3. Relativistic Wave Equations

- The Schrödinger equation $i\hbar\dot{\psi} = H\psi = (\nabla^2 + V)\psi$

  is <u>not</u> relativistically invariant.

  x & t have to appear as x^μ, p^μ & ∂^μ etc.

- Moreover, as $E > m_x + m_{\bar{x}}$, particle-anti-particle pairs are created & number of particles is <u>not</u> conserved, one has to deal with multi-particle states.

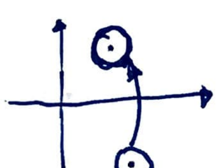
  e.g. $\gamma \to e^+ e^-$

3.1. The Klein-Gordon Equation

$\varphi(x^\mu)$ ... solution, then $\Lambda\varphi$ ... also a solution

$$\varphi(x^\mu) \longrightarrow \varphi'(x^\mu) = \varphi(\Lambda^{-1\,\mu\nu} x_\nu)$$

↑ canonical (usual) choice



a^μ ... max of $\varphi(x^\mu)$ then $\Lambda a^\mu = a^{\mu\prime}$

$$\boxed{\varphi'(x') = \varphi(x)} \quad \text{or} \quad \varphi'(x) = \varphi(\Lambda^{-1}x)$$

$$= \varphi(\Lambda^{-1}\underbrace{\Lambda x}_{x'}) = \varphi(x)$$

- If φ solves the eqn, then φ' should, too.

• Simplest scalar field equation that does this is

$$\underbrace{\partial^\mu \partial_\mu}\varphi + m^2\varphi = 0 \qquad \text{KLEIN-GORDON EQ.}$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2\right)\varphi + m^2\varphi = 0 \Rightarrow \boxed{(\partial^2 + m^2)\varphi = 0}$$

∂^μ transforms as x^μ $\quad \partial^\mu = \frac{\partial}{\partial x_\mu}$

& $\partial^\mu \partial_\mu$ is Lorentz invariant

• What are the solutions to $y'' = -ay$, $y = e^{iax}$

$$\varphi(x^\mu) = e^{\pm i p^\mu x_\mu} \qquad p^\mu = (E, \vec{p})$$

$$= e^{\pm i(Et - \vec{p}\vec{x})} \quad \& \quad p^\mu p_\mu = m^2, = \sqrt{E^2 - \vec{p}^2}$$

Now: $\varphi'(x^\mu) = \varphi(\Lambda^{-1}x^\mu) = e^{ip\Lambda^{-1}x} = e^{i(\Lambda p)x}$

because: $x \cdot p = \Lambda x \, \Lambda p$ is L.I. $\quad /\Lambda^{-1}$

$$\Lambda^{-1}xp = x\Lambda p$$

$$(\Lambda p)(\Lambda p) = p^2$$

• Note that the on-shell condition has two solutions: $E^2 - \vec{p}^2 = m^2$, $E = \pm\sqrt{\vec{p}^2 + m^2}$.

• The QFT interprets this as particles & anti-particles.

• A more transparent way to derive the e.o.m. is by starting from a L.I. scalar function = action and require it to be extremized, similar to Lagrangian in classical mechanics: $S[x, \dot{x}, t] = \int dt\, L(x, \dot{x})$

• Instead of t only, we need to integrate over the entire 4-space

$$S(\varphi(x^\mu)) = \int d^4x\, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

Lorentz-invariant (LI) integral

L.I. Action, scalar

Consider a single real scalar field then

$$\mathcal{L} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$ is a good

candidate. One can add higher order terms, like $\lambda \varphi^4$ to get interactions.

• Extremizing S gives $\delta S = \frac{\delta \mathcal{L}}{\delta \varphi} \delta \varphi + \frac{\delta \mathcal{L}}{\delta \partial_\mu \varphi} \delta \partial_\mu \varphi$

$$= \delta\varphi \left(\frac{\delta \mathcal{L}}{\delta \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \right) + \underbrace{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} \delta \varphi \right)}_{=0}$$

E.O.M: $$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \varphi} - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

$$\Rightarrow \quad \partial_\mu \partial^\mu \varphi + m^2 \varphi = 0 \qquad \text{OK}$$

- Because we extremized the action $S[\varphi]$ by finding the solution, and because S is LI, it follows that $S[\varphi'] = S[\varphi]$ and φ' also extremizes S and thereby solves the equations of motion.

3.2 FIELDS AND PARTICLES

- We have to associate the field operator $\varphi(x^\mu)$ to the particles that we observe (their charges, spin, momentum, energy, mass and other properties).

- In QFT this is done by introducing a set of states (multiparticle) called the Fock space.

e.g. $|0\rangle$ ... vacuum, empty state, no particles

$|\varphi(p)\rangle$ ... state that represents a single particle with momentum p

$\langle$ BRA | OPERATOR | KET $\rangle$ ... function, solution

$$\langle 0 | \; \varphi(x^\mu) \; | \varphi(p) \rangle = e^{-ipx}$$

final state, creates a vacuum — field operator at x^μ — initial state to be annihilated

BRA = FINAL STATE — FIELD OPERATOR — KET = INITIAL STATE

$$\underbrace{\langle 0 |}_{\text{vacuum state, empty - no particles}} \; \underbrace{\varphi(x^\mu)}_{\text{real scalar field operator acting at } x^\mu} \; \underbrace{| \varphi(p) \rangle}_{\text{real scalar particle state with momentum } p} = \underbrace{e^{-ipx}}_{\text{wave function of the annihilated scalar particle w. momentum } p \text{ \& } E_p^2 - p^2 = m^2}$$

vacuum state, empty - no particles

real scalar particle state with momentum p

wave function of the annihilated scalar particle w. momentum p & $E_p^2 - p^2 = m^2$

real scalar field operator acting at x^μ

this operator annihilates $|\varphi(p)\rangle$ into the vacuum

- taking the charge conjugated equation (this is a real scalar field) gives

$$\langle \varphi(p) | \quad \varphi^+(x^\mu) \quad | 0 \rangle = e^{+ipx}$$

* Here, the final state is the bosonic particle with momentum p. It was created from the vacuum by the creation operator $\varphi^+(x^\mu)$ at the position x^μ from the initial vacuum state $|0\rangle$. The RHS is ψ^* of the particle when it was created, it has $E = E_p = \sqrt{\vec{p}^2 + m^2}$ & mass m^2.

* If we are dealing with a complex scalar field
$$\mathcal{L} = \partial_\mu \varphi (\partial^\mu \varphi)^+ - m^2 \varphi^+ \varphi, \quad \varphi = \tfrac{1}{\sqrt{2}}(\varphi_1 \pm i \varphi_2)$$

Then we have $\varphi^{+} \neq \varphi$

anti-particle state

$$\langle 0 | \varphi^{+} | \varphi'(p) \rangle = e^{-ipx}$$

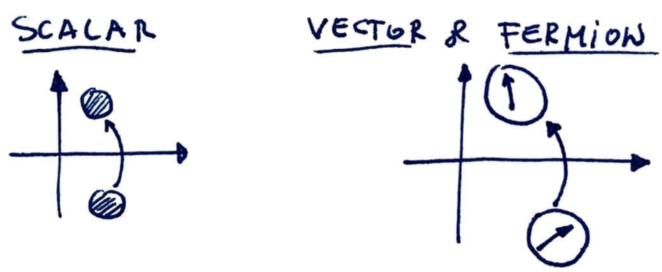
$$\langle 0 | \varphi | \varphi(p) \rangle = e^{-ipx}$$

3.3. MAXWELL'S EQUATIONS

- Scalar fields have masses & carry charges. How do we describe gauge bosons with spin 1 & fermions with spin $1/2$?

$\boxed{\text{SPIN } 1}$: Photons, Weak bosons W, Z, mesons, like ρ^0, ..

- This is the <u>vector</u> representation of the rotation group



- Vectors are v^i, r^i, $\Rightarrow$ x^μ, p^μ & transform as

$$r^{i} = R^{ij} r_j \quad \text{or} \quad x^{\mu'} = \Lambda^{\mu\nu} x_\nu$$

so the field goes as $A'^{\mu}(x'^{\mu}) = \Lambda A(x)$

or : $A^{\mu'}(x^{\mu}) = \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x)$

in short : $A' = \Lambda A(\Lambda^{-1}x)$

- So we <u>rotate</u> the field & transform the argument

$$A'(x) = \Lambda A(\Lambda^{-1}x)$$

(in general we have one Λ for each index, i.e. 0 for the scalar field)

- As for the scalar field, we have the states that characterize the mass, momentum & now also spin → polarization ε.

3 polarizations

$$3D: \quad \underbrace{\langle 0}_{\text{vacuum}} | \underbrace{V^i(x)}_{\substack{\text{gauge boson, or}\\ \text{vector operator}\\ \text{for each polarization}}} | \underbrace{v(p,\varepsilon)\rangle}_{\substack{\text{vector}\\ \text{particle that}\\ \text{gets annihilated}}} = \varepsilon^i e^{-ipx}$$

solves the K.G. eqn

- of course, the ε^i is a vector that also rotates $\varepsilon'^i = R^{ij}\varepsilon^j$
- higher spins have more degrees of freedom related to spin s, i.e. $2s+1$ .. degeneracy factor = # of degrees of freedom so R generalizes $R^{(s)\,ij}$

- generalization to Minkowski space-time in 4D has to be done carefully because $\mu = 0,1,2,3$ .. 4D & $s=1$, thus $2s+1 = 3 < 4$, which means that we have too many d.o.f. s in ε^μ than needed.

- Let us generalize to 4D by the ansatz

$$\langle 0 \mid V^\mu(x) \mid v(p,\varepsilon)\rangle = \varepsilon^\mu e^{-ipx}$$

- Now ε^μ is a 4-vector that rotates under Lorentz transformations, as usual $\varepsilon'^\mu = \Lambda^\mu{}_\nu \, \varepsilon^\nu$

- For momenta we have $(E,\vec{p}) = p^\mu$ & we normalize to $p^\mu p_\mu = E^2 - \vec{p}^2 = m^2 > 0$

- For polarization vectors we have

$\varepsilon^{(0)\mu} = (1,0,0,0)$ time-like pol. mode

$\varepsilon^{(\lambda)\mu} = (0, \vec{\varepsilon}^{(\lambda)}(p))$ $\lambda = 1,2$ ... transverse modes

$\lambda = 3$ ... longitudinal mode

- we can align $p^\mu = (E,0,0,p)$ with $\lambda = 3$

$$\Rightarrow p^\mu \varepsilon^{(1,2)}_\mu = 0 \Rightarrow \text{transverse modes}$$

- Normalization for ε_μ

$$\lambda = 0 \qquad \varepsilon^{(0)}_\mu \varepsilon^{\mu(0)} = \varepsilon^{(0)} \cdot \varepsilon^{(0)} = -1$$

$$\lambda = 1,2,3 \qquad \varepsilon^{(\lambda)}_\mu \varepsilon^{\mu(\lambda)} = 1$$

- Sum over polarizations

$$\sum_\lambda \varepsilon^{(\lambda)}_\mu \varepsilon^{(\lambda)}_\nu = -g_{\mu\nu}$$

for a $m=0$ gauge boson, massless photon

- The timelike $\varepsilon^{(0)}_\mu = (1,0,0,0)$ is unphysical in a sense that it does not affect the EM interactions

- 3D Maxwell's equations give $\vec{\varepsilon}$ in the solution

$$\vec{A}(x) = \underbrace{\vec{\varepsilon}\, e^{-iEt + ip\vec{x}}}$$

this is the wave-function of $\vec{A}$, the photon

$$p^\mu = (E,0,0,E) \Rightarrow E = p$$

- we also know that $\vec{E} \perp \vec{B} + \vec{p} \Rightarrow \underline{\underline{\vec{\varepsilon}\cdot\vec{p} = 0}}$

- As we saw, the $p\cdot x$ remains unchanged under LT, $\rightarrow$ $\Lambda p\, \Lambda p = xp$

$$x\, \Lambda p = \Lambda^{-1} x p$$

but e.g. $\varepsilon = (0, \varepsilon^1, \varepsilon^2, 0) \rightarrow (\#, \#, \varepsilon^2, 0)$ under a boost in the x-direction. This timelike component has no impact on the interactions, as long as the em current $\bar{\psi}\gamma^\mu\psi = j^{\mu\, em}$ is conserved: $\partial_\mu j^\mu = 0$

- For the massive g.b. fields one has additional fields (Goldstone bosons) that enter in the longitudinal modes.

• As for the scalars, we can cast the Maxwell's eom in a variational form by defining

$$A^\mu = (\phi, \vec{A}), \qquad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

A^μ: 4D E.M. vector potential

$F^{\mu\nu}$: relativistic anti-symmetric E.M. tensor, field strengt tensor

$$x^\mu = (\underset{0}{ct}, \vec{x}) \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\underset{0}{\frac{d}{c\,dt}}, -\nabla\right)^\mu$$

$$\Rightarrow \quad F^{i0} = (-\nabla^i \phi - \partial_0 A^i) = E^i \quad \text{electric field}$$

$$F^{ij} = (-\nabla^i A^j + \nabla^j A^i) = -\varepsilon^{ijk} B^k \quad \text{magnetic field}$$

• By contracting the Lorentz indices, we have

$$\mathcal{L} = \underbrace{-\frac{1}{4}}_{\text{canonical normalization}} \underbrace{F^{\mu\nu} F_{\mu\nu}}_{\text{Lorentz scalar}} - j^\mu A_\mu$$

j^μ: any external current, e.g. $\bar{\psi}\gamma^\mu\psi$, that satisfies $\partial_\mu j^\mu = 0$.

$j^\mu = (\rho, \vec{j})$ — ρ: charge distribution, $\vec{j}$: current density

$$\partial_0 \rho - \vec{\nabla}\vec{j} = 0$$

• By requiring $\delta\mathcal{L} = 0$ we extremize the action and get the coupled e.o.m. (i.e. Maxwell's eqs.)

$$\mathcal{L} = -\frac{1}{4}(\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu) - A_\mu j^\nu \quad / \delta \text{ on } A$$

$$= -\frac{1}{2}(\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) F^{\mu\nu} - \delta A_\mu j^\nu = 0$$

$$= -\frac{1}{2} \delta A_\nu \{ \partial_\mu F^{\mu\nu} + \frac{1}{2}\partial_\nu \delta A_\mu F^{\mu\nu} - \delta A_\nu j^\mu$$

$$= +\frac{1}{2} \delta A_\nu (\partial_\mu F^{\mu\nu} + \partial_\mu F^{\mu\nu}) - \delta A_\nu j^\nu$$

$$= \delta A_\nu (\partial_\mu F^{\mu\nu} - j^\nu) = 0$$

$\Rightarrow$ $\boxed{\partial_\mu F^{\mu\nu} = j^\nu}$; if we insert the $\begin{pmatrix} 0 & E_x & E_y & \dots \\ & & & \end{pmatrix}$

we get: $\vec{\nabla} \cdot \vec{E} = \rho$, $\varepsilon^{ijk} \partial_j B^k - \frac{d}{dt} E^i = j^i$

3.4 THE DIRAC EQUATION

- spin $\frac{1}{2}$ electron needs $2s+1 = 2$ states, right?
- Which Lorentz representation behaves like that and how do we write down the e.o.m. / action and how does it couple to gauge bosons A^μ & scalars φ?
- Dirac introduced a set of 4 matrices γ^μ that, when combined with the <u>spinorial</u> representation ψ, transforms as a Lorentz vector,

$$\bar{\psi} \gamma^\mu \psi \rightarrow \Lambda^\mu{}_\nu \bar{\psi} \gamma^\nu \psi .$$

- These matrices obey the Clifford algebra, which is the anti-commuting Minkowskian implementation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

- There are infinitely many representations (actual numbers) of these matrices, connected by LT. These two are ~~..~~ very commonly used:

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$ NON-RELATIVISTIC

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \begin{matrix} \sigma^\mu = (1, \sigma^i) \\ \bar{\sigma}^\mu = (1, -\sigma^i) \end{matrix}$$ RELATIVISTIC CHIRAL REP.

- of course $\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ are the usual Pauli matrices.

- We can try to contract γ^μ with ∂_μ to build a LI object and solve its e.o.m.

DIRAC EQUATION

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

four-component Dirac spinor

- As the scalars & gauge bosons, these objects obey the K.G. equation such that $\psi \propto \underbrace{\xi^s}_{\sim \text{fermionic polarization}} \cdot e^{-ipx}$

- We will see that ψ : - i) obeys the K-G eq.
  - ii) gives a conserved current $\bar{\psi}\gamma^\mu\psi$
  - iii) has $E_p > 0$ & $E_p < 0$ solutions when $\vec{p}=0$ (at rest) : particles & anti-particles

i) To see that ψ obeys the KG eq. let's take the "absolute value" of the Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \quad / \, (i\gamma^\mu\partial_\mu + m)$$

$$= \left(i^2\gamma^\mu\partial_\mu\gamma^\nu\partial_\nu \cancel{\pm} m^2\right)\psi$$

$$= \Big(-\tfrac{1}{2}\underbrace{(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)}_{2g^{\mu\nu}}\partial_\mu\partial_\nu - m^2\Big)\psi = \underbrace{-(\partial_\mu\partial^\mu + m^2)\psi = 0}_{\text{KLEIN-GORDON EQ.}}$$

QED.

ii) To analyze the solution at rest, let's go to the rest frame with $\vec{p}=0$, where $e^{-ipx} \to e^{-iEt}$

and, as we said, the NON-REL. representation is useful $\psi = \begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} e^{-iEt}$, $\gamma^0 E - \gamma^i \overset{=0}{p_i} = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} E$

$$\Rightarrow (\gamma^0 E - m)\psi = \left(\begin{array}{cc|cc} E-m & & 0 & \\ & E-m & & \\ \hline & & -E-m & \\ 0 & & & -E-m \end{array}\right)\begin{pmatrix} a\\ b\\ c\\ d\end{pmatrix} e^{-iEt} = 0$$

a) $E = m$ $\quad \psi_+ = \begin{pmatrix} \overbrace{\begin{pmatrix} a\\ b\end{pmatrix}}^{=\xi} \\ 0\end{pmatrix} e^{-imt}$, $\quad \psi_- = \begin{pmatrix} 0 \\ \underbrace{\begin{pmatrix} c\\ d\end{pmatrix}}_{\shortparallel}\end{pmatrix} e^{+imt}$

ξ ... TWO-COMPONENT SPINOR

• Clearly, we have two disjoint solutions

$$\psi_+ = \begin{pmatrix} \xi \\ 0 \end{pmatrix} e^{-imt} \qquad , \qquad \psi_- = \begin{pmatrix} 0 \\ \eta \end{pmatrix} e^{+imt}$$

PARTICLES (Electrons) ANTI-PARTICLES (positrons)

for a massive Dirac fermion at rest.

• We can perform a Lorentz boost from the rest frame and find a general solution

$$(i\gamma^\mu \partial_\mu - m)\psi = \begin{pmatrix} E-m & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -E\mp m \end{pmatrix} \begin{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \\ \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} e^{-iEt+\vec{p}\vec{x}} = 0.$$

• The positive energy solution is where $s=1,2$ is the spin index

$$\psi_+ = u^{(s)}(p)\, e^{-ipx}$$

$$\psi_- = v^{(s)}(p)\, e^{+ipx}$$

$\xi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ ... spin up e^- $\quad \xi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ ... spin down, etc.

$u^{(s)}(p)$ ... particle spin orientation

– $v^s(p)$ ... anti-particle ———

• EXAMPLE

$$\psi_+ = \begin{pmatrix} \xi \\ 0 \end{pmatrix} e^{-imt} \xrightarrow{\text{BOOST}} \underbrace{\begin{pmatrix} \xi \\ -\frac{\sigma\cdot p}{2m}\xi \end{pmatrix}}_{+\,\mathcal{O}(p^2)} e^{-iE_p + i\vec{p}x}, \qquad E_p = \sqrt{p^2+m^2}$$

• Now we can introduce the proper Fock states.

- FERMIONIC STATES & OPERATORS

$$\underbrace{\langle 0}_{\text{vacuum}} | \psi(x) | \underbrace{e^-(p,s)\rangle}_{\substack{\text{electron}\\ \text{state with}\\ \text{momentum}\\ p \,\&\, \text{spin } s}} = \overbrace{\underbrace{u^{(s)}(p)}_{\substack{\text{spinorial}\\ \text{part} \sim\\ \text{polarization}}} \underbrace{e^{-ipx}}_{\substack{\text{the usual}\\ \text{plane-wave}\\ \text{part}}}}^{\text{the wave function of the annihilated } e^-(s,p)}$$

$\psi(x)$: annihilates an e^- at x^μ

$p^\mu = (E_r, \vec{p})$

The conjugate part represents the creation of the e^-

$$\langle e^-(p,s) | \psi^+(x) | 0 \rangle = u^{(s)+}(p)\, e^{+ipx}$$

- Similarly, we have creation & annihilation operators for the anti-particle, by convention:

$$\langle 0 | \psi^+(x) | e^+(p,s)\rangle = v^{(s)+}(p)\, e^{-ipx}$$

$$\langle e^+(p,s) | \psi(x) | 0 \rangle = v^{(s)}(p)\, e^{ipx}$$

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iii) CONSERVED GAUGE CURRENT j^μ from FERMIONS

- $(\gamma^\mu)^+ = ?$ $\quad (\gamma^0)^+ = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T = \gamma^0$

$$(\gamma^i)^+ = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}^+ = \begin{pmatrix} 0 & -\sigma^{i*} \\ \underset{\sigma^i}{\overset{}{\sigma^{i+}}} & 0 \end{pmatrix} = -\gamma^i$$

anti-Hermitian

- note that $\gamma^0\gamma^i = -\gamma^i\gamma^0$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix} = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix} \quad - \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = - \begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$$

- The $\gamma^0\gamma^i = -\gamma^i\gamma^0$ identity means that we can form $(\gamma^0\gamma^\mu)$ which is <u>Hermitian</u>, i.e.

$$(\gamma^0\gamma^\mu)^\dagger = \gamma^{\mu\dagger}\gamma^{0\dagger} = \left\{\begin{matrix} \gamma^0\gamma^0 = \gamma^0\gamma^0 \\ -\gamma^i\gamma^0 = \gamma^0\gamma^i \end{matrix}\right\} = \gamma^0\gamma^\mu$$

- Therefore $\psi^\dagger(\gamma^0\gamma^\mu)\psi$ will also be Hermitian

$$(\psi^\dagger(\gamma^0\gamma^\mu)\psi)^\dagger = \psi^\dagger(\gamma^0\gamma^\mu)^\dagger\psi = \psi^\dagger\gamma^0\gamma^\mu\psi$$

- For convenience we call $\psi^\dagger\gamma^0 = \bar{\psi}$ such that

$\bar{\psi}\gamma^\mu\psi$ is a Hermitian object, a good candidate for $j^\mu = \bar{\psi}\gamma^\mu\psi$, for which $\partial_\mu j^\mu = 0$

- Let's work out the e.o.m. for this current

$$i\gamma^\mu\partial_\mu\psi - m\psi = 0 \quad |^\dagger$$

$$-i\,\partial_\mu\psi^\dagger\gamma^{\mu\dagger} - m\,\psi^\dagger = 0 \quad |\cdot\gamma^0$$

$$-i\,\partial_\mu(\psi^\dagger\gamma^0)\gamma^\mu - m\,\psi^\dagger\gamma^0 = -i(\partial_\mu\bar{\psi})\gamma^\mu - m\bar{\psi} = 0$$

such that

$$-i\,\partial_\mu(\bar{\psi}\gamma^\mu\psi) = -i\,\partial_\mu\bar{\psi}\gamma^\mu\psi - i\bar{\psi}\gamma^\mu\partial_\mu\psi$$

$$= m\bar{\psi}\gamma^\mu\psi - m\bar{\psi}\gamma^\mu\psi = 0. \quad \text{very good.}$$

- Remember how we introduced the coupling to the EM field in non-relativistic QM

$$\hat{p} \to \hat{p} + e\hat{A}$$

or $\partial_\mu \to \partial_\mu + ieA_\mu$, therefore the Dirac equation becomes

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\Downarrow$$

covariant derivative

$$(i\gamma^\mu D_\mu - m)\psi, \quad D_\mu = \partial_\mu + ieA_\mu$$

The Lagrangian should also be modified

FREE DIRAC $\quad \mathcal{L} = \bar{\psi}(i\not{\partial} - m)\psi$

QED : FERMION & A_μ : $\mathcal{L} = \underbrace{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}}_{\text{MAXWELL'S EQUATIONS}} + \underbrace{\bar{\psi}(i\not{D} - m)\psi}_{\text{FREE FERMION \& EM CURRENT}}$

$$\mathcal{L} \ni i(ieA_\mu\bar{\psi}\gamma^\mu\psi) = -A_\mu\underbrace{[e\bar{\psi}\gamma^\mu\psi]}_{j^\mu}$$

So we identified the QED four-current j^μ from fermions : $\quad j^\mu = e\bar{\psi}\gamma^\mu\psi$

3.5. Relativistic Normalization of States

Our equations of motion are LI because we derived them from the action, which is a Lorentz scalar.

To describe physical processes (decays and scatterings) we have to introduce the states in a LI way.

- NR QM : $\langle p_1 | p_2 \rangle = (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$

This is not invariant under LT, e.g. under a boost in the z-direction. However

REL : $\langle p_1 | p_2 \rangle = 2 E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$

- Let's see in what sense this definition is LI.

$$\begin{pmatrix} E_p' \\ p^{3'} \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta \\ \gamma\beta & \gamma \end{pmatrix} \begin{pmatrix} E_p \\ p^3 \end{pmatrix} = \begin{pmatrix} \gamma E_p + \gamma\beta p^3 \\ \gamma p^3 + \gamma\beta E_p \end{pmatrix}$$

Let's check :

$$\langle p_1' | p_2' \rangle = 2 E_{p_1}' (2\pi)^3 \delta^{(3)}(\vec{p}_1{}' - \vec{p}_2{}')$$

$$= 2\gamma (E_{p_1} + \beta p_1^3)(2\pi)^3 \delta(\gamma(p_1^3 + \beta E_{p_1}) - p_2^{3'})$$

$$\delta(p_1^{1'} - p_2^{1'}) \, \delta(p_1^{2'} - p_2^{2'})$$

- Relativistic normalization of states

$$\langle p_1 | p_2 \rangle = 2 E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

- Boost along the 3-direction : $E_p' = \gamma (E_p + \beta p^3)$

$$p^{1'} = p^1 \quad , \; p^{2'} = p^2$$

$$p^{3'} = \gamma (p^3 + \beta E_p)$$

- The normalization then becomes

$$\langle p_1' | p_2' \rangle = 2 E_{p_1}' (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

$$= 2 \gamma (E_p + \beta p^3)(2\pi)^3 \; \delta(p_1^{1'} - p_2^{1'}) \, \delta(p_1^{2'} - p_2^{2'}) \quad \delta(p_1^1 - p_2^1) \times \delta(p_1^2 - p_2^2)$$

$$\delta(\gamma(p_1^3 + \beta E_{p_1}) - p_2^{3'})$$

- The δ function of an arbitrary $g(x)$ is given by

$$\delta(g(x)) = \frac{1}{|g'(x)|} \qquad E_{p_1} = \sqrt{p_1^2 + m_1^2}$$

$$\delta(g(p_1^3)) = \frac{1}{\gamma + \gamma\beta \frac{dE_{p_1}}{dp_1^3}} \times \delta(p_1^3 - p_2^3) \qquad \frac{dE_{p_1}}{dp_1^3} = \frac{1}{\cancel{2}} \frac{1}{E_{p_1}} \cdot \cancel{2} p_1^3$$

$$= \frac{E_{p_1}}{\gamma (E_{p_1} + \beta p_1^3)} \delta(p_1^3 - p_2^3) = \frac{E_{p_1}}{E_{p_1}'} \delta(p_1^3 - p_2^3)$$

$$\Rightarrow \langle p_1' | p_2' \rangle = 2 \cancel{E_{p_1}'} (2\pi)^3 \frac{E_{p_1}}{\cancel{E_{p_1}'}} \delta^{(3)}(\vec{p}_1 - \vec{p}_2) = 2 E_{p_1} (2\pi)^3 \delta^{(3)}(\vec{p}_1 - \vec{p}_2)$$

- Very good, this is indeed Lorentz invariant. Let's check the overall normalization.

- PHASE SPACE INTEGRAL

$$\int \frac{d^3p}{(2\pi)^3} \quad \ldots \text{ this is } SO(3) \text{ invariant but not LI.}$$

$$\int \frac{d^4p}{(2\pi)^4} 2\pi\, \delta(p^2-m^2)$$

on-shell condition.

$$\int dp^0\, \delta(p^2-m^2) = \int dp^0 \left(\delta(p^{0\,2} - \vec{p}^{\,2} - m^2) \right)$$

$$= \frac{1}{|2p^0|} \delta(p^0 - E_p) = \frac{1}{2E_p}$$

$$\int \frac{d^4p}{(2\pi)^{\cancel{4}3}} \cancel{2\pi}\, \delta(p^2-m^2) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p}$$

- The sum over all states becomes a properly relativistically normalized integral

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle\langle p| = 1$$

- When applied to a particular state, we have

$$\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} |p\rangle \underbrace{\langle p | k\rangle}_{2E_p (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{k})} = \int d^3p\, \delta^{(3)}(\vec{p}-\vec{k}) |p\rangle\langle p|k\rangle = |k\rangle.$$

- Since $\langle p_1 | p_2\rangle = 2E_{p_1} \delta^{(3)}(\vec{p}_1 - \vec{p}_2)(2\pi)^3$

$$2\left[|p\rangle\right] = 1 - 3 = -2 \implies \left[|p\rangle\right] = -1 \text{ or GeV}^{-1}.$$

3.6. Spin & Statistics Theorem / Connection

$| \quad \rangle$ ... states carry information on spin.

In QFT this information is brought in by the appropriate quantization conditions, required by unitarity & positivity. (Dirac, Feynman, Wightman & Streater)

$S \in \mathbb{N} \Rightarrow$ BOSE - EINSTEIN STATISTICS

$S \in \frac{\mathbb{N}}{2} \Rightarrow$ FERMI - DIRAC STATISTICS