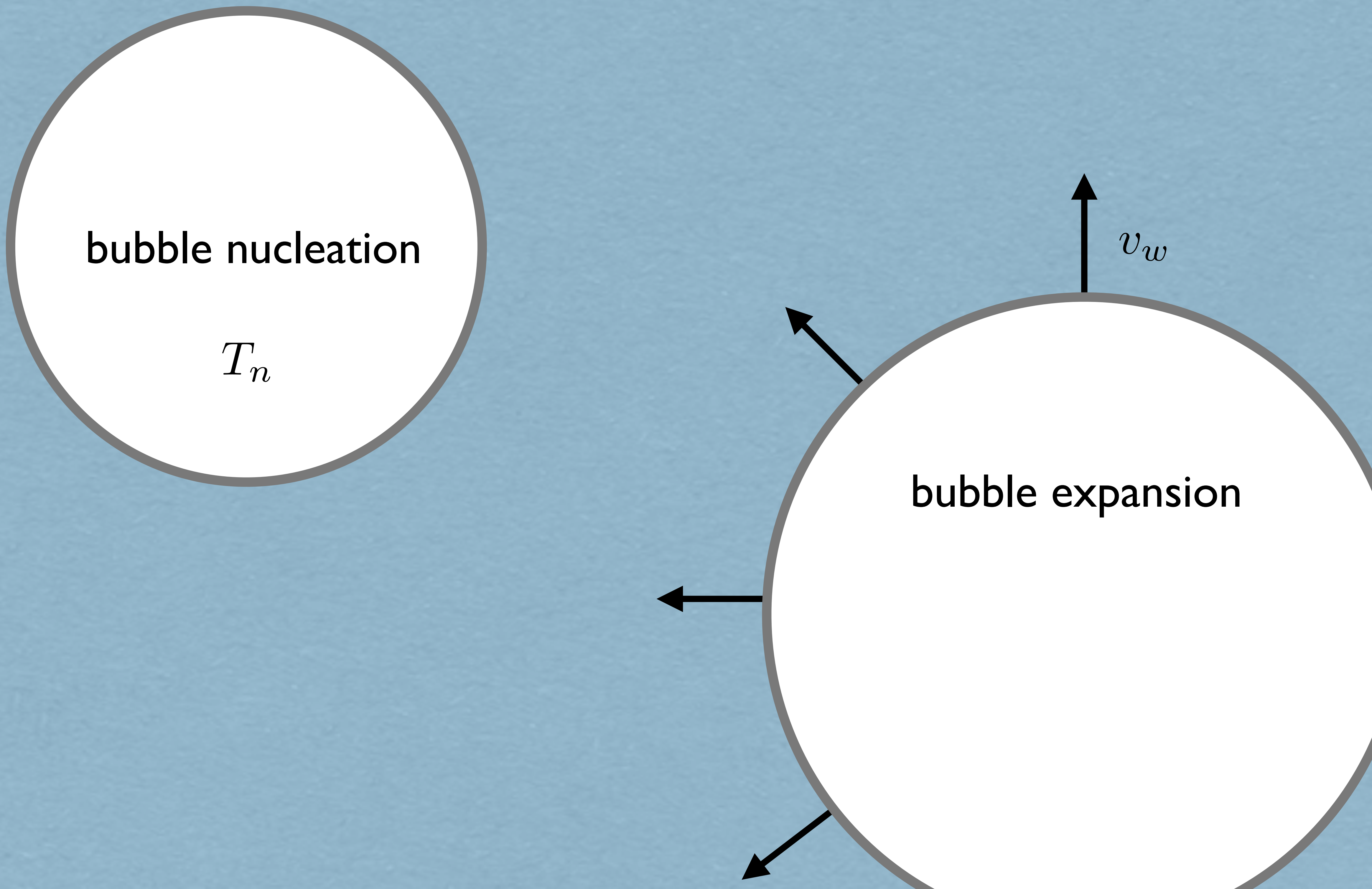
$$V_3(T) = \left(\frac{2\zeta M_{\rm Pl}}{T}\right)^3 \qquad \Rightarrow \qquad \int_{T_n}^{\infty} \frac{dT}{T}\left(\frac{2\zeta M_{\rm Pl}}{T}\right)^4 e^{-S_3/T} = \mathcal{O}(1)$$

Nucleation happens at T_n

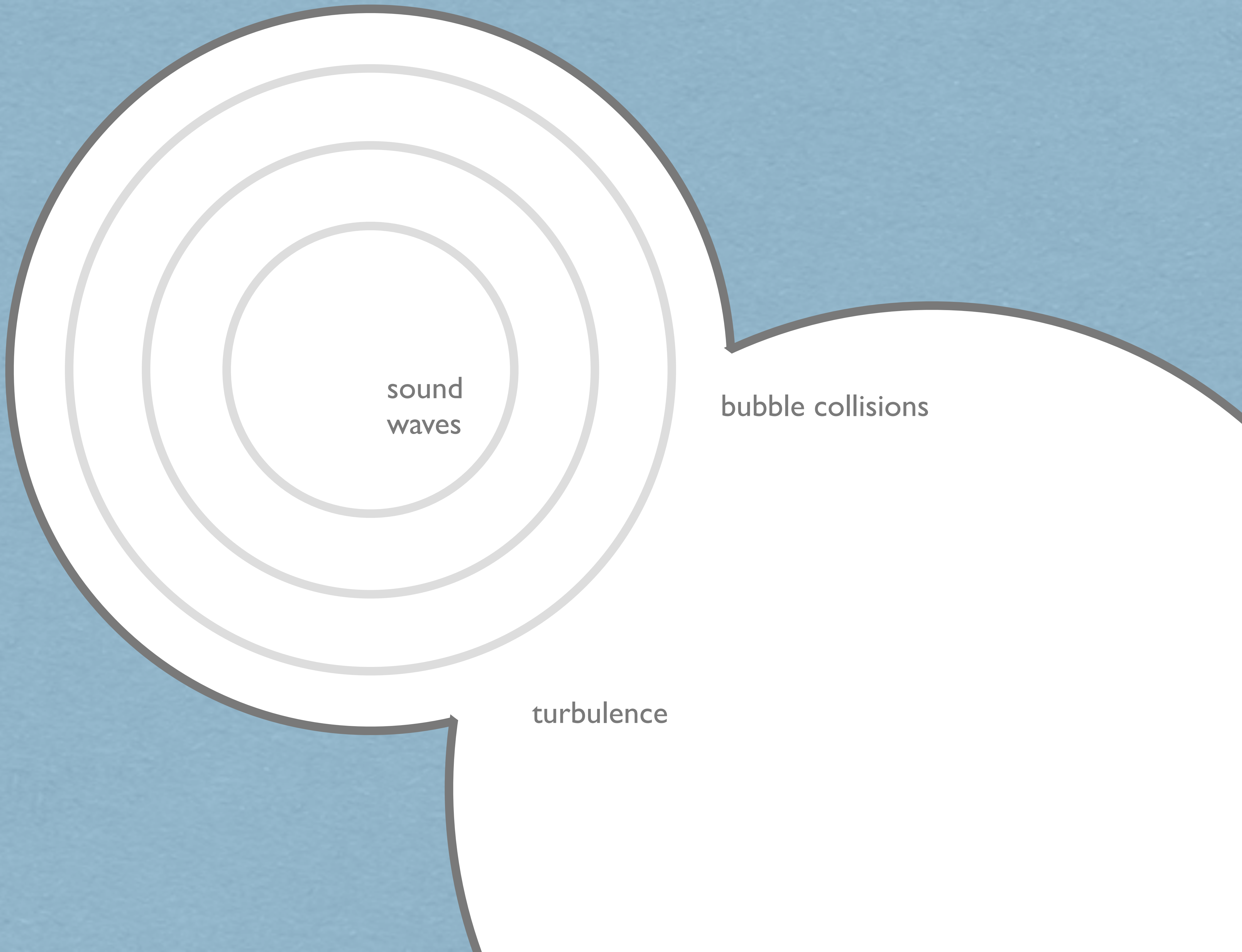
$$P_{\rm FV}(T_n) \simeq \mathcal{O}(1) \qquad S_3(T) \sim 137 + \log\frac{(10\,E)^2}{\lambda D} + 4\log\frac{100\ {\rm GeV}}{T_n}$$

Gravitational Waves

Gravitational waves



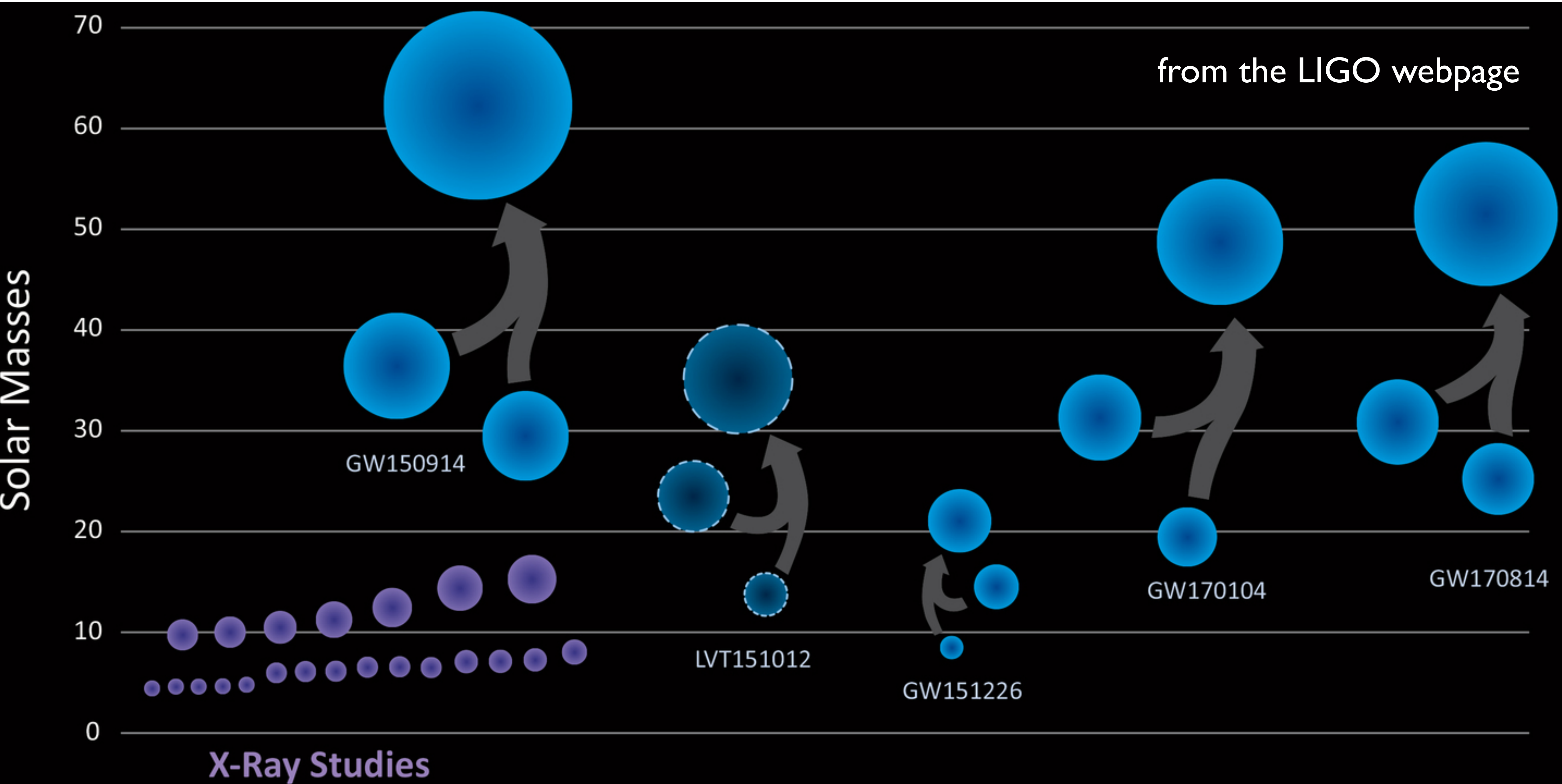
Gravitational wave sources



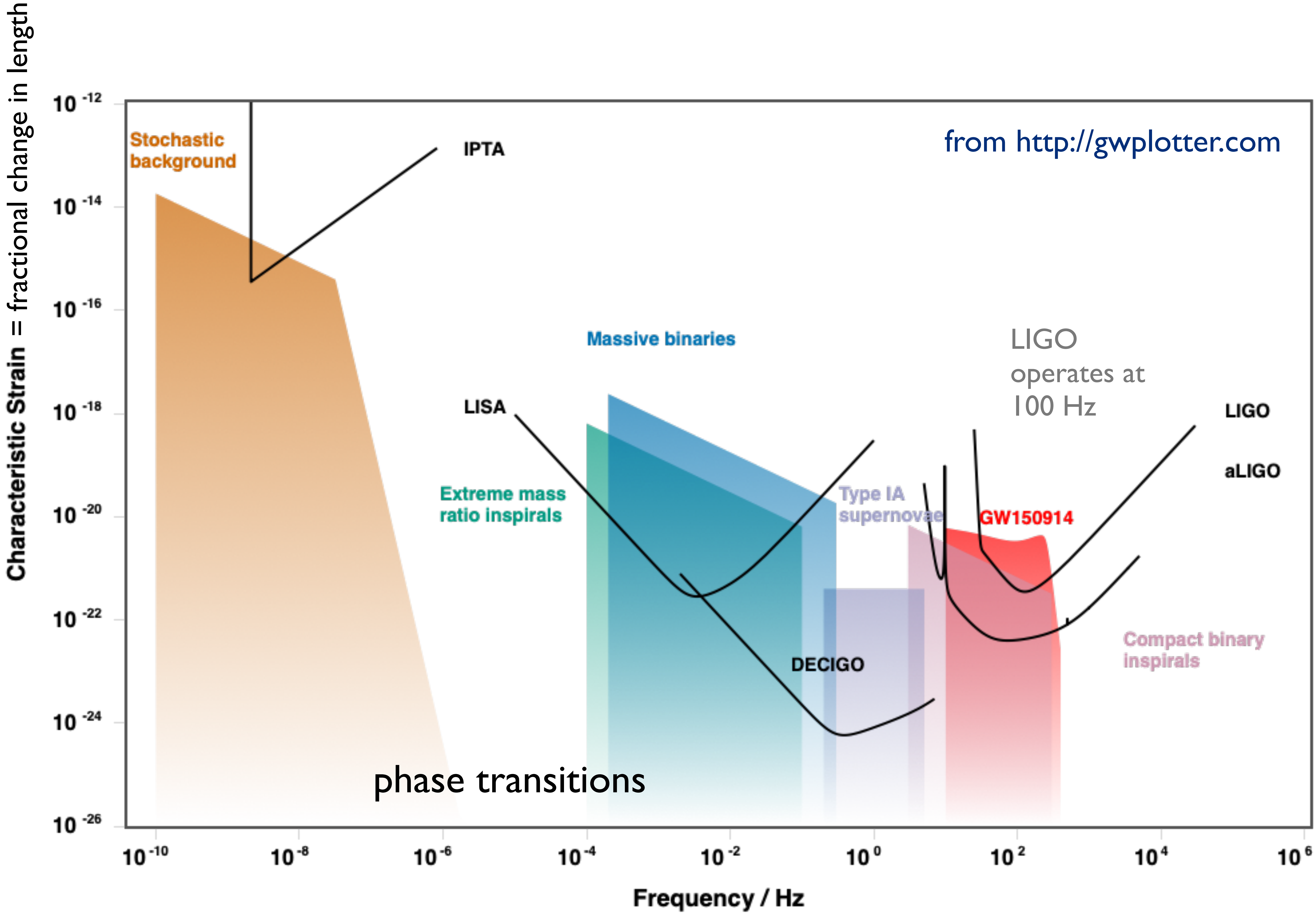
Gravitational wave sources

Current ‘point sources’ LIGO/VIRGO events

mergers of BH with O(10) solar masses, neutron star mergers



Experimental status



Gravitational wave redux

see e.g. notes by Sean Carroll, section 6

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

We are interested in the graviton excitations above the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$ metric tensor

$$\eta_{\mu\nu} = \mathrm{diag}\{1,-1,-1,-1\}$$ flat Minkowski metric

$h_{\mu\nu}$ graviton degrees of freedom, massless, need to fix the gauge (headaches)

$|h_{\mu\nu}| \ll 1$ weak field limit

$$\begin{aligned} G_{\mu\nu} &= R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}R \\ &= \frac{1}{2}\left(\partial_\sigma\partial_\nu h^\sigma_\mu + \partial_\sigma\partial_\mu h^\sigma_\nu - \partial_\mu\partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\mu\partial_\nu h^{\mu\nu} + \eta_{\mu\nu}\Box h\right) \end{aligned}$$

Gravitational wave redux

see e.g. notes by Sean Carroll, section 6

$$G_{\mu\nu} = 8\pi G_N T_{\mu\nu}$$

trace-reversed perturbation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h$$

Einstein equation becomes

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

in vacuum

$$\Box \bar{h}_{\mu\nu} = 0$$

$$k_\mu k^\mu = 0$$

plane waves, propagates at the speed of light $c_h = c$

plane waves

$$h^{TT}_{\mu\nu} = C_{\mu\nu} e^{ik_\mu x^\mu}$$

$$C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

two polarizations, as any massless particle

Point source solution

see e.g. notes by Sean Carroll, section 6

take a distant point source, like BH/NS mergers

$$\Box_x G\left(x^\sigma - y^\sigma\right) = \delta^{(4)}\left(x^\sigma - y^\sigma\right)$$

solution to graviton propagation

$$\bar{h}_{\mu\nu}\left(x^\sigma\right) = -16\pi G \int G\left(x^\sigma - y^\sigma\right) T_{\mu\nu}\left(y^\sigma\right) d^4y$$

$$\bar{h}_{ij}(t, \mathbf{x}) = \frac{2G}{3R}\frac{d^2 q_{ij}}{dt^2}$$

R…distance to the binary source

q_{ij} quadrupole moment :

$$q_{ij}(t) = 3\int y^i y^j T^{00}(t, \mathbf{y}) d^3y$$

M…mass of the stars

$$T^{00}(t, \mathbf{x}) = M\delta\left(x^3\right)\left[\delta\left(x^1 - r\cos\Omega t\right)\delta\left(x^2 - r\sin\Omega t\right) + \delta\left(x^1 + r\cos\Omega t\right)\delta\left(x^2 + r\sin\Omega t\right)\right]$$

Primordial sources

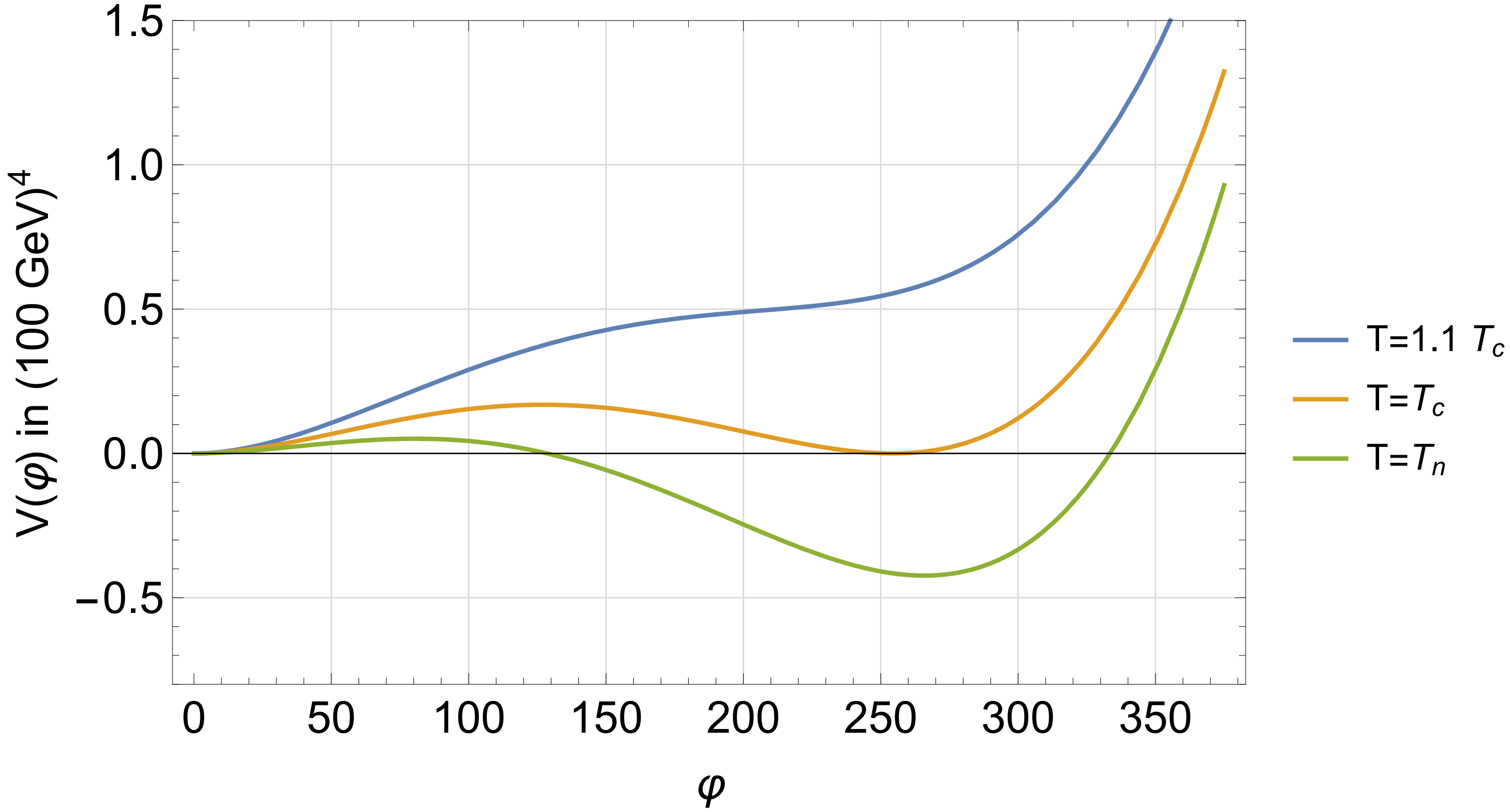
inflation

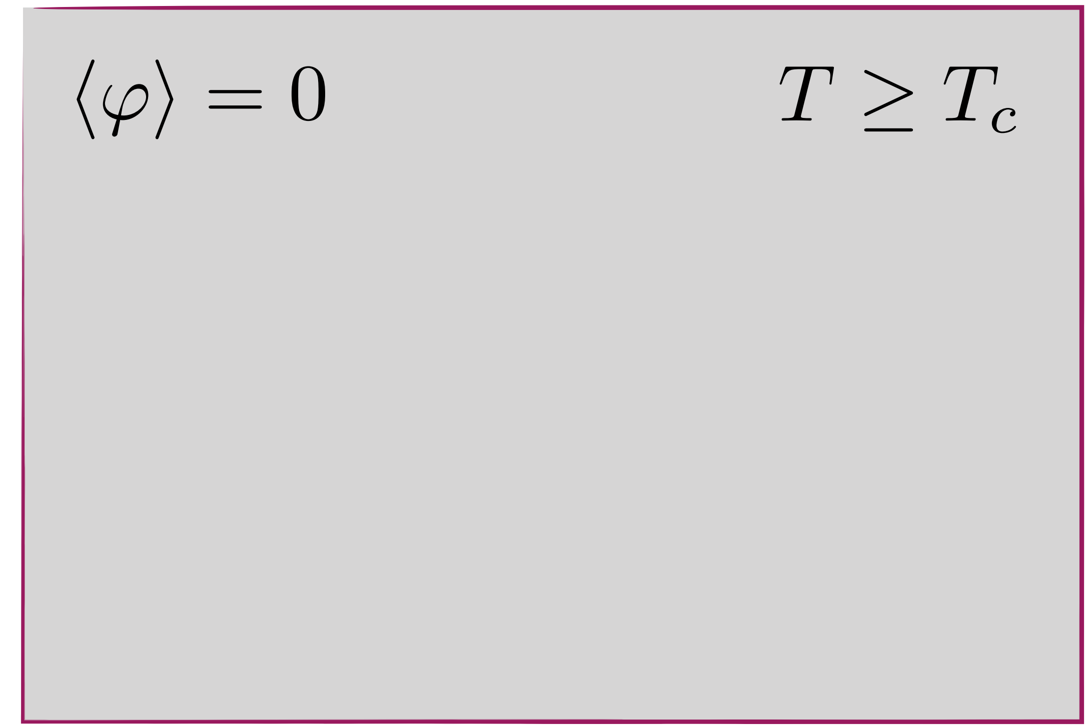
defects, strings, domain walls

first order phase transitions

axions, BH evaporation, …

Physical picture

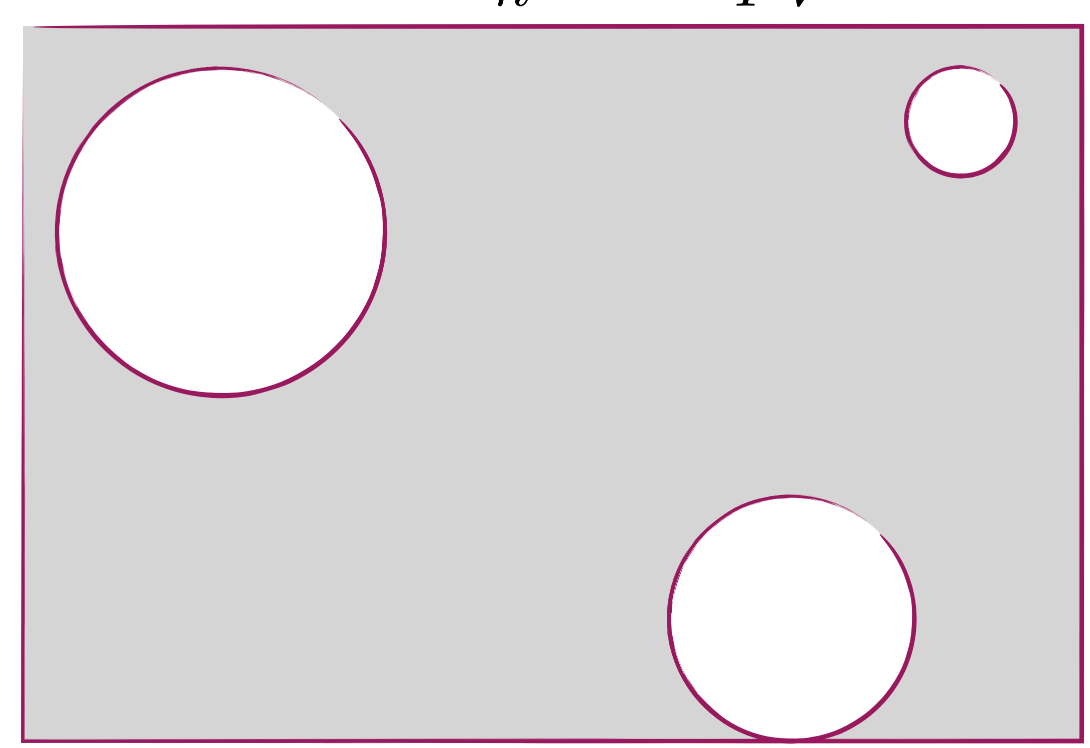




$$@T = T_c: \quad \Gamma = 0$$

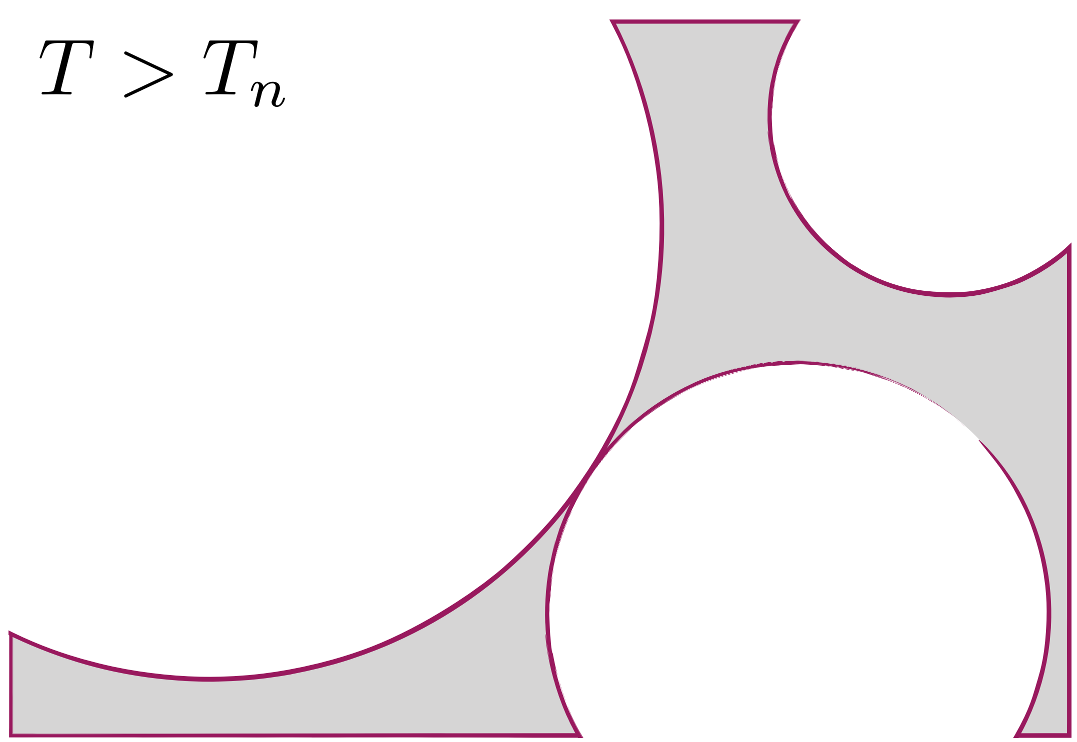
bubbles nucleate and expand

$$@T = T_n: \quad P_{FV} \simeq 1$$

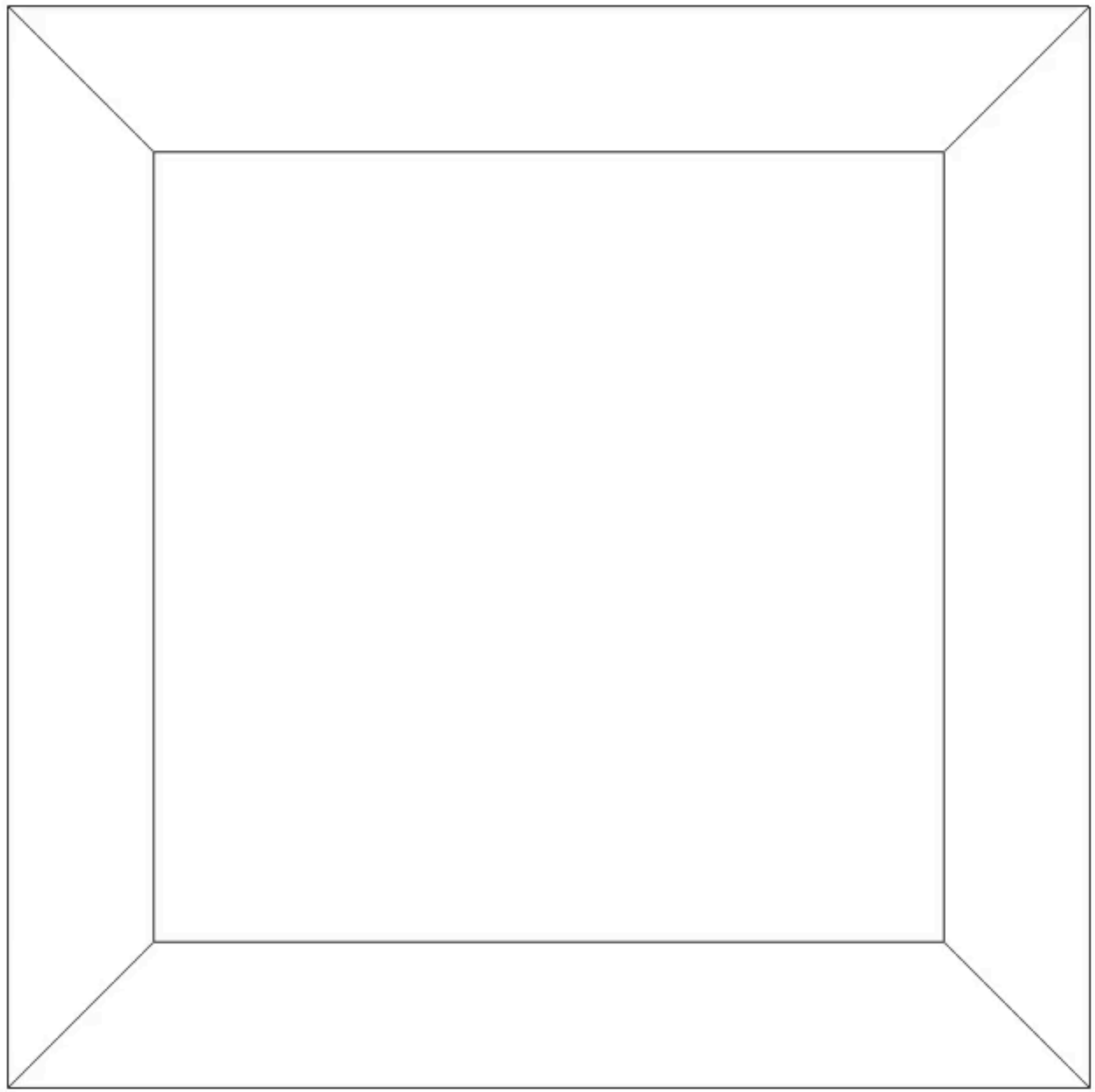


bubbles collide, sound waves and
turbulence is formed afterwards

$T > T_n$



Simulation of bubble collisions



from David Weir's webpage: https://saoghal.net/pages/older-visualisations.html

Parameters characterizing the GW spectrum

nucleation temperature T_n obtained by solving $\frac{S_3(T_n)}{T_n} \simeq 140$

$T_n \sim v \to f \sim \mathrm{mHz}$ perfect e.g. for LISA

typically at v, but could be lower with supercooling

vacuum energy fraction

$$\alpha_{T_n} = \frac{\Delta V(T_n)}{\rho_\gamma(T_n)} = \frac{\text{vacuum}}{\text{total density}}$$

$\alpha_{T_n} \geq 1$ strong

$\alpha_{T_n} \ll 1$ weak

inverse duration

$$\frac{\beta}{H} = T \frac{dS_3}{dT}\bigg|_{T_n}$$

bubble wall velocity v_w

runaway in vacuum (Euclidean -> Minkowski)

complicated at finite *T*

Combine sources

$$\Omega_{GW} = \Omega_\varphi + \Omega_{\rm sw} + \Omega_{\rm turb}$$

LISA working group summary reports 1512.06239, 1910.13125 and 2003.07360

$\Omega_\varphi \ldots$ bubble collisions

$$\Omega_\varphi\, h^2 = 5\times 10^{-6}\,\sqrt[3]{\frac{100}{g_*}}\left(\frac{v_w}{\beta}\right)^2\left(\frac{\kappa_\varphi \alpha_{T_n}}{1+\alpha_{T_n}}\right)^2\left(\frac{f}{f_\varphi}\right)^3\left(1+2\left(\frac{f}{f_\varphi}\right)^{2.07}\right)^{-2.18}$$

$f \ldots$ redshifted frequency today $\quad f = \frac{a_{T_n}}{a_0} f_{T_n} \simeq \frac{T_0}{T_n} f_{T_n} \simeq \frac{T_0 T_n}{M_{\rm Pl}}\frac{\beta}{H_{T_n}} \sim 10^{-5}\ {\rm Hz}\left(\frac{T_n}{100\ {\rm GeV}}\right)\left(\frac{\beta}{H_{T_n}}\right)$

$$f_\varphi = 1.2\times 10^{-5}\ {\rm Hz}\left(\frac{T_n}{100\ {\rm GeV}}\right)\left(\frac{g_*}{100}\right)^{1/6}\frac{\beta}{v_w}$$

characteristic collision frequency

$\kappa_\varphi \ldots$ energy transfer efficiency

see e.g. 2003.07360

Combine sources

$$\Omega_{GW} = \Omega_\varphi + \Omega_{\rm sw} + \Omega_{\rm turb}$$

LISA working group summary reports 1512.06239, 1910.13125 and 2003.07360

$\Omega_{\rm sw} \ldots$ sound waves

$$\Omega_{\rm sw}\, h^2 = 4 \times 10^{-6} \sqrt[3]{\frac{100}{g_*}}\, \frac{v_w}{\beta}\, \min\left(1, H_{T_n}\tau_{\rm sw}\right) \left(\frac{\kappa_{\rm sw}\alpha_{T_n}}{1+\alpha_{T_n}}\right)^2 \left(\frac{f}{f_{\rm sw}}\right)^3 \left(1 + \frac{3}{4}\left(\frac{f}{f_{\rm sw}}\right)^2\right)^{-7/2}$$

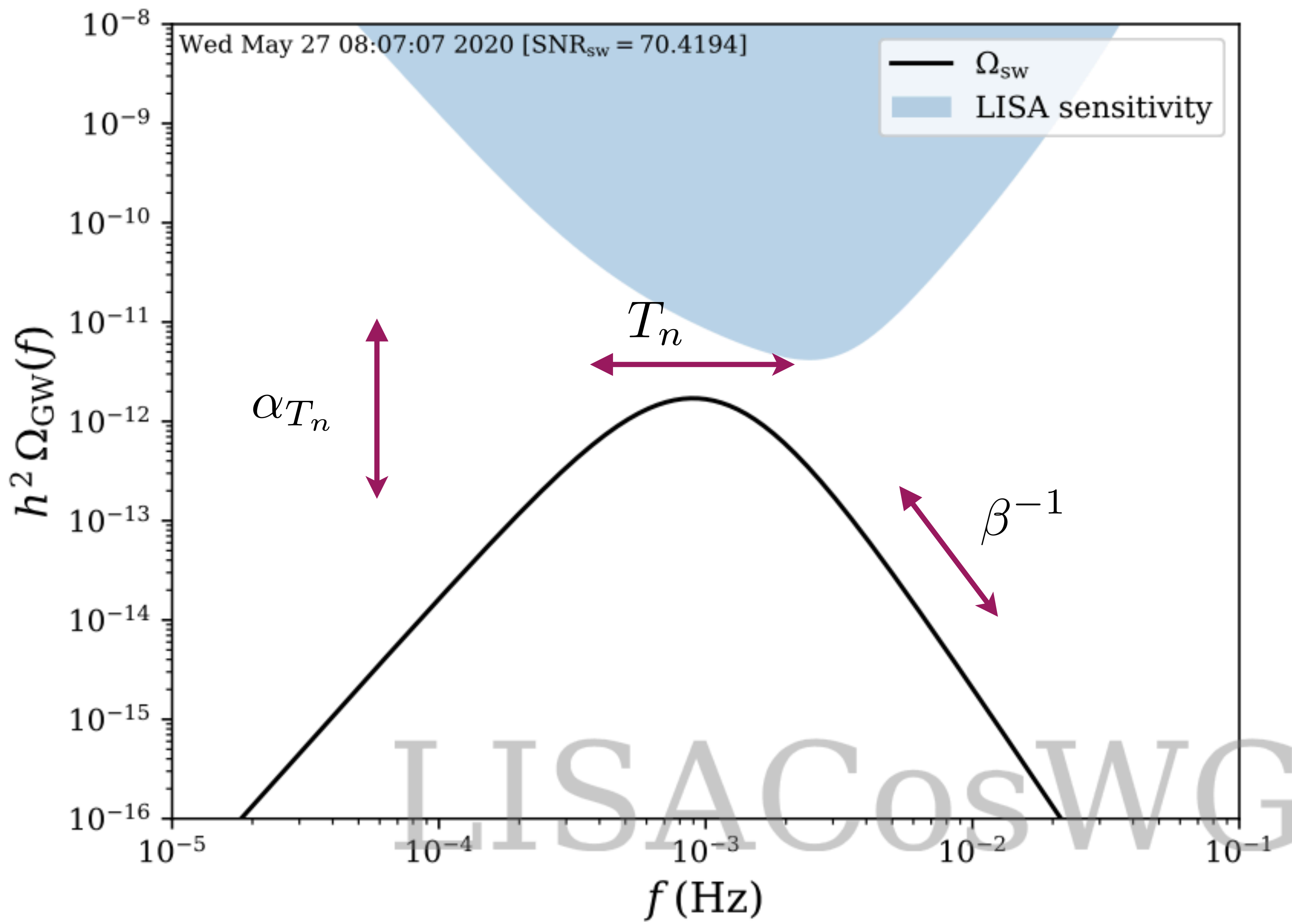
sound waves have a finite duration associated to $H_{T_n}\tau_{\rm sw} = \dfrac{v_w}{\beta} \dfrac{\sqrt[3]{8\pi}}{\sqrt{\frac{3}{4}\frac{\kappa_{\rm sw}\alpha_{\rm eff}}{1+\alpha_{\rm eff}}}}$ $\quad \alpha_{\rm eff} = \alpha_{T_n}\left(1-\kappa_\varphi\right)$

and a characteristic frequency $\quad f_{\rm sw} = 1.9 \times 10^{-5}\ {\rm Hz} \left(\dfrac{T_n}{100\ {\rm GeV}}\right) \left(\dfrac{g_*}{100}\right)^{1/6} \dfrac{\beta}{v_w}$

as well as the energy efficiency coefficient $\kappa_{\rm sw} = \left(1-\kappa_\varphi\right) \dfrac{\alpha_{\rm eff}}{0.73 + 0.083\sqrt{\alpha_{\rm eff}} + \alpha_{\rm eff}}$

See PTPlot

$$\Omega_{GW} = \Omega_\varphi + \Omega_{\rm sw} + \Omega_{\rm turb}$$



Combine sources

$$\Omega_{GW} = \Omega_{\varphi} + \Omega_{\rm sw} + \Omega_{\rm turb}$$

LISA working group summary reports 1512.06239, 1910.13125 and 2003.07360

$\Omega_{\rm turb} \ldots$ turbulence

$$\Omega_{\rm turb}\, h^2 = 3 \times 10^{-5} \sqrt[3]{\frac{100}{g_*}} \frac{v_w}{\beta} \min\left(1, H_{T_n} \tau_{\rm sw}\right) \left(\frac{\kappa_{\rm sw} \alpha_{T_n}}{1+\alpha_{T_n}}\right)^2 \times \left(\frac{f}{f_{\rm turb}}\right)^3 \left(1 + \left(\frac{f}{f_{\rm turb}}\right)\right)^{-11/3} \left(1 + \frac{8\pi f}{h_*}\right)^{-1}$$

$$h_* = 1.65 \times 10^{-5}\ {\rm Hz} \left(\frac{T_n}{100\ {\rm GeV}}\right) \left(\frac{g_*}{100\ {\rm GeV}}\right)^{1/6}$$

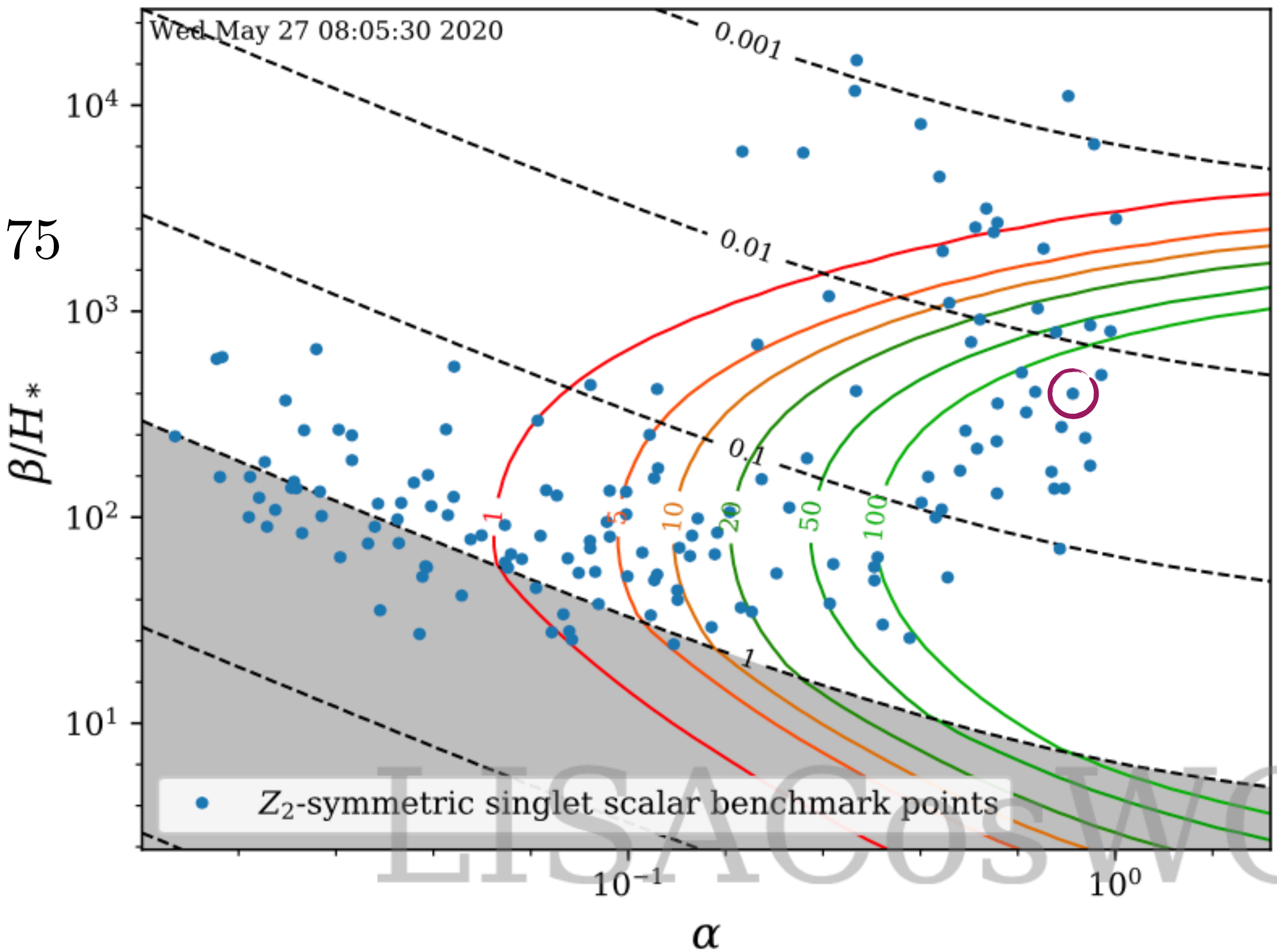
$$f_{\rm turb} = 3 \times 10^{-5}\ {\rm Hz} \left(\frac{T_n}{100\ {\rm GeV}}\right) \left(\frac{g_*}{100}\right)^{1/6} \frac{\beta}{v_w}$$

A simple example with a scalar singlet *S* $Z_2 : S \to -S$

$$\Delta V = \frac{a_2}{2}\,|H|\,S^2 + \frac{b_2}{2}S^2 + \frac{b_4}{4}S^4$$

$v_w = 1$

$g_* = 106.75$



input

$m_S = 90\ \text{GeV}$

$a_2 = 0.9$

GW parameters

$\alpha_{T_n} = 0.8$

$\frac{\beta}{H_{T_n}} = 398$

$T_n = 32\ \text{GeV}$

Recent update on SW sensitivities

see Kai Schmitz's 2005.10789

Outlook for the future

upgrades

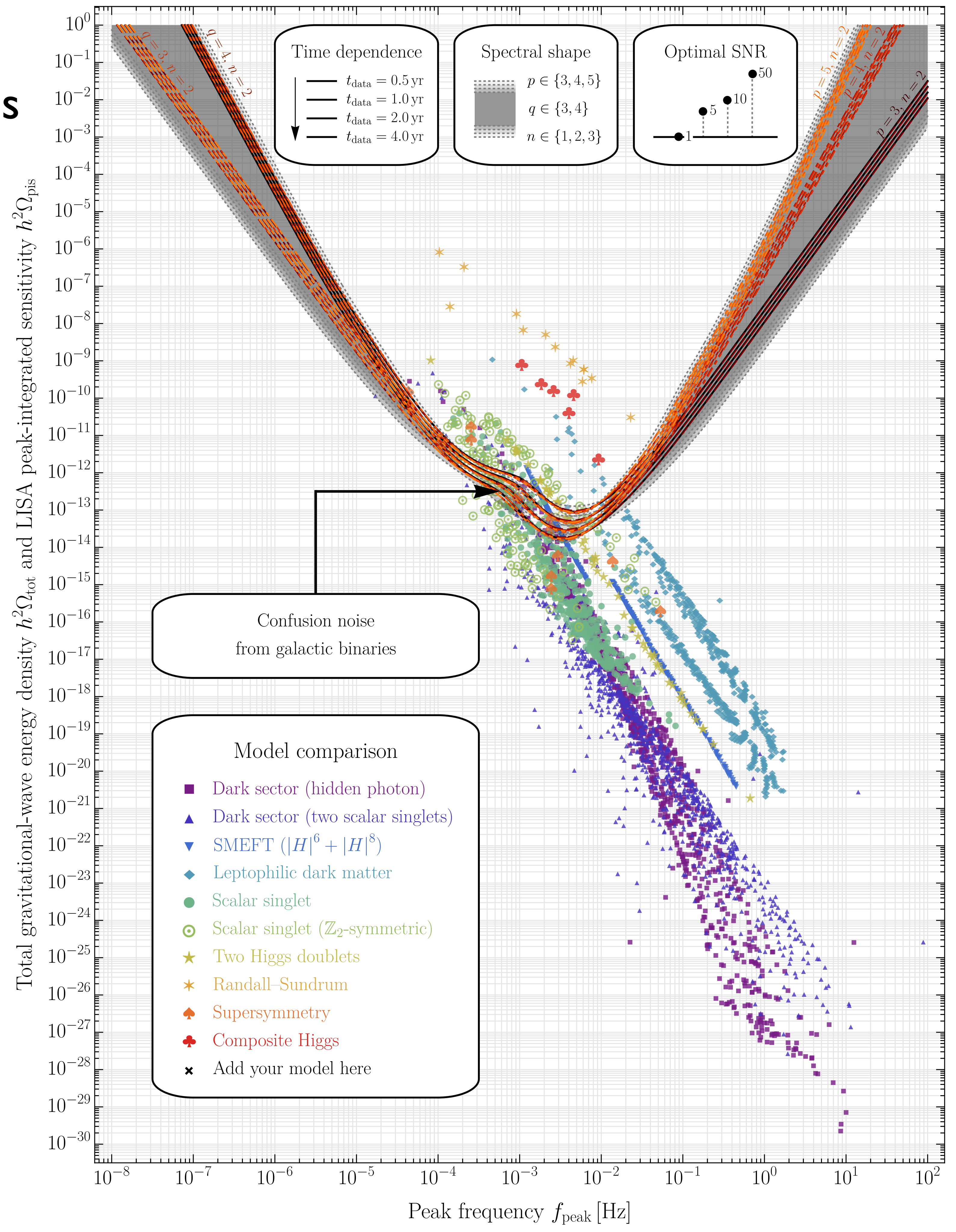
Adv LIGO, Adv VIRGO, GEO Hi-freq, IndIGO

coming on-line, underground

KAGRA

space-based, satellites

LISA, DECIGO, Taiji, Tian Qin



Thank you