2. SYMMETRIES OF SPACE-TIME

- Laws of nature are invariant under relativistic space-time transformations. (We do not worry about curved-space-times and gravity here)

$$x^\mu = (ct, \vec{x}) \quad c = 1$$

we measure both, space and time in units of $c = 3 \cdot 10^8 \frac{m}{s}$

- We will consider only those theories that are invariant under rotations, boosts and space-time translation (Poincaré group).

- Moreover, we will be interested in phenomena at very short distances, when quantum mechanics plays the dominant role.

$$[\, \delta x \, \delta p \sim \hbar \,]$$

This means we can measure distances and times in units of $\hbar c = 197 \text{MeV fm} \simeq 0.2 \text{ GeV fm}$.

N.R. classical	REL. classical
N.R. quantum	REL. quantum

2.1. Relativistic Particle Kinematics

- We will use $x^\mu = (t, \vec{x})$ "mostly minus" metric

$$p^\mu = (E, \vec{p}) \qquad g^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

$$p_\mu = g_{\mu\nu} p^\nu$$

- Then, the scalar products are given by

$$p^\mu p_\mu = p^\mu p^\nu g_{\mu\nu} = (E^2 - \vec{p}^{\,2}) = m^2 > 0$$

- Lorentz transformation contains rotations (12, 13, 23) and boosts (1, 2, 3)

$$(E, p, 0, 0) \xrightarrow{R_{12}} \begin{pmatrix} 1 & & & \\ & c & s & \emptyset \\ & -s & c & \\ & & & 1 \\ & & & & 1 \end{pmatrix} \begin{pmatrix} E \\ p \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E \\ cp \\ -sp \\ 0 \end{pmatrix}$$

($c = \cos\Theta_{12}$)

Boost
$$\begin{pmatrix} E \\ p \\ 0 \\ 0 \end{pmatrix} \longrightarrow \gamma \begin{pmatrix} \beta & 1 & & \\ 1 & \beta & & \\ & & 1 & \emptyset \\ & & & 1 \end{pmatrix} \begin{pmatrix} E \\ p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(E + \beta p) \\ \gamma(p + \beta E) \\ 0 \\ 0 \end{pmatrix}$$

$$\beta = \frac{v}{c} = v, \quad (c = 1) \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \cosh\eta \leftarrow \text{rapidity}$$

note
$$(\gamma\beta)^2 = \frac{\beta^2}{1-\beta^2} = \gamma^2 - 1 = \frac{1-1+\beta^2}{1-\beta^2}$$
$$\cosh^2\eta - 1 = \sinh^2\eta = (\gamma\beta)^2$$

$$R_{12}: \begin{pmatrix} \cos\Theta_{12} & \sin\Theta_{12} \\ -\sin\Theta_{12} & \cos\Theta_{12} \end{pmatrix} \quad \text{3 rotations}$$

$$B_1: \begin{pmatrix} \cosh\eta_1 & \sinh\eta_1 \\ \sinh\eta_1 & \cosh\eta_1 \end{pmatrix} \quad \text{3 boosts}$$

} 6 generators

- In the rest frame of the particle, we have

$$p^\mu = (E_0, \vec{p}) \longrightarrow (m, 0) \qquad (\vec{p} = 0)$$

- When we boost it, we get: $p'^\mu = (\gamma m, \gamma \beta m)$

$$= (E_p, \vec{p})^\mu$$

- Note that: $\frac{p}{E_p} = \frac{\gamma \beta m}{\gamma m} = \beta$, $\frac{E_p}{m} = \gamma$ ... boost factor

- This means we can easily determine the boost factor by measuring m & $\vec{p}$.

<u>EXAMPLE</u>: $M \rightarrow m_1 m_2$ $(\eta \rightarrow \pi^+\pi^-)$

i) $m_1 = m_2 = 0$ $\quad h \rightarrow \gamma\gamma \quad m_1 = m_2 = 0$ (simplest)

rest frame $\quad (M, 0) = \left(\frac{M}{2}, \frac{M}{2}\right) + \left(\frac{M}{2}, -\frac{M}{2}\right)$

ii) $m_1 = 0,\ m_2 \neq 0$

$$(M, 0) = (E_p, p) + (p, -p) \quad \text{(because it's } m_2 = 0\text{)}$$

$$E_p^2 - p^2 = m^2 \Rightarrow E_p = \sqrt{p^2 + m^2}$$

$$(M - p)^2 = E_p^2 = p^2 + m^2$$

$$M^2 - 2Mp + p^2 = p^2 + m^2 \Rightarrow p = \frac{M^2 - m^2}{2M}$$

$$E_p = M - p = \frac{M^2 + m^2}{2M}.$$

iii) $m_1 \neq 0,\ m_2 \neq 0,\ m_1 \neq m_2$

$$(M, 0) = (E_1, p) + (E_2, -p)$$

$$E_1^2 - p^2 = m_1^2 \quad , \quad E_2^2 - p^2 = m_2^2$$

$$M = \sqrt{p^2 + m_1^2} + \sqrt{p^2 + m_2^2}$$

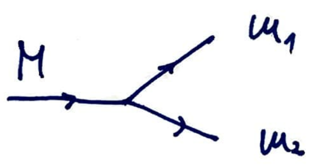
$$M^2 - 2M\sqrt{p^2 + m_1^2} + \cancel{p^2} + m_1^2 = \cancel{p^2} + m_2^2$$

$$\left(\frac{M^2 + m_1^2 - m_2^2}{2M}\right)^2 = p^2 + m_1^2$$

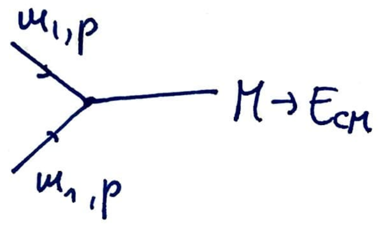
$$p^2 = \frac{M^4 + 2M^2(m_1^2 - m_2^2) + (m_1^2 - m_2^2)^2 - 4M^2 m_1^2}{(2M)^2}$$

$$p = \frac{\left(M^4 - 2M^2(m_1^2 + m_2^2) + (m_1^2 - m_2^2)^2\right)^{1/2}}{2M} = \frac{\lambda(M, m_1, m_2)}{2M}$$

(see also PDG - kinematics)



or :



2.2. NATURAL UNITS

- since $c = 1$, $\hbar = 1$ we say $m_e = 511\,keV \sim 0.5\,MeV$
- conversions to typical distances & times:

e.g. $m_\pi = 140\ MeV$, $\frac{\hbar c}{m_\pi c^2} = \frac{\overset{200}{\cancel{0,2}}\ \cancel{MeV}\ fm}{140\ \cancel{MeV}} = \frac{fm}{0{,}7\cancel{00}} \sim \cancel{1}\ fm$

$$\frac{\hbar c}{m_\pi c^2}\frac{1}{c} = 10^{-15}\frac{\cancel{m}\ s}{3\cdot 10^8\ \cancel{m}} \simeq 10^{-23}\ s$$

↳ typical range of strong interactions

- Remember the Compton wavelength

$$\lambda_c = \frac{2\pi\hbar c}{m_e c^2} = \frac{\overset{200}{\cancel{0,2}}\cdot 6\cdot eV\,nm}{0{,}5\ MeV} = 2.4\cdot 10^{-3}\ nm$$

- Another part of dimensionless units involves setting $\varepsilon_0 = \mu_0 = 1$, such that Maxwell's equations & the Coulomb potential both simplify: $\vec{\nabla}\cdot\vec{E} = \rho$, ...

$$V_{em} = \frac{e^2}{4\pi r} = \frac{\alpha\ \overset{1}{\overset{\shortparallel}{(\hbar c)}}}{r} \qquad \left[\frac{200\,eV\,nm}{r}\right]$$

$\alpha = \frac{1}{137}$ ⟸ units of eV ✓

QED is PERTURBATIVE $\left\{ e \sim \sqrt{4\pi\alpha} \simeq \frac{1}{\sqrt{10}} \sim 0.31 < 1 \right.$

2.3. GROUP THEORY REVIEW

It is quite remarkable that natural phenomena should obey mathematical laws defined by internal symmetries. But experimentally this is confirmed:

SO(1,3)	LORENTZ
SO(3)	ROTATIONS
SU(2)	SPIN, ISOSPIN
SU(3)	STRONG INTERACTIONS

Moreover, some interactions conserve certain space-time symmetries, like parity & charge conjugation.

Therefore, we have to be familiar with both, continuous and discrete symmetry groups.

Basics : $G = \{g_1, g_2, \dots, g_n\}$ ~ finite group

(g_1: a single group element)

i) Group operation e.g. $g_1 g_2 = g_x$, $g_x \in G$ INTERNAL OPERATION

ii) ASSOCIATION $g_1(g_2 g_3) = (g_1 g_2) g_3$

iii) IDENTITY $g_1 I = I g_1 = g_1$

iv) INVERSE $\forall g_i \; \exists g_i^{-1} : g_i g_i^{-1} = I$

vi) $g_1 g_2 = g_2 g_1$ — ABELIAN GROUP

Hilbert space

- $G \longleftrightarrow \hat{O}$ acting on a set of states $| \rangle$
  ↳ e.g. $\hat{H}$ or $\hat{L}$ or $\hat{p}, \hat{p}^2, \hat{x}, \ldots$

- $| \rangle \xrightarrow{g} | \rangle'$ such that the entire space and norms are preserved ⇒ we describe the group action by a unitary operator, one for each group element g_i : U_{g_i}

$$U_g | \rangle = | \rangle'$$

$$U_{g_1} U_{g_2} = U_{g_1 g_2}$$

this is a UNITARY representation

- If U_g commutes with $\hat{O}$, it is conserves the properties of the system, e.g. when $\hat{O} = \hat{H}$

$$[U, H] = 0$$

then $\hat{H} |\psi\rangle = E |\psi\rangle$

& $\hat{H} \hat{U}_g |\psi\rangle = U_g H |\psi\rangle = E U_g |\psi\rangle$

or $\hat{H} |\psi'\rangle = E |\psi'\rangle$ ... group operation conserves the eigenvalues

SIMPLE EXAMPLE: $\mathbb{Z}_2$ symmetry (DARK MATTER stabilization)

$$\mathbb{Z}_2 = \{1, -1\} \qquad 1^2 = 1, \; 1(-1) = -1, \; (-1)^2 = 1$$
$$1^{-1} = 1, \; (-1)^{-1} = -1$$

- This is how parity & charge conjugation work. We can represent it by a 2×2 matrix acting on a Hilbert space. Concretely $U_g = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = C$

$$| \; \rangle = \begin{pmatrix} |\pi^+\rangle \\ |\pi^-\rangle \end{pmatrix} \qquad C \begin{pmatrix} |\pi^+\rangle \\ |\pi^-\rangle \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} |\pi^+\rangle \\ |\pi^-\rangle \end{pmatrix} = \begin{pmatrix} |\pi^-\rangle \\ |\pi^+\rangle \end{pmatrix}$$

- Note $C^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $C(C|\pi^+\rangle) = C|\pi^-\rangle = |\pi^+\rangle$

With $\{C, C^2 = 1\}$ we get a 2×2 unitary representation of the $\mathbb{Z}_2$ group.

- For Abelian groups, the matrices commute, therefore we can diagonalize them simultaneously.

$$C' = R^T C R = \begin{pmatrix} c & -s \\ +s & c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} -s & c \\ c & s \end{pmatrix}\begin{pmatrix} c & s \\ -s & c \end{pmatrix} = \begin{pmatrix} -2sc & * \\ c^2 - s^2 & * \end{pmatrix}$$
$$c^2 - s^2 = 0$$

$\Rightarrow c = s \Rightarrow \theta = 45° \quad c = s = \frac{1}{\sqrt{2}}$

eigenvalues: $\mathrm{tr}\, C = 0$, $\det C = -1 \Rightarrow C' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

states:
$C = 1 : \frac{1}{\sqrt{2}}(|\pi^+\rangle + |\pi^-\rangle)$; $C|\pi_1\rangle = |\pi_1\rangle$
$C = -1 : \frac{1}{\sqrt{2}}(|\pi^+\rangle - |\pi^-\rangle)$ $\quad C|\pi_{-1}\rangle = -|\pi_{-1}\rangle$ (ok).

- For NON-ABELIAN groups, we cannot diagonalize all the group elements simultaneously & speak of numbers only, we will have non-trivial (non-diagonal) matrix representations. $G = \{U_g\}$

However, many representations can be partially diagonalized (or reduced) into blocks.

$$U_R = \begin{pmatrix} U_1 & & \phi & \\ & \boxed{U_2} & & \\ \phi & & \ddots & \\ & & & U_n \end{pmatrix}$$

U_R: reducible representation

U_2: irreducible sub-blocks ($d_i \times d_i$ matrix, d_i ... dimension)

- For a finite discrete group G with n elements

$$\sum_i d_i^2 = n$$

- Breaking up the Hilbert space (i.e. constructing appropriate states) using the symmetry groups helps us to solve physics problems defined by $\hat{H}$ (energy levels, degeneracies, eigenstates, transitions).

EXAMPLE : Reducible representation : Permutations

Π_3 ... permutation group with 3 elements [123]

$\Pi_3 = \{ [123], [213], [132], [231], [312], [321] \}$

I (= [123])

6 elements. Multiplication : $[123][213] = [123]$

$[123]|123\rangle = |123\rangle \qquad [213]|123\rangle = |213\rangle$

$[123]|213\rangle = |213\rangle$

$[231][231] = [312]$

$[132][312] = [213]$

$\Pi_3 = \{ [123], [231], [312], [132], [321], [213] \}$

$[123] = I$

$6 = 2^2 + 1 + 1$ (EXERCISE 2.3)

2.4. CONTINUOUS GROUPS $G = \{ g_1, g_2, \dots, \infty \}$

↳ parameters e.g. $\theta_{12}, \theta_{23}, \theta_{1\ldots}$ in $SO(3)$

a) Translations $x^\mu \to x^\mu + a^\mu$

$U(a^\mu) = e^{-i a^\mu P_\mu} \qquad \mu = 0,1,2,3 \Rightarrow 4$ generators

$\langle x | p \rangle = e^{ipx}$

displace the wave-function by a^μ

$\langle x | U(a) | p \rangle = \langle x | p \rangle e^{-iap} = e^{ip(x-a)}$

$\{U(a^0), U(a^1), \dots U(a^3)\}$ ... set of 4 unitary operators

$\{U(a^\mu)\}$ ... unitary representation of space translations

$$\hat{U}(a^\mu) = e^{-ia^\mu \hat{P}_\mu}$$

↙ Hermitian operator... generator of translations

↑ parameter of translations

$$\left.\frac{d\hat{U}}{da^\mu}\right|_{a^\mu=0} = -i\hat{P}_\mu \left. e^{-ia^\mu P_\mu}\right|_{a^\mu=0} = -i\hat{P}_\mu$$

$$\Rightarrow P_\mu = i\frac{dU}{da^\mu}$$

$$\underbrace{U^+ = U}_{\text{unitary}}, \quad U^+ = e^{iaP^+} = e^{iaP} \Rightarrow \underbrace{P = P^+}_{\text{Hermitian}}$$

$[P^\mu, P^\nu] = \delta^{\mu\nu}$ ... different space-time directions commute, this is an Abelian group with 4 continuous parameters a^μ, $\mu = 0, 1, 2, 3$.

- if $[U, H] = 0 \Rightarrow$ Energy is conserved, as we saw earlier. For each generator $\hat{P}^\mu$ we get an eigenvalue p^μ, which is a conserved quantity

This is the Noether's theorem: for each symmetry of a system, we get a conserved quantity.

<u>NON-ABELIAN</u> SO(3) ... three directions $\theta_{12}, \theta_{23}, \theta_{13}$
$\quad \| \quad \| \quad \|$
$\quad \alpha_3 \quad \alpha_1 \quad \alpha_2$

$$U(\vec{\alpha}) = e^{-i\vec{\alpha}\cdot\vec{J}}$$

↓ 3 parameters of rotations; ← three generators $\Rightarrow J^i = i \left.\frac{dU}{d\alpha^i}\right|_{\alpha^i=0}$

Hermitian: $J^{3+} = J^3$

$$U(\alpha^3) = \begin{pmatrix} c & -s & 0 \\ +s & c & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad J^3 = i \left.\begin{pmatrix} -s & -c & 0 \\ +c & -s & 0 \\ 0 & 0 & 0 \end{pmatrix}\right|_{\alpha^3=0} = i\begin{pmatrix} 0 & -1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

↑ unitary

- Opposite to P^μ which commute with each other, the J^i don't. : $[J^i, J^j] = i\varepsilon^{ijk}J^k$

  $U(\vec{\alpha})U(\vec{\beta}) = U(\vec{\gamma})$

  ~~group~~ structure constants of the algebra

- The J^i are Hermitian operators and if $[U(\alpha^i), H] = 0$ they correspond to 3 conserved charges, i.e. conservation of angular momentum in 3D.

- If we have a spherically symmetric problem, like $V(r)$ of a Hydrogen atom $V = \frac{\alpha}{r}$, we can use the group operators to construct, classify and move between the states : $J^2|j, j_z\rangle = j(j+1)|j, j_z\rangle$

  $J_z|j, j_z\rangle = j_z|j, j_z\rangle$

• GROUP GENERATORS : $\hat{J}^2$, $\hat{J}_z$, $\hat{J}_\pm = \hat{J}_1 \pm i\hat{J}_2$

• EXAMPLE : SO(3) representations $\mathcal{U}(\alpha) = e^{-i\alpha J}$

i) SCALAR SPIN-0 REP. = Identity $\Rightarrow$ $J^i = 0$
1D

$$U(\alpha) = e^{-i\alpha^i J^i} = e^0 = 1$$

trivially satisfies $[J^i, J^j] = i\,\varepsilon^{ijk} J^k$

ii) TWO DIMENSIONAL SPINOR REPRESENTATION

Pauli matrices : $J^i = \frac{\sigma^i}{2}$ satisfy: $\underbrace{\left[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}\right] = i\,\varepsilon^{ijk}\frac{\sigma^k}{2}}_{\text{VERIFY}}$

THESE ACT ON 2D STATES WITH A SPINOR INDEX.
WE ALSO SAY SO(3) ~ SU(2) ← σ's are fundamental generators of SU(2)

~~iii)~~

iii) THREE DIMENSIONAL ~~RE~~ - VECTOR REPRESENTATION

$$J^1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad J^2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}, \quad J^3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Again $[J^i, J^j] = i\,\varepsilon^{ijk} J^k$

OR: $(J^{ij})_{kl} = -i\left(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k\right)$, $J^1 = J^{23}_{kl} = i\left(\delta^2_{k=2}\delta^3_{l=3} - \delta^{23}_{l=2,3}\right)$

iv) HIGHER SPINS j : $J^i_{(2j+1)\times(2j+1)}$ MATRICES

- Again, when $[J^i, H] = 0$, generators commute with H and the equations of motion obey the symmetry set by $\mathcal{U} = e^{-i\alpha J}$

Short Review

$SO(n)$ ... special orthonormal groups

↳ set of $n \times n$ orthonormal matrices w. $\det O = 1$

we have $\frac{n(n-1)}{2}$ generators (anti-symmetric $n \times n$) & $\left[\frac{n}{2}\right]$ diagonal (Casimir) operators that define all the quantum numbers.

$SO(2)$	$\frac{2 \cdot 1}{2} = 1$ generator, 1 diagonal  $\sim U(1)$ therefore Abelian
$SO(3)$	$\frac{3 \cdot 2}{2} = 3$ generators, $\left[\frac{3}{2}\right] = 1$ diagonal
$SO(4)$	$\frac{4 \cdot 3}{2} = 6$ generators, think of the Lorentz group $\sim SO(1,3)$ w. 3 rotations & 3 boosts

- Another important group acts on complex vector spaces & is defined by unitary $n \times n$ matrices called $U(n)$ ... elements $U = e^{-i\alpha^a t^a}$

$$U^+ \cdot U = 1 \Rightarrow e^{-i\alpha^a t^a} \cdot e^{+i\alpha t^{a+}} = 1$$

$t^0 = 1$ can be removed $\Rightarrow t^{a+} = t^a$

$\Downarrow$

$SU(n)$ — Hermitian generators

size is n^2-1, of which $n-1$ are diagonal

$U(1)$ ... Abelian phase rotation

$SU(2)$ ... spin matrices : $U = e^{-i\alpha\sigma/2}$

$2^2-1 = 3$ generators σ^i

$n-1 = 1$ diagonal σ^3

$SU(3)$ ... $3^2-1 = 8$ generators λ^i (Gell-Mann matrices)

$3-1 = 2$ diagonals λ^3, λ^8

these classify all the states

IN GENERAL, t^a OBEY $[t^a, t^b] = i f^{abc} t^c$

Lie algebra — structure constants

Lie ALGEBRA $[t^a, t^b] = i f^{abc} t^c$

f^{abc} ... structure constants, can be chosen anti-symmetric, e.g. SU(2) ~ SO(3)

$$\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\,\varepsilon^{abc}\,\frac{\sigma^c}{2} \qquad f^{abc} = \varepsilon^{abc}$$

ε^{abc}: totally anti-symmetric

---

5. DISCRETE SPACE-TIME SYMMETRIES

SPACE-TIME : LORENTZ $(J^{\mu\nu})_{\alpha\beta} = i(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha)$ ← 6 generators

continuous symmetries: TRANSLATIONS P^μ ← 4 generators

There are two extra discrete symmetries

- Parity P & Time reversal T

Clearly $P^2 = 1$ & $T^2 = 1$ $\Rightarrow$ $P = \pm 1$, $T = \pm 1$

How does P act on space-time

$$x^\mu = (t, \vec{x}) \xrightarrow{P} (t, -\vec{x})$$

$$x^\mu = (t, \vec{x}) \xrightarrow{T} (-t, \vec{x})$$

$$x^{\mu\prime} = \Lambda^\mu{}_\nu x^\nu$$

Note that rotations & boosts have $\det \Lambda = +1$

while $\det P = \det T = -1$. Therefore, the two cannot be connected; we don't get P or T from Λ by some continuous deformation.

Particles, (which are represented by states, do have <u>intrinsic</u> parity (just like they have intrinsic charge & spin). Thus, their states can have $\pm$ eigenvalues of P:

$$P\,|A(\vec{p})\rangle = \pm\,|A(-\vec{p})\rangle$$

$$T\,|A(E)\rangle = \pm\,|A(E)\rangle$$

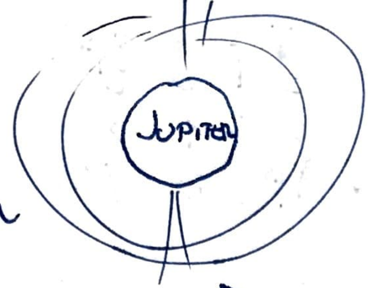
Finally, (charged) states can have intrinsic charge, C, which is conserved by E.M. and strong interactions:

$$C\,|A(E,\vec{p})\rangle = \pm\,|A(E,\vec{p})\rangle$$

- C, P, T may be violated (and they are) but CPT ought to be conserved on very general grounds, foundational to QFT.

2.2 $m_\gamma \geq 0$

a) $r_j = 7 \cdot 10^4 \, m$



dipole up to several radii of Jupiter

$$\lambda_c = \frac{hc}{m_\gamma c^2} = 10 \, r_j$$

$$\Rightarrow m_\gamma c^2 < \frac{hc}{10 r_j} = \frac{1240 \, eV \, nm}{10^5 \, m} = 10^3 \cdot 10^{-9-5} \, eV$$

$$< 10^{-11} \, eV$$

b) $M_W = 80 \, GeV$ $\quad \lambda_W = \frac{hc}{M_W} = \frac{1240 \, MeV \, fm}{80 \, GeV} = \frac{1{,}2 \, GeV}{80 \, GeV} fm$

$$\simeq \underline{0{,}1 \, fm}$$

c) size of the e^- from e^-e^- & e^+e^- scattering, which is valid up to 200 GeV. $\Rightarrow$ e^- is point-like up to $\lambda_e \sim \frac{hc}{E} = \frac{1{,}24 \, GeV \, fm}{200 \, GeV} = 6{,}2 \cdot 10^{-18} \, m$

$$= 6{,}2 \cdot 10^{-16} \, cm$$

2.4. a) $\Lambda_3(\beta) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$

$$\Lambda_1(\beta) = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

b) $\Lambda_c = \Lambda_1 \Lambda_3 = \begin{pmatrix} \gamma^2 & \gamma\beta & 0 & \gamma^2\beta \\ \gamma^2\beta & \gamma & 0 & \gamma^2\beta^2 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}$

$$\Lambda^0_{c\ 0} = \gamma_c = \frac{1}{\sqrt{1-\beta_c^2}} = \frac{1}{1-\beta^2} \Rightarrow 1-\beta_c^2 = (1-\beta^2)^2$$

$$-\beta_c^2 = -2\beta^2 + \beta^4$$

$$\beta_c^2 = \beta^2(2-\beta^2)$$

c) $\Lambda_c \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma^2 \\ \gamma^2\beta \\ 0 \\ \gamma\beta \end{pmatrix}$ $\quad \vec{\beta} = \left(\beta, 0, \frac{\beta}{\gamma}\right)$

$$|\beta|^2 = \beta^2 + \frac{\beta^2}{\gamma^2} = \frac{\gamma^2\beta^2 + \beta^2}{\gamma^2}$$

$$= \beta^2 + \beta^2(1-\beta^2)$$

$$= 2\beta^2 - \beta^4$$

d) Expand Λ_c for small β

$\gamma \sim 1 + \frac{1}{2}\beta^2, \quad \gamma^2 \sim 1+\beta^2$

$$\Lambda_c \simeq \begin{pmatrix} 1+\beta^2 & \beta & 0 & \beta \\ \beta & 1+\frac{1}{2}\beta^2 & 0 & \beta^2 \\ 0 & 0 & 1 & 0 \\ \beta & 0 & 0 & 1+\frac{1}{2}\beta^2 \end{pmatrix}$$

e) $\beta_c^i = (\beta, 0, \beta/\gamma)$ $\beta_c^1 = \beta,\ \beta_c^2 = 0,\ \beta_c^3 = \frac{\beta}{\gamma}$

$\beta_c^2 = \beta^2 + \frac{\beta^2}{\gamma^2}$

$$\Lambda_{boost} = \left(\begin{array}{c|c} & \\ \hline & 1 + (\gamma_c - 1)\frac{\beta^2}{\beta_c^2}(1 + 1/\gamma^2) \ \ldots \\ & \end{array}\right)$$