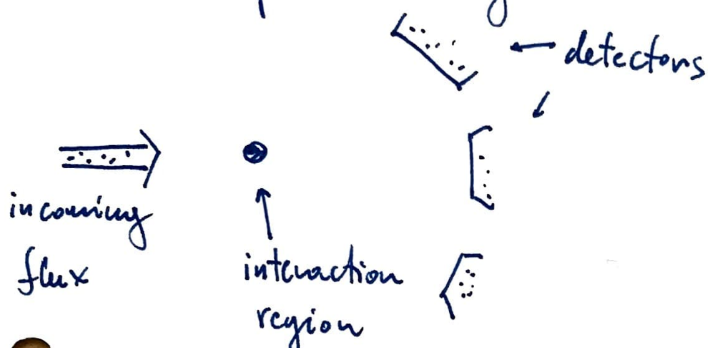
7. Tools for Calculation

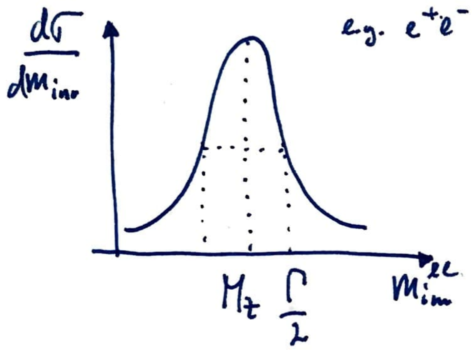
- What do we measure? How do we calculate these observables?

- Scatterings & decay widths



COUNTING

- angular distributions $\frac{d\sigma}{d\theta}$
- momentum distributions $\frac{d\sigma}{dp_T}$
- invariant masses, etc.

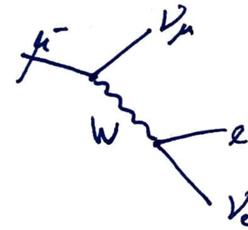


7.1. Observables in Particle Experiments

- stochastic decay rate of A : $\frac{dP}{dt} = -\frac{P}{\tau_A}$

$$P(t) = \frac{1}{\tau} e^{-t/\tau} \qquad \Downarrow \qquad \ln P = e^{-t/\tau_A}$$

$\frac{1}{\tau}$ ... total width Γ for the unstable particle



$$\text{e.g.: } \Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3} = \frac{10^{-10}\,\cancel{\text{GeV}^{-4}}\,(0.1)^5\,\text{GeV}^{\cancel{5}}}{10^3} \simeq 10^{-18}\,\text{GeV},$$

$$\tau_\mu = \frac{\hbar}{\Gamma} = \frac{\hbar c}{\Gamma_\mu c} = \frac{0{,}2\,\cancel{\text{GeV}}\,\text{fm}\;\;\text{s}}{10^{-18}\,\cancel{\text{GeV}}\,3\cdot 10^8\,\text{m}} = \frac{2\cdot 10^{-16}}{3\cdot 10^{-10}}\,\text{s} \approx \mu\text{s}.$$

- <u>More decay channels</u>

$$A \to x_1\bar{x}_1,\ x_2\bar{x}_2,\ x_3\bar{x}_3 m, \dots$$

these are distinguishable final states, e^+e^-, $q\bar{q}$, $\mu^-\bar{\nu}_\mu\gamma$, ...

$$\Gamma^{tot} = \sum_i \Gamma_i$$

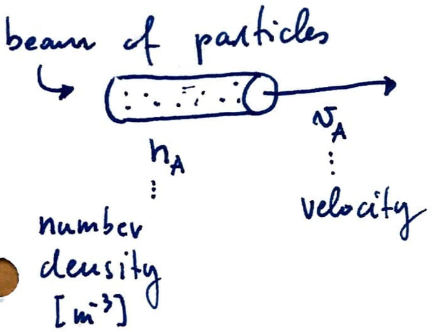
partial decay widths

totale decay width is summed over all channels

- Lifetime : $\tau_A = \dfrac{\hbar}{\Gamma_A}$

- Branching ratio : $Br_i = \dfrac{\Gamma_i}{\Gamma^{tot}}$ * these are counting experiments

How do we define the CROSS-SECTION ?



B

cross-section

then: $\dfrac{N_{events}}{s} = n_A v_A \sigma$

$[\frac{1}{s}] = eV = 1$

$[n_A] = [m^{-3}] = 3$

$[v] = 0$

$[\sigma] = 1-3 = -2 \ \dots \ GeV^{-2}$

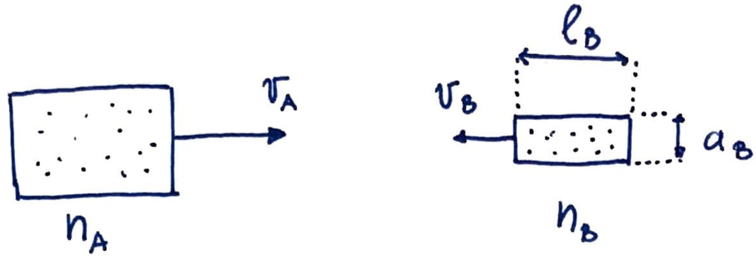
$= cm^2$

- geometric interpretation of σ ... the effective area (in cm^2) that the beam A "sees".

$$\boxed{\frac{\#}{s} = n v \sigma}$$

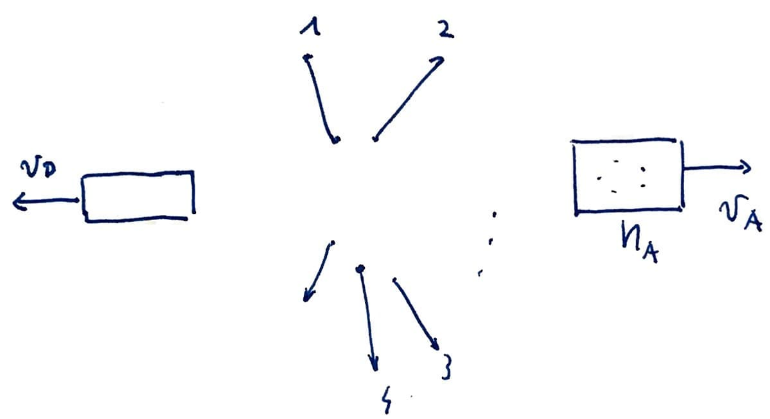
* exercise work out the units & size of barn, mb, fb ...

BEGINNING :



$$\frac{\#}{\text{sec}} = \underbrace{n_A n_B}_{\substack{\text{density} \\ \text{of scatterers} \\ m^{-6}}} \underbrace{l_B A_B}_{\substack{\text{volume} \\ \text{of} \\ B \\ m^3}} \underbrace{|v_A - v_B|}_{\substack{\text{relative} \\ \text{velocity} \\ \frac{m}{s}}} \underbrace{\sigma}_{\text{the cross-section } [cm^2]}$$

END :



$$A + B \to 1 + 2 + 3 + \underbrace{\ldots}_{\text{may include } A \& B} + n$$

- Typically we measure momenta & charges, so

$$\frac{d\sigma}{d^3p_1\, d^3p_2 \ldots d^3p_n} \quad \ldots \text{ differential cross-section}$$

$$\int \frac{d\sigma}{d^3p_i}\, d^3p_i = \sigma^{\text{total}} \quad \ldots \text{ total cross section}$$

7.2. MASTER FORMULAE FOR PARTIAL WIDTH & CROSS-SECTIONS

- Fermi's golden rule (QM textbooks)

- We start with a Lorentz invariant transition matrix element $\mathcal{M}$ for Γ_A $(A \to 1 + 2 + \dots)$

$$\underbrace{\langle 1\, 2 \dots |}_{\text{final state}} \; \underbrace{\hat{T}}_{\substack{\text{transition operator} \\ \text{(time evolution operator)}}} \; \underbrace{| A(p_A) \rangle}_{\text{initial state, possibly moving with } p_A} = \underbrace{\mathcal{M}(A \to 1 + \dots + n)}_{\text{invariant matrix element}} \; \overbrace{(2\pi)^4 \, \delta^{(4)}\Big(p_A - \sum_j p_j\Big)}^{\text{4-momentum conservation}}$$

Dimensionality of $\mathcal{M}$

$$[T] = 0$$

$$[\,|\;\rangle] = \text{GeV}^{-n+1}$$

$$[\delta^{(4)}] = \text{GeV}^{-4}$$

$$\Rightarrow -n + 0 + (-1) = [\mathcal{M}] - 4 \quad \Rightarrow \quad [\mathcal{M}] = -n - 1 + 4$$

$$= 3 - n$$

$$\boxed{[\mathcal{M}] = 3 - n}$$

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to get the total rate $\Gamma_{A \to 1 \dots n}$, we need to integrate over all the final state momenta

$$\int d\Pi_n = \int \underbrace{\prod_i \frac{d^3 p_i}{(2\pi)^3 \, 2E_i}}_{\text{Lorentz-invariant phase space integral}} \; (2\pi)^4 \, \delta^{(4)}\Big(\overbrace{P}^{\text{total 4-momentum}} - \sum_i p_i\Big)$$

• Dimensionality of the phase space

$$\propto \int \underbrace{\prod_i \frac{d^3 p_i}{2E_i}}\ \delta^{(4)}$$

$$[\Pi_n] = [3-1] \cdot n - 4 = 2n-4 \quad ; \quad \underline{GeV^{2n-4}}$$

• now, the Fermi golden rule states: (see also PDG)

$$\underline{\Gamma(A \to f) = \frac{1}{2M_A} \int d\Pi_n \, |\mathcal{M}(A \to f)|^2}$$

• units: $[\Gamma] = 1$
$$= -1 + \cancel{2n} - 4 + 2(3 - \cancel{n})$$
$$= 6 - 5 = 1 \quad \text{ok}$$

• the derivation of the formula & $\mathcal{M}$ requires QFT.

• if we have more channels (in the theory / model) that give the same final state, we have to sum them <u>inside</u> of $|\mathcal{M}|$, i.e. ~~co~~ <u>coherently</u>.

• Another sum one has to do is over <u>all</u> degrees of freedom. This can be <u>spin</u> for fermions, vector boson, it can be SU(3)$_c$ colour, or other quantum numbers.

• one also has to <u>average</u> (if the initial state is not) polarized (specified in a certain direction) over the initial states.

Now we move on to the scattering rates

$$\underbrace{\langle 12\ldots n|}_{\text{final}} T | \underbrace{A(p_A) B(p_B)\rangle}_{\text{initial}} = \mathcal{M}(A+B \to 1+\ldots+n)\,(2\pi)^4 \delta^{(4)}(p_A+p_B-\sum_i p_i)$$

- all the states were defined in a Lorentz-invariant way, $\Rightarrow \mathcal{M}$ is L.I.

- dimensions $\quad -n+0-2 = [\mathcal{M}]-4$

$$[\mathcal{M}] = 2-n \quad \text{for } 2\to n \text{ scattering}$$

$$[\mathcal{M}] = \cancel{4} - n_i - n_f \quad \text{in general}$$

- now the cross-section is given by

$$\sigma(A+B\to f) = \frac{1}{2E_A 2E_B |v_A - v_B|} \int d\Pi_n |\mathcal{M}(A+B\to f)|^2$$

$$[\sigma] = -1-1+\cancel{2n-4}+2(\cancel{2-n}) = -2 \quad \text{or cm}^2 \quad \text{ok} \checkmark$$

7.3. PHASE SPACE

* if $\mathcal{M}$ is constant, then particles are distributed along the kinematically allowed phase space.

* measurements in the phase space reveal the underlying structure of the interactions in $\mathcal{M}$.

- Let us work out the two body phase space in detail. This is useful for many, if not most, physical situations. $\underbrace{(2\pi)\,\delta(E_{cm}-E_1-E_2)\,(2\pi)^3\,\delta^{(3)}(\overset{\overset{0}{\shortparallel}}{\vec{P}}-\vec{p}_1-\vec{p}_2)}$

$$\int d\Pi_2 = \int \frac{d^3p_1}{(2\pi)^3\,2E_1}\int \frac{d^3p_2}{(2\pi)^3 2E_2}\,(2\pi)^4\delta^{(4)}(P-p_1-p_2)$$

- CMS frame $\vec{p}_1+\vec{p}_2=0$; $\delta^{(3)}(\vec{p}_1+\vec{p}_2) \Rightarrow \vec{p}_1=-\vec{p}_2$

$$P=(E_{cm},0)\quad,\quad p_1=(E_1,\vec{p}\,),\quad p_2=(E_2,-\vec{p}\,)$$

$$E_1^2-\vec{p}^{\,2}=m_1^2\quad,\quad E_2^2-\vec{p}^{\,2}=m_2^2\,.$$

- When we do away with three integrations over $\vec{p}_2$, we have a single simple integral over $\vec{p}_1\to\vec{p}$:

$$\int d\Pi_2 = \int \frac{d^3p}{(2\pi)^3\,2E_1\,2E_2}\,(2\pi)\,\delta(E_{cm}-E_1-E_2)$$

- Often we will be interested in angular distribution and integrate over angles to get the total rate

$$d^3p = dp\,p^2\,d(\cos\theta)\,d\varphi = dp\,p^2\,d\Omega$$

$$\int dp\,\delta(E_{cm}-E_1(p)-E_2(p)) = \frac{1}{\left|\frac{dE_1}{dp}+\frac{dE_2}{dp}\right|} =$$

- Integrating over the energies, gives

$$\int dp\, \delta(E_{CM} - E_1 - E_2) = \frac{1}{\left|\frac{dE_1}{dp} + \frac{dE_2}{dp}\right|} = \frac{1}{\frac{p}{E_1} + \frac{p}{E_2}} = \frac{E_1 E_2}{\underbrace{p(E_1+E_2)}_{\substack{\shortparallel \\ E_{CM}}}}$$

$$E_1 = \sqrt{p^2 - m_1^2}\,, \quad \frac{dE_1}{dp} = \cancel{\frac{1}{2}}\frac{1}{E_1}\cdot\cancel{2}p$$

$$\Rightarrow \int d\Pi_2 = \int \frac{p^{\cancel{2}}\, d\Omega}{16\pi^2 \cancel{E_1 E_2}} \frac{\cancel{E_1 E_2}}{\cancel{p} E_{CM}} = \frac{1}{8\pi}\left(\frac{2p}{E_{CM}}\right)\int \frac{d\Omega}{4\pi}$$

- of course $[d\Pi_2] = \underset{\substack{\shortparallel \\ 2}}{2n} - 4 = 0$, which can also be seen in the $p \gg m$ limit $p = E_1 = E_2 = E_{CM} \Rightarrow \frac{2p}{E_{CM}} = 1$, when the $\int d\Pi_2 = \frac{1}{8\pi}\underbrace{\int \frac{d\Omega}{4\pi}}$

$\downarrow$

1 if no angular dependence

$\Downarrow$

$$\int d\Pi_2 = \frac{1}{8\pi}$$

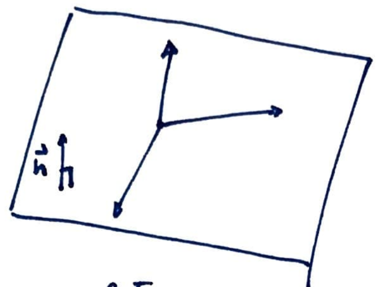
- Final subtlety in any Γ or σ calculation is taking care of the symmetry factor $\frac{1}{n!}$ for any n-indistinguishable particles in the final state, e.g. $h \to \gamma\gamma$ or $\pi^0 \to \gamma\gamma$ or $pp \to 3\pi^0 + pp$ etc.

e.g $|\pi^0(p)\,\pi^0(\vec{p})\rangle = |\pi^0(-p)\,\pi^0(p)\rangle$

$\boxed{\text{3 - BODY}}$ phase space

- can also be simplified in the CM frame.

$$\underbrace{\vec{p}_1 + \vec{p}_2 + \vec{p}_3}_{\text{these lie in the event plane}} = 0 \quad , \quad Q = (E_{CM}, 0) \quad , \quad Q^0 = E_{CM}$$



- let us introduce $x_1 = \dfrac{2E_1}{E_{CM}}$, x_2, x_3

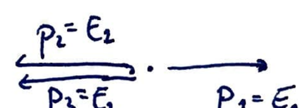
since $E_1 + E_2 + E_3 = E_{CM} \;\Big|\; \dfrac{2}{E_{CM}}$

$$x_1 + x_2 + x_3 = 2$$

- $\boxed{\text{do the Exercise 7.2}}$

$$\int d\Pi_3 = \frac{E_{CM}^2}{128\,\pi^3} \int dx_1\, dx_2$$

- The integration limits on x_1 & x_2 are not easy to write down. However in the relativistic $E_i \gg m_i$ limit, we have $E_i^{max} = \dfrac{E_{CM}}{2}$



$E_1 = E_2 + E_3 = E_{CM} - E_1$
$\Downarrow$
$E_{CM} = 2E_1$

or : $x_i^{MAX} = 1$

then $\displaystyle\int_0^1 dx_1 \int_{1-x_1}^{1} dx_2$ ← min for a fixed x_1

- for massive particles / low energy collisions

$$\vec{p}_i \cdot \vec{p}_j = |p_i|^2 + |p_j|^2 + 2\, p_i\, p_j \underbrace{\cos\theta_{ij}}_{\in(-1,1)}$$

$\Downarrow$

implicit contraint on x_1, x_2

- it is very useful to define two Lorentz invariant

parameters : $m_{12}^2 = (p_1+p_2)^2$, $m_{23}^2 = (p_2+p_3)^2$,

such that $\displaystyle\int d\Pi_3 = \frac{1}{128\pi^3 E_{cm}^2} \int dm_{12}^2\, dm_{23}^2$

<u>Exercise</u>: Phase space dimensionality of

- β decay
- $0\nu 2\beta$ decay

7.4. EXAMPLE : $\pi^+\pi^-$ SCATTERING @ ρ RESONANCE

- resonances are quasi-particles, unstable states that can amplify the cross-section. In QM they are described by a Breit-Wigner form

$$\mathcal{M} \sim \frac{1}{E - E_R + i\Gamma/2}$$

$$\mathcal{M} \sim \frac{1}{E - E_R + i\Gamma/2}$$

energy transfer during scattering

position (mass) of the resonance

width of the resonance

$$\psi(t) = \frac{1}{2\pi}\int dE \, \underbrace{\frac{e^{-iEt}}{E - E_R + i\Gamma/2}}_{}$$ $\Rightarrow$ will be generalized to QFT

$f(E)$ ... has a simple pole $\frac{1}{0}$ when

$$E_0 = E_R - i\Gamma/2$$

. by Cauchy theorem $\frac{1}{2\pi i}\int dx\, f(x) = \sum \mathrm{res}\, f(x)$

$$\Rightarrow \psi(t) = i\, e^{-iE_0 t} = i\, e^{-iE_R t}\, \underbrace{e^{-i(-i)\Gamma/2\, t}}_{e^{-\Gamma/2 t}}$$

thus $|\psi(t)|^2 \cancel{\propto} e^{-\Gamma/t}$ & $\tau_R = \frac{\hbar}{\Gamma_R}$

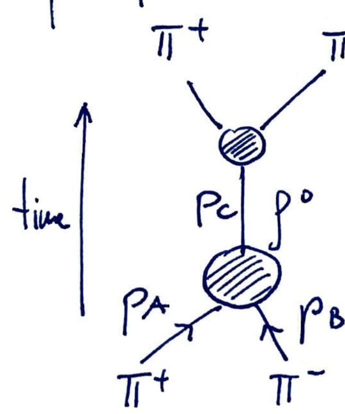
* Let us consider a specific case of $\pi^+\pi^- \to \rho^0 \to \pi^+\pi^-$

$m_{\pi^+} = 139,6$ MeV

$m_{\rho^0} = 770$ MeV

$\downarrow$

spin 1 - vector boson



$p_A + p_B = p_C$

- production given by $\mathcal{M}(\pi^+\pi^- \to \rho^0)$

rest frame $p_c \cdot \varepsilon = 0$ valid in all frames

$\Downarrow$

$\varepsilon^\mu = (0, \vec{\varepsilon})$

$$\mathcal{M}(\pi^+\pi^- \to \rho^0) = g_\rho\, \varepsilon^*(p_A - p_B) + g_\rho \underbrace{\varepsilon^*(\overset{p_c}{\overset{\shortparallel}{p_A + p_B}})}_{\overset{\shortparallel}{0}}$$

$[\mathcal{M}] = 1 \Rightarrow [g_\rho] = 0$ ... dimensionless

$$\sigma(\pi^+\pi^- \to \rho^0) = \frac{1}{2E_A\, 2E_B\, |v_A - v_B|} \int \frac{d^3p_c}{(2\pi)^3} \frac{1}{2E_c} \times$$

use the master formula from above

$$\times\, |\mathcal{M}|^2 (2\pi)^4 \delta^{(4)}(p_A + p_B - p_c)$$

Let's focus on the CM frame and define our kinematical variables to be

$p_A = (E, p)$ $\quad E_{CM} = m_\rho \quad , \quad E = \frac{m_\rho}{2}$

$p_B = (E, -p)$ $\quad E^2 - p^2 = m_\pi^2 \Rightarrow p = \sqrt{\frac{m_\rho^2}{4} - m_\pi^2}$

ε is in the space direction $(0, \vec{\varepsilon})$

$$\varepsilon^*(p_A - p_B) = (0, \vec{\varepsilon}^*)(0, 2p) = -2\, \vec{\varepsilon}^* \vec{p} =$$

single power of momentum $\Rightarrow$ P-wave $(-1)^\ell = -1$

$$\int \frac{d^3p_c}{(2\pi)^3}\,\frac{1}{2E_c} = \int \frac{d^4p_c}{(2\pi)^4}\,\delta^{(4)}(p_c^2-m_\rho^2)\,2\pi$$

$$\Rightarrow \sigma(\pi^+\pi^- \to \rho^0) = \frac{1}{\cancel{4}\cdot\left(\frac{m_\rho^2}{\cancel{4}}\right)\frac{\cancel{4}p}{m_\rho}}\,2\pi\,\delta\left((p_A+p_B)^2-m_\rho^2\right)$$

$$\times\; g_\rho^2\,\cancel{4}\,|\vec{\varepsilon}\cdot\vec{p}|^2$$

* now we sum over the polarizations in $\vec{\varepsilon}$

$$\sigma(\pi^+\pi^- \to \rho^0) = \frac{g_\rho^2\,p}{m_\rho}\,(2\pi)\,\delta\left((p_A+p_B)^2-m_\rho^2\right)$$

DECAY RATE OF $\rho^0 \to \pi^+\pi^-$

$$\Gamma_\rho = \frac{1}{2m_\rho}\int d\Pi_2\,|\mathcal{M}|^2$$

$$= \frac{1}{\cancel{2}m_\rho}\,\frac{1}{8\pi}\,\frac{\cancel{2}p}{\underset{m_\rho}{\overset{\shortparallel}{E_{cm}}}}\int\frac{\cancel{d\Omega}}{\cancel{4\pi}}\cdot\left|g_\rho\,\varepsilon^*(p_A-p_B)\right|^2$$

$$= \frac{g_\rho^2}{8\pi}\,\frac{p}{m_\rho^2}\,\underbrace{\langle\, 4|\vec{\varepsilon}^{\,*}\cdot\vec{p}\,|^2\rangle}_{=\,4\,\frac{p^2}{3}}$$ ← average over outgoing pions

$$= \frac{g_\rho^2}{6\pi}\,\frac{p^3}{m_\rho^2} = \frac{g_\rho^2}{6\pi}\,\frac{m_\rho^3}{m_\rho^2}\sqrt{\frac{m_\rho^2}{\cancel{4}}\cancel{\times}1-\frac{4m_\pi^2}{m_\rho^2}}^{\,3}\cdot\frac{\cancel{4}^{3/2}1}{8}$$

$$= g_\rho^2\,\frac{m_\rho}{48\pi}\cdot\left(1-\left(\frac{2m_\pi}{m_\rho}\right)^2\right)^{3/2} = g_\rho^2\cdot$$

Let us treat the whole process via a relativistic Breit-Wigner resonance

$$\mathcal{M} \sim \frac{1}{p^2 - m_R^2 + i m_R \Gamma_R}$$

- The matrix element can be expanded close to the resonance peak $P \sim (m_R + \Delta E, 0)$

$$p^2 \sim m_R^2 + 2 m_R \Delta E - 0$$

then $$\mathcal{M} \sim \frac{1}{2 m_R (\Delta E + i \Gamma_R / 2)}$$, as earlier

- let us compute the full expression for σ

$$\sigma(\pi^+(p_A) + \pi^-(p_B) \to \rho^0 \to \pi^+(p_A') + \pi^-(p_B'))$$

$$= \frac{1}{2E_A 2E_B |v_A - v_B|} \int d\Pi_2 \left| \sum_\varepsilon \frac{\mathcal{M}(\pi^+\pi^- \to \rho_0(\varepsilon)) \mathcal{M}(\rho^0(\varepsilon) \to \pi^+\pi^-)}{(p_A + p_B)^2 - m_\rho^2 + i m_\rho \Gamma_\rho} \right|^2$$

$p_A = \vec{p}$, $p_A' = \vec{p}\,'$

$$= \frac{1}{m_\rho^{\cancel{2}} \left(\frac{4\cancel{p}}{\cancel{m_\rho}}\right)^2} \cdot \frac{1}{8\pi} \frac{\cancel{2p}}{m_\rho} \int \frac{d\Omega}{4\pi} \frac{1}{3} \sum_\varepsilon \frac{|\sum_\varepsilon 2 g_\rho \vec{\varepsilon} \cdot \vec{p}\, 2 g_\rho \vec{\varepsilon} \cdot \vec{p}\,'|^2}{(E_{cm}^2 - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2}$$

$$= \frac{g_\rho^4}{16\pi} \frac{1}{m_\rho^2} \frac{1}{(E_{cm}^2 - m_\rho^2)^2 + m_\rho^2 \Gamma_\rho^2} \underbrace{\int \frac{d\Omega}{4\pi} |\vec{p} \cdot \vec{p}\,'|^2}$$

$\Rightarrow \frac{d\sigma}{d\Omega} \propto |\vec{p} \cdot \vec{p}\,'|^2$ ... P-wave $L=1$

- NWA = Narrow width approximation

$$\rho^0 \to f$$

$$\sigma(\pi^+\pi^- \to \rho^0 \to f) = \frac{1}{4m_\rho p} \int d\Pi_f \left| \frac{\sum_\varepsilon 2g_\rho\, \vec{\varepsilon}^* \cdot \vec{p}\, \mathcal{M}(\rho^0_\varepsilon \to f)}{(p_A+p_B)^2 - m_\rho^2 + i m_\rho \Gamma} \right|^2$$

- the final phase space contains the delta function

$$(2\pi)^4 \delta^{(4)}(p_A+p_B-\sum p_f) = \int \frac{d^4 p_c}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_A+p_B-p_c)$$
$$(2\pi)^4 \delta^{(4)}(p_c - \sum p_f)$$

$$\Gamma(\rho \to f) = \frac{1}{2m_\rho} \int d\Pi_f |\mathcal{M}|^2$$

$$\sigma(\pi^+\pi^- \to \rho \to f) = \frac{1}{m_\rho p} \int \frac{d^4 p_c}{(2\pi)^4} (2\pi)^4 \delta^{(4)}(p_A+p_B-p_c)$$
$$\sum_\varepsilon |2g_\rho\, \vec{\varepsilon}^* \cdot \vec{p}_A|^2 \frac{1}{(p_c^2-m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2} 2m_\rho\, \Gamma(\rho \to f)$$

$$= \frac{1}{m_\rho p} 4 g_\rho^2 p^2 \frac{\frac{m_\rho}{\pi} \Gamma(\rho \to f)}{(p_c^2-m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2}$$

$$\sum_f \sigma(\pi\pi \to \rho \to f) = \frac{4 g_\rho^2 p}{m_\rho} \underbrace{\frac{m_\rho \Gamma^{tot}/\pi}{(p_c^2-m_\rho^2)^2 + (m_\rho\Gamma_\rho)^2}}_{\propto\, \delta \text{ function in } p_c}$$

$$\int d^2 p_c \frac{m_\rho \Gamma_\rho/\pi}{(p_c^2-m_\rho^2)^2 + (m_\rho\Gamma)^2} = \frac{m_\rho\Gamma_\rho}{\pi} \underbrace{\int \frac{dx}{x^2+a^2}}_{\frac{1}{a}\tan^{-1}\frac{x}{a}} = \frac{\cancel{m_\rho\Gamma_\rho}}{\pi} \frac{1}{\cancel{m_\rho\Gamma}} \cdot \tan^{-1}\left(\frac{p_c^2 - m_\rho^2}{m_\rho\Gamma_\rho}\right)$$

$$x = p_c^2 - m_\rho^2$$

$$dx = dp_c^2$$

$$\to 1$$

$$= \frac{1}{\pi} \tan^{-1}\left(\frac{p_c^2-m_\rho^2}{m_\rho\Gamma_\rho}\right)\Bigg|_{-\infty}^{\infty} = \frac{1}{\pi}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

- so we showed that $\int dp_c^2 \frac{m_\rho \Gamma_\rho / \pi}{(p_c^2 - m_\rho^2) + (m_\rho \Gamma_\rho)^2} = 1$

- then : $\sigma(\pi^+\pi^- \to \rho \to f) = \frac{g_f^2 p}{m_\rho} 2\pi \, \delta((p_A + p_B)^2 - m_\rho^2)$

Ex 7.1 $\quad m_\sigma = 500$ GeV , spin 0 , isospin 0

$S = 0 \quad , \quad I_3 = 0$

- broad , $\sigma \to \pi^+\pi^-$ $\qquad J^{PC}(\pi^+) = \qquad \pi^+ : I^G = 1^-$

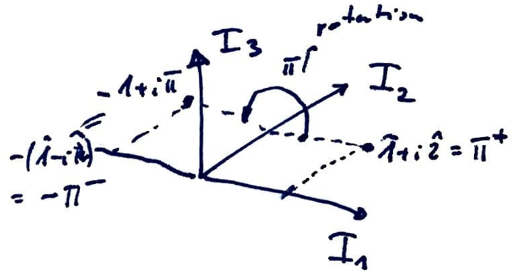
$J^P = 0^-$

$\mathcal{M}(\pi^i \pi^j \to \sigma) = G \delta^{ij} \quad , \quad i = 1,2,3$

isospin state $\qquad \pi^\pm = \pi^1 \pm i \pi^2$

$\pi^+ + \pi^- = 2\pi^1$

$\pi^+ - \pi^- = 2i\pi^2$



P and G parity of σ?

$P(\pi^+) = -1 \Rightarrow P(\pi^+\pi^-) = -1^2 = 1 \Rightarrow P(\sigma) = 1$

G parity : $G(\sigma) = +1$, $\quad \sigma$ can be 0 or 1

$C|\pi^+\rangle = |\pi^-\rangle$

$G = C e^{i\pi I_2} = C R_2(\pi) \Rightarrow C R_2(\pi)|\pi^+\rangle = -C|\pi^-\rangle = -|\pi^+\rangle$

$G|\pi^+\rangle = -|\pi^+\rangle \qquad G|\pi^+\pi^-\rangle = |\pi^+\pi^-\rangle \Rightarrow G(\sigma) = +1$

$G|\pi^-\rangle = -|\pi^-\rangle$

$G|\pi^0\rangle = -|\pi^0\rangle \qquad 3 \times 3 =$

$G|\pi\rangle = -|\pi\rangle$

$$\mathcal{M}(\pi^i\pi^j \to \sigma) = g\,\delta^{ij} \qquad \pi^1 = \frac{1}{2}(\pi^+ + \pi^-)$$

$$\mathcal{M}(\pi^+\pi^- \to \sigma) = \mathcal{M}((\pi^1 + i\pi^2)(\pi^1 - i\pi^2)) \to \sigma)$$

$$= \mathcal{M}(\pi^1\pi^1 + \pi^2\pi^2 \to \sigma) = 2g \qquad [\mathcal{M}] = 4 - n_1 - n_i$$

$$\mathcal{M}(\pi^0\pi^0 \to \sigma) = \mathcal{M}(\pi^3\pi^3 \to \sigma) = g$$

$$\Gamma(\sigma \to \pi^+\pi^-) = \frac{1}{2m_\sigma}\int d\Pi_2\, |\mathcal{M}|^2 \qquad \sigma \to \pi^1\pi^1 + \sigma \to \pi^2\pi^2$$

$$= \frac{1}{\cancel{2}m_\sigma}\,\frac{1}{\cancel{2}8\pi}\,\frac{\cancel{2}p}{\underset{m_\sigma}{\underset{\shortparallel}{E_{cm}}}}\cdot\cancel{4}g^2$$

$$p = \sqrt{m_\sigma^2/4 - m_\pi^2} \quad ; \quad 2p = \sqrt{m_\sigma^2 - 4m_\pi^2}$$

$$= \frac{1}{2\pi m_\sigma^2}\sqrt{m_\sigma^2/4 - m_\pi^2}\; g^2 = \frac{g^2}{4\pi m_\sigma}\sqrt{1 - \frac{4m_\pi^2}{m_\sigma^2}}$$

$$= \frac{1}{2m_\sigma}\cdot\frac{1}{8\pi}\,\frac{2p}{E_{cm}}\cdot\cancel{4}g^2 = \frac{g^2}{16\pi m_\sigma}\,\frac{2p}{\underset{m_\sigma}{\underset{\shortparallel}{E_{cm}}}} = \frac{g^2 m_\sigma}{\cancel{16}\,8\pi m_\sigma^2}\sqrt{1 \ldots}$$

$$\mathrm{Br}(\sigma \to \pi^0\pi^0) = \frac{1}{3}$$

$$\Gamma_{tot} = \underbrace{\Gamma_{\sigma\to\pi^1\pi^1} + \Gamma_{\sigma\to\pi^2\pi^2}}_{\Gamma_{\sigma\to\pi^+\pi^-}} + \Gamma_{\sigma\to\pi^0\pi^0} = \left(\frac{2}{3} + \frac{1}{3}\right)\Gamma_{tot}$$

$$\Rightarrow \Gamma_{\sigma\to\pi^+\pi^-} = \frac{2}{3}\,\Gamma_{tot}$$

Exercise 7.2. : Derivation of the 3 body final state

$1, 2, 3 \quad d\Pi_3 \quad ; \quad m_1, m_2, m_3 \quad ; \quad Q = p_1 + p_2 + p_3 \qquad E_{cm} = E_1 + E_2 + E_3$

CM system : $Q = (E_{cm}, 0)$; $p_1 = (E_1, \vec{p}_1) \Rightarrow$, $p_2 = (E_2, \vec{p}_2)$, $p_3 = (E_3, \vec{p}_3)$

$\Rightarrow Q^2 = E_{cm}^2$

(a) $x_1 = \dfrac{2Q \cdot p_1}{Q^2} = \dfrac{2(E_{cm} E_1)}{E_{cm}^2} = \dfrac{2E_1}{E_{cm}}$, $x_2 = \dfrac{2E_2}{E_{cm}}$, $x_3 = \dfrac{2E_3}{E_{cm}}$

$E_1 = \frac{x_1}{2} E_{cm}$

$E_i = \frac{x_i}{2} E_{cm} \qquad x_1 + x_2 + x_3 = 2 \dfrac{E_1 + E_2 + E_3}{E_{cm}} = \underline{\underline{2}}.$

(b)

$E_i^2 = p_i^2 - m_i^2 = \left(\frac{E_{cm}}{2} x_i\right)^2$ or : $p_i^2 = \left(\frac{E_{cm}}{2} x_i\right)^2 + m_i^2$

(c) invariant mass of 1 & 2 : $m_{12}^2 = (p_1 + p_2)^2$ is related

to x_3 : $\quad m_{12}^2 = ((E_1 + E_2)^2 - (p_1 + p_2)^2)$

$\Downarrow$

$m_{12}^2 = (p_1 + p_2)^2$

$= (1 - x_3) Q^2 + m_3^2$

$= E_1^2 + E_2^2 + 2E_1E_2 - p_1^2 - p_2^2 - 2p_1p_2$

$= m_1^2 + m_2^2 + 2x_1x_2 \dfrac{E_{cm}^2}{4} - 2\sqrt{\left(\left(\frac{E_{cm}}{2} x_1\right)^2 + m_1^2\right)\left(\left(\frac{E_{cm}}{2} x_2\right)^2 + m_2^2\right)}$

$p_1 + p_2 = Q - p_3$

$(p_1 + p_2)^2 = Q^2 - 2Qp_3 + p_3^2 = Q^2 - 2(E_{cm} E_3) + m_3^2$

$= Q^2 - \cancel{2} E_{cm} x_3 \dfrac{E_{cm}}{\cancel{2}} + m_3^2 = Q^2(1 - x_3) + m_3^2 .$

$m_{23}^2 \left(= (p_2 + p_3)^2\right) = (Q - p_1)^2 = Q^2 - 2Qp_1 + m_1^2$

$= Q^2(1 - x_1) + m_1^2$

$m_{13}^2 = Q^2(1 - x_2) + m_2^2$

(d) Θ_{12} ... angle between $\vec{p}_1$ & $\vec{p}_2$ in the CM frame

Show $m_{12}^2 = (p_1+p_2)^2 = (1-x_3)Q^2 + m_3^2$ determines $\Theta_{12}(x_i)$

$$= m_1^2 + m_2^2 + 2p_1 p_2 = m_1^2 + m_2^2 + 2(E_1E_2 - \vec{p}_1\vec{p}_2)$$

$$= m_1^2 + m_2^2 + \cancel{2}\frac{x_1 x_2}{\cancel{4}2}\underset{\overset{\parallel}{Q^2}}{E_{cm}^2} - 2\underset{\overset{\parallel}{\sqrt{\left(\frac{Q^2x_1^2}{4}+m_1^2\right)\left(\left(\frac{Qx_2}{2}\right)^2+m_1^2\right)}}}{p_1 p_2}\cos\Theta_{12}$$

$$= m_1^2 + m_2^2 + \frac{x_1x_2}{2}Q^2 - 2\sqrt{}\cos\Theta_{12} = (1-x_3)Q^2 + m_3^2$$

$$\Rightarrow \boxed{\cos\Theta_{12} = \frac{m_1^2+m_2^2-m_3^2-\left(1-x_3-\frac{x_1x_2}{2}\right)Q^2}{\sqrt{(Q^2x_1^2+4m_1^2)((Q^8x_2)^2+4m_1^2)}}}$$

- similar for Θ_{23}, Θ_{13} from m_{13}^2 & m_{23}^2

$$m_{12}^2 + m_{23}^2 + m_{13}^2 = (1-x_3)Q^2 + m_3^2 + (1-x_1)Q^2 + m_1^2 + (1-x_2)Q^2 + m_2^2$$

$$m_{23}^2 = (p_2+p_3)^2 = (1-x_1)Q^2 + m_1^2$$

$$\parallel$$

$$(\underbrace{3-x_1-x_2-x_3}_{3-2=1})Q^2$$

$$Q^2 + m_1^2 + m_2^2 + m_3^2$$

(e) $\int d\Pi_3$ in the CM frame, 9 integrals, 4 δ functions

$$\parallel$$

$$\int \frac{d^3p_1}{(2\pi)^3 2E_1}\frac{d^3p_2}{(2\pi)^3 2E_2}\frac{d^3p_3}{(2\pi)^3 2E_3}(2\pi)^4\delta^{(4)}(Q-p_1-p_2-p_3)$$

$$\delta(E_{CM}-E_1-E_2-E_3)$$

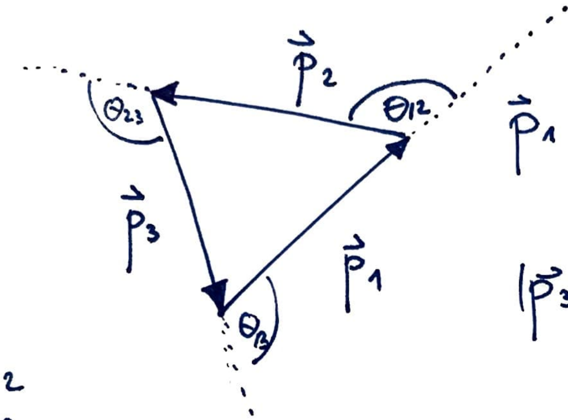
$p_3 = -(\vec{p}_1+\vec{p}_2)$

$$\delta^{(3)}(\vec{p}_1+\vec{p}_2+\vec{p}_3 = 0)$$

$$d^3p_1 = p_1^2\, dp_1\, d\theta_1\, d\varphi_1 \sin\theta_1 = p_1^2\, dp_1\, d(\cos\theta_1)\, d\varphi_1$$

$$(\vec{p}_1 + \vec{p}_2)^2 = \vec{p}_1^{\,2} + \vec{p}_2^{\,2} + 2|\vec{p}_1||\vec{p}_2| \cos\theta_{12}$$

$$\int \frac{d^3p_1}{(2\pi)^3 2E_1} \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)\,\delta(E_{cm} - E_1 - E_2 - E_3), \quad \vec{p}_3 = -\vec{p}_1 - \vec{p}_2$$



$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 0$$

$$|\vec{p}_3|^2 = |\vec{p}_1|^2 + |\vec{p}_2|^2 + 2|\vec{p}_1||\vec{p}_2| \cos\theta_{12}$$

$$E_3^2 - \underbrace{|\vec{p}_3|^2}_{= |-\vec{p}_1 - \vec{p}_2|^2} = m_3^2$$

$$\Rightarrow E_3^2 = |\vec{p}_1 + \vec{p}_2|^2 + m_3^2 = E_1^2 + E_2^2 - m_1^2 - m_2^2 + m_3^2 - 2\sqrt{\cdots}\cos\theta_{12}$$

$$E_1^2 = |p_1|^2 + m_1^2, \quad E_2^2 = |p_2|^2 + m_2^2$$

$$\theta_{12} + \underline{\alpha_{12}} + \theta_{23} + \underline{\alpha_{23}} + \theta_{13} + \underbrace{\alpha_{13}}_{= \pi} = 3\cdot\pi$$

$$\Rightarrow \theta_{12} + \theta_{23} + \theta_{13} = 2\pi$$

$$\int \frac{dp_1\, dp_2\, p_1^2 p_2^2\, \overset{2}{d(\cos\theta_{12})}\, d\cos\theta\, \overbrace{\overset{2\pi}{d\varphi_{12}}\, \overset{2\pi}{d\varphi}}^{8\pi^2}}{(2\pi)^3 (2\pi)^3\, \underset{\sqrt{(p_1^2 - m_1^2)(p_2^2 - m_2^2)}}{2E_1 2E_1}} 2\pi\, \delta(E_{cm} - E_1 - E_2 - E_3)$$

$$E_3^2 = |\vec{p}_1 + \vec{p}_2|^2 + m_3^2 = |\vec{p}_1|^2 + |\vec{p}_2|^2 + 2|\vec{p}_1||\vec{p}_2| \cos\theta_{12} + m_3^2$$

$$= p_1^2 + p_2^2 + 2p_1p_2\cos\theta_{12} + m_3^2 \qquad E_3 = E_{cm} - E_1 - E_2$$

$$E_1^2 = p_1^2 - m_1^2$$
$$E_2^2 = p_2^2 - m_2^2$$

$$E_3^2 = (E_{cm} - E_1 - E_2)^2 = p_1^2 + p_2^2 + 2p_1p_2 \underline{\underline{\cos\theta_{12}}} + m_3^2$$

$$\int \frac{dp_1\, dp_2\; p_1^2\, p_2^2}{(2\pi)^3 4\pi \sqrt{(p_1^2 - m_1^2)(p_2^2 - m_2^2)}} = \int \frac{\cancel{d}p_1 \cancel{d}p_2}{(2\pi)^3\pi} \underbrace{\frac{p_1 dp_1}{2E_1}}_{\frac{E_{cm}}{4}dx_1} \underbrace{\frac{p_2 dp_2}{2E_2}}_{\frac{E_{cm}}{4}dx_2}$$

$$x_1 = \frac{2Q\cdot p_1}{Q^2} = \frac{2\cancel{E_{cm}}E_1}{E_{cm}^{\cancel{2}}} = \frac{2}{E_{cm}}\sqrt{p_1^2 + m_1^2}$$

$$dx_1 = \frac{4}{E_{cm}} \frac{1}{\cancel{2}} \frac{\cancel{2}p_1 dp_1}{2E_1} \Rightarrow \frac{E_{cm}}{4} dx_1 = \frac{p_1 dp_1}{2E_1}$$

$$\sqrt{\frac{x_1^2 E_{cm}^2}{4} - m_1^2} = p_1$$

• go to m_{23}^2 & $m_{13}^2 = (1-x_2)Q^2 + m_2^2$

$$\Rightarrow dm_{13}^2 = -dx_2 Q^2$$

$$m_{23}^2 = (p_2+p_3)^2 = (1-x_1)Q^2 + m_1^2$$

$$\Rightarrow dm_{23}^2 = -dx_1 Q^2$$

$$\Rightarrow \int \frac{p_1\, p_2}{(2\pi)^3\pi} \frac{\cancel{E_{cm}}}{4} \frac{dm_{23}^2}{\cancel{Q^2}} \cdot \frac{\cancel{E_{cm}}}{4} \frac{dm_{13}^2}{Q^2}$$

$$m_{23}^2 = (1-x_1)Q^2 + m_1^2 \;, \quad x_1 = \frac{2}{E_{cm}}\sqrt{p_1^2 + m_1^2} = \frac{Q^2 + m_1^2 - m_{23}^2}{Q^2}$$

$$-m_{23}^2 + m_1^2 + Q^2 = x_1 Q^2$$

$$x_1 = 1 + \frac{m_1^2 - m_{23}^2}{Q^2}$$

$$\frac{\cancel{4}}{\cancel{Q^2}}(p_1^2 + m_1^2) = \frac{(Q^2 + m_1^2 - m_{23}^2)^2}{4Q^{\cancel{4}2}}$$

Let's review

$$\int d\Pi_3 = \int \frac{d^3p_1 \; d^3p_2 \; d^3p_3}{(2\pi)^3 2E_1 \; (2\pi)^3 2E_2 \; (2\pi)^3 2E_3} (2\pi)^3 \delta(E_{cm} - E_1 - E_2 - E_3) \; (2\pi)^3 \delta^{(3)}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3)$$

INTEGRATE OVER $\vec{p}_3$

$$= \int \frac{p_1^2 dp_1 \; p_2^2 dp_2 \; d(\cos\theta_{12}) \; \overbrace{d(\cos\theta) \, d\varphi_1 \, d\varphi_2}^{\cancel{8\pi^2}}}{(2\pi)^3 2E_1 \; \cancel{(2\pi)^3} \cancel{2} E_2} \cancel{2\pi} \delta(E_{cm} - E_1 - E_2 - E_3)$$

$$\int dx \, f(x) \, \delta(g(x)) = \frac{f(g(x)=0)}{|g'(x)|} \qquad E_{cm} - E_1 - E_2$$

$$\vec{p}_3 = -\vec{p}_1 - \vec{p}_2 \qquad |\vec{p}_3|^2 = E_3^2 - m_3^2 = |\vec{p}_1 + \vec{p}_2|^2$$

$$= p_1^2 + p_2^2 + 2 p_1 p_2 \cos\theta_{12}$$

$$E_3 = \sqrt{m_3^2 + p_1^2 + p_2^2 + 2 p_1 p_2 \cos\theta_{12}}$$

$$\frac{dE_3}{d\cos\theta_{12}} = \frac{1}{\cancel{2}} \cdot \frac{1}{E_3} \cdot \cancel{2} p_1 p_2 = \frac{p_1 p_2}{E_3}$$

$$\int d\cos\theta_{12} \; \delta(E_{cm} - E_1 - E_2 - E_3(\cos\theta_{12}))$$

$$= \frac{1}{\left| \frac{dE_3}{d\cos\theta_{12}} \right|} = \frac{E_3}{p_1 p_2} \qquad ; \qquad x_1 = \frac{2E_1}{E_{cm}}$$

$$= \int \frac{p_1 dp_1 \; p_2 dp_2}{16\pi^3 E_1 E_2} E_3 \quad \cdots\cdots \rightarrow \quad \frac{Q^2}{128\pi^3} \int dx_1 \, dx_2$$

$$= \frac{\cancel{Q^2}}{128\pi^3} \int \frac{dm_{13}^2 \, dm_{23}^2}{\cancel{Q^2} Q^2}$$

$$\int d\Pi_3 = \frac{1}{128\pi^3 Q^2} \int dm_{13}^2 \, dm_{23}^2$$