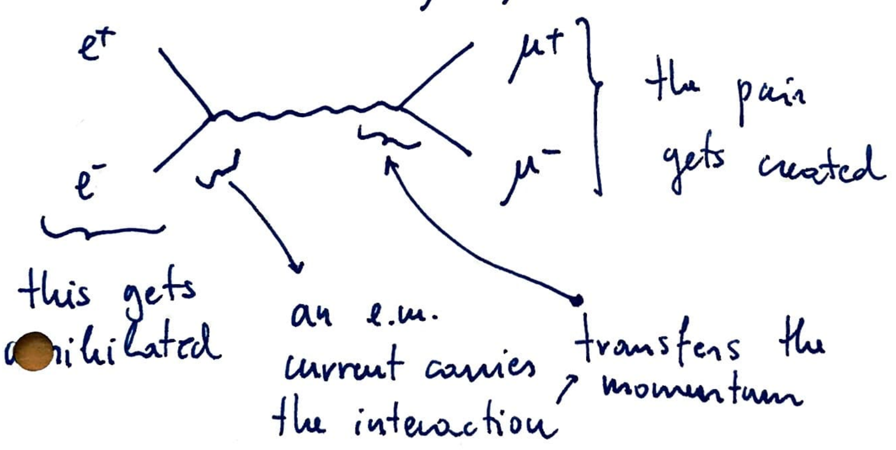
8. e^+e^- ANNIHILATION

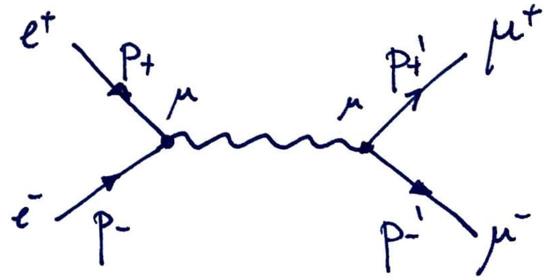
- For studying strong interactions it is actually most convenient to use clean leptonic probes, such as e^- & μ^-.

Also, to develop the techniques for scattering amplitudes & cross-sections with fermions, let's start with processes involving e & μ only.

8.1. $e^+e^- \to \mu^+\mu^-$



MOMENTUM FLOW :



$\underbrace{j^\mu j_\mu}_{\text{current-current int.}}$

$$\mathcal{A} = \underbrace{(-e)}_{\substack{\text{charge,}\\ \text{strength of e.m. int.}}} \langle \mu^+\mu^- | j^\mu | 0 \rangle \underbrace{\frac{1}{q^2}}_{\substack{\text{zero mass}\\ \text{BW = propagator}}} \langle 0 | \underbrace{j_\mu}_{\text{e.m. current}} | \underbrace{e^+e^-}_{\substack{\text{two particle}\\ \text{initial state}}} \rangle (-e)$$

8.2. Describing massless spin $\frac{1}{2}$ fermions

- high energies $E \gg m_e$ the e^- mass can be neglected. The Dirac eq. becomes $i\not\partial \Psi = 0$, which is easy to study using $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix}$, $\sigma^\mu = (1, \sigma^i)$
$\bar\sigma^\mu = (1, -\sigma^i)$

The 4-component Dirac spinor Ψ can be decomposed into 2 sub-spaces with 2-component Weyl spinors $\psi_{L,R}$, such that $\Psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$

$$i \begin{pmatrix} 0 & \sigma^\mu \partial_\mu \\ \bar\sigma^\mu \partial_\mu & 0 \end{pmatrix} \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} i\sigma^\mu \partial_\mu \psi_R \\ i\bar\sigma^\mu \partial_\mu \psi_L \end{pmatrix} = 0.$$

These two states are called left and right-handed, or chiral states. They do not mix under LT, unless masses are involved to flip the chirality, e.g. $e_L \to e_R$ only w. $\frac{m_e}{E}$ factors.

Because $D_\mu = \partial_\mu + ieA_\mu$, the same holds for the e.m. interactions, they do not mix the L, R states.

$$\psi_R : \quad i(\underset{1}{\sigma^0} \tfrac{d}{dt} + \sigma^i \tfrac{d}{dx^i})\psi_R \Rightarrow \psi_R = u_R(p)\, e^{-iEt + i\vec{p}\vec{x}}$$

$$m = 0 \Rightarrow E = |\vec{p}|$$

$u_R(p)$ is a 2-component Weyl spinor. Let's choose $\vec{p} = p\,\hat{3}$ and look for $u_R(p)$

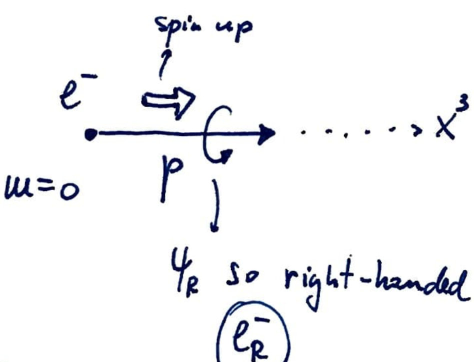
$$(E - p\sigma^3)\,u_R(p) = \begin{pmatrix} E-p & 0 \\ 0 & E+p \end{pmatrix}\begin{pmatrix} u_{R1} \\ u_{R2} \end{pmatrix} = 0$$

clearly there are $E>0$ & $E<0$ solutions.

$E = p$ $\Rightarrow$ $\begin{pmatrix} 0 & 0 \\ 0 & 2E \end{pmatrix}\begin{pmatrix} u_{R1} \\ u_{R2} \end{pmatrix} = 0 = \begin{pmatrix} 0 \\ 2E\,u_{R2} \end{pmatrix} \Rightarrow u_{R2} = 0$

$E > 0$

$$\psi_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-iEt + iEx^3}$$



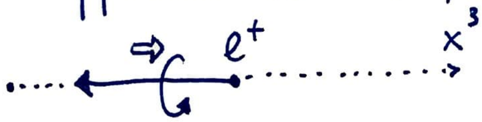
this spinor has $\sigma_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = +1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ therefore it is spin up. along x^3, which is its direction of motion

The operator ψ_R (ψ_R^+) destroys (creates) this state.

The second solution $E < 0 = -p$

$$\psi_R = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{+iEt + iEx^3}$$

This is a <u>positron</u> moving along $-x^3$ with spin ~~down~~ opposite to S_3, thus spin up along x^3



this is a LEFT HANDED e_L^+ state.

The $E=-p$ solution $\psi_R = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{iEt+iEx^3}$ creates a positron state.

<u>HELICITY</u> operator $h = \hat{p}\cdot S$, $\hat{p} = \frac{p^i}{|\vec{p}|}$

$\Rightarrow$ ψ_R w $E>0$ has $h=+\frac{1}{2}$ e_R^-

ψ_R w $E<0$ has $h=-\frac{1}{2}$ e_L^+

SIMILARLY, we have for ψ_L along $\vec{p} = p\,\hat{3}$

$E>0$ $\psi_L = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-iEt+iEx^3}$ LH e^- $m=0$

$E<0$ $\psi_L = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{+iEt+iEx^3}$ RH e^+ $m=0$

Thus we get the relevant matrix elements.

$$\langle 0 \,|\, \psi_R \,|\, \underset{\text{FINAL}}{e_R^-} \rangle = u_R(p)\, e^{-ipx}$$

$$\langle e_L^+ \,|\, \psi_R \,|\, 0 \rangle = v_L(p)\, e^{ipx}$$

- CONVENTION : $u(p)$ annihilates fermions, $v(p)$ creates anti-fermions

- <u>SUMMARY</u>:

with proper $\sqrt{2E}$ normalization

	ANNIHILATION		CREATION
e_R^-	$u_R = \sqrt{2E}\,\xi_+$	e_L^+	$v_L = \sqrt{2E}\,\xi_+$
e_L^-	$u_L = \sqrt{2E}\,\xi_-$	e_R^+	$v_R = \sqrt{2E}\,\xi_-$

spin up

spin down along $\vec{p}$

· INCLUDING THE FERMION MASS WILL MIX ψ_L & ψ_R

$$(i\gamma^\mu\partial_\mu - m)\psi = 0$$

$$i\bar\sigma\partial\psi_L - m\psi_R = 0$$

$$i\sigma\partial\psi_R - m\psi_L = 0$$

In the Lagrangian language, we have

$$\mathcal{L} = \bar\psi(i\not\partial - m)\psi$$

$$= \psi_R^\dagger i\sigma\partial\psi_R + \psi_L^\dagger i\bar\sigma\partial\psi_L - m(\psi_L^\dagger\psi_R + \psi_R^\dagger\psi_L)$$

8.3. EVALUATION OF MATRIX ELEMENTS $e^+e^- \to \mu^+\mu^-$

· The amplitude & cross-section depends on the spin states of the initial / final particles. With the solutions from above we can work out the $\langle j^\mu\rangle$ matrix elements.

$$\langle 0 | j^\mu | e_R^-(p_-)\, e_L^+(p_+)\rangle$$

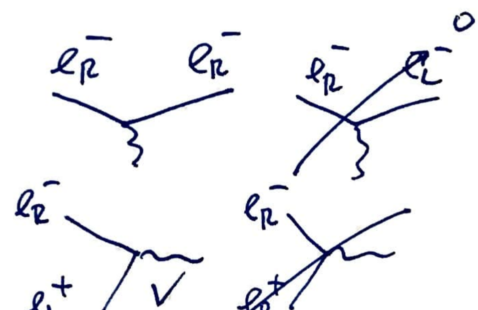
helicities

· where $j^\mu = \bar\psi\gamma^\mu\psi = \psi^\dagger\gamma^0\begin{pmatrix}0 & \sigma\\ \bar\sigma & 0\end{pmatrix}^\mu\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix} = (\psi_L^\dagger\ \psi_R^\dagger)\begin{pmatrix}\bar\sigma^\mu & 0\\ 0 & \sigma^\mu\end{pmatrix}\begin{pmatrix}\psi_L\\ \psi_R\end{pmatrix}$

$$= \underbrace{\psi_L^\dagger\bar\sigma^\mu\psi_L}_{\text{L-L pice}} + \underbrace{\psi_R^\dagger\sigma^\mu\psi_R}_{\text{R-R-pice}}$$ } THIS IS A VECTOR current

HELICITY CONSERVATION : when $m=0$:

when $m \neq 0$ the flip is suppresed by $\frac{m}{E}$



ME (matrix element) can be evaluated in the CMS

$$\xrightarrow{e^-} \quad \xleftarrow{e^+} \qquad m=0 \Rightarrow |p| = E$$

along $\hat{3}$

$$p_- = (E, E) \qquad p_+ = (E, -E)$$
$$= (E,0,0,E) \qquad = (E,0,0,-E)$$

$$\langle 0 | j^\mu | e_R^- e_L^+ \rangle = \langle 0 | \psi_R^+ \sigma^\mu \psi_R + \psi_L^+ \bar\sigma^\mu \psi_L | \underbrace{e_R^- e_L^+}_{\text{to be annihilated}} \rangle$$

$$= \langle 0 | \psi_R^+ \sigma^\mu \psi_R | e_R^- e_L^+ \rangle = v_L^+(p_+) \sigma^\mu u_R(p_-)$$

$$= \sqrt{2E} \begin{pmatrix} 0 \\ 1 \end{pmatrix}^+ (1, \vec\sigma)^\mu \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 2E \,(0 \;\, 1) \left(\cancel{1}^{0}, \sigma^1, \underset{\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}}{\underset{\shortparallel}{\sigma^2}}, \cancel{\sigma^3}^{0} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= 2E\,(0, 1, i, 0)^\mu$$

- This agrees with angular momentum conservation.

The $\varepsilon_+ = \frac{1}{\sqrt2}(\hat 1 + i \hat 2)$ is the $J_3 = +1$ polarization.

Thus the two spins $\quad \underset{e_R^-}{\xrightarrow{s_3 = 1/2 \;\Rightarrow}} \quad \underset{e_L^+}{\xleftarrow{s_3 = 1/2 \;\Rightarrow}} \quad \Rightarrow J_3 = \frac12$

Such that $\langle 0 | j^\mu | e_R^- e_L^+ \rangle = 2E\sqrt2 \underset{}{\overset{(0, \vec\varepsilon_+) \,\shortparallel}{\varepsilon_+^\mu}}$ ✓

$$[j^\mu] = \Big[\overset{j^0 = \rho}{\frac{1}{cm^3}}, \overset{j^i}{\frac{1}{cm^2 s}}\Big] = [GeV^3] = 3 \quad , \quad [|p_1 p_2\rangle] = [GeV^{-2}] = -2$$

$$\Rightarrow [\langle 0 | j | p_1 p_2 \rangle] = 3 - 2 = 1$$

$$\langle 0 | j^\mu | e_L^- e_R^+ \rangle = \langle 0 | \psi_L^+ \bar\sigma^\mu \psi_L | e_L^- e_R^+ \rangle = v_R^+(p_+) \bar\sigma^\mu u_L(p_-)$$

The L- e^- states give $\xrightarrow[e_L^-]{\Leftarrow}\xleftarrow[e_R^+]{\Leftarrow}$ $\Rightarrow J_3 = -1$

$$\langle 0 | j^\mu | e_L e_R \rangle = \langle 0 | \psi_L^+ \bar\sigma^\mu \psi_R \rangle e_L^- e_R^+ \rangle$$

$$= v_R^+(p_+) \bar\sigma^\mu u_L(p_-) = 2E \,(1\ 0)(1, -\sigma^1, -\sigma^2, -\sigma^3)\begin{pmatrix}0\\1\end{pmatrix}$$

$$= -2E\,(0, 1, -i, 0)^\mu = -2\sqrt{2}\,E\,\varepsilon_-^\mu$$

- The other combinations are suppressed by $\frac{m}{E}$.

- Similarly we have for the (massless) final state muons:

$$\langle \mu_R^- \mu_L^+(p_+') | j^\mu | 0 \rangle = 2\sqrt{2}\,E\,\varepsilon_+^{\prime *\mu}$$

$$\langle \mu_L^-(p_-') \mu_R^+(p_+') | j^\mu | 0 \rangle = 2\sqrt{2}\,E\,\varepsilon_-^{\prime *\mu}$$

energy is conserved & $m_\mu \cong 0$

- Putting it all together, we have

$$e_R^- e_L^+ \to \mu_R^- \mu_L^+ \qquad \mathcal{M} = -\frac{e^2}{q^2}\, 2\,(2E)^2\, \vec\varepsilon_+^{\,\prime *}\,\vec\varepsilon_+ = -2e^2\, \vec\varepsilon_+^{\,\prime *}\vec\varepsilon_+$$

$$p_+ + p_- = q = (2E, 0, 0, 0) \Rightarrow q^2 = (2E)^2$$

- To get explicit angular dependencies, we define

$$p_- = (E, 0, 0, E) \qquad p_-' = (E, Es, 0, Ec)$$

$$p_+ = (E, 0, 0, -E) \qquad p_+' = (E, -Es, 0, -Ec)$$

$$\underbrace{\varepsilon_\pm^\mu = (0, 1, \pm i, 0)\tfrac{1}{\sqrt{2}}}_{\text{INCOMING } e^- \,\&\, e^+} \qquad \varepsilon_\pm^{\mu\prime} = (0, c, \pm i, -s)/\sqrt{2}$$

$$\varepsilon' p' = 0, \quad \varepsilon^2 = 1$$

$$\Rightarrow \quad \vec\varepsilon_+^{\,\prime *}\vec\varepsilon_+ = \vec\varepsilon_-^{\,\prime *}\vec\varepsilon_- = \tfrac{1}{2}\left(c + (-i)(i) + -s\cdot 0\right) = \tfrac{1}{2}(1+c)$$

$$\vec\varepsilon_+^{\,\prime *}\vec\varepsilon_- = \vec\varepsilon_-^{\,\prime *}\vec\varepsilon_+ = \tfrac{1}{2}(1-c)$$

$$\mathcal{M}_{e_R^- e_L^+ \to \mu_R^- \mu_L^+} = -2e^2 \vec{\varepsilon}_+^{\,1*} \vec{\varepsilon}_+ = -\cancel{2} e^2 (1+\cos\theta) \tfrac{1}{\cancel{2}}$$

$$\mathcal{M}_{e_L^- e_R^+ \to \mu_L^- \mu_R^+} = -2e^2 \vec{\varepsilon}_-^{\,1*} \vec{\varepsilon}_- = -e^2(1+\cos\theta)$$

$$\mathcal{M}_{e_L^- e_R^+ \to \mu_R^- \mu_L} = -2\varepsilon^2 \varepsilon_+ \varepsilon_- = e^2(1-\cos\theta)$$

$$\vdots$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_R^- \mu_L^+)|^2 = e^4(1+\cos\theta)^2$$

$$|\mathcal{M}(e_R^- e_L^+ \to \mu_L^- \mu_R^+)|^2 = e^4(1-\cos\theta)^2$$

DONE, we have all the ME.

8.4. The $e^+e^- \to \mu^+\mu^-$ CROSS-SECTION

$e_R^- e_L^+ \to \mu_R^- \mu_L^+$ & $e_L^- e_R^+ \to \mu_L^- \mu_R^+$:

$$\sigma = \frac{1}{\underbrace{2E}_{E_{CM}}\, 2E\, 2} \int d\Pi_2 \, |\mathcal{M}|^2$$

$$= \frac{1}{2E_{CM}^2} \cdot \frac{1}{8\pi} \int_{-1}^{1} \frac{d(\cos\theta)}{2} e^4 (1+\cos\theta)^2$$

$$\frac{e^2}{4\pi} = \alpha$$

$$\frac{d\sigma}{dc} = \frac{\pi\alpha^2}{2E_{CM}^2}(1+c)^2$$

- for the other two : $\frac{d\sigma}{dc} = \frac{\pi\alpha^2}{2E_{CM}^2}(1-c)^2$

- average over all initial spin states & sum over final states

$$\Rightarrow \frac{d\sigma}{dc} = \frac{\pi\alpha^2}{2E_{CM}^2}\left(\frac{1}{2}\left((1+c)^2+(1-c)^2\right)\right) = \frac{\pi\alpha^2}{2E_{CM}^2}\cdot(1+c^2)$$

$$\sigma_{tot} = \frac{\pi\alpha^2}{2E_{CM}^2}\cdot \underbrace{\int_{-1}^{1} dc\,(1+c^2)}_{2+\frac{2}{3}} = \frac{4\pi\alpha^2}{3E_{CM}^2}, \quad [\sigma] = -2 \quad \checkmark$$

• a note on the σ units, barn $= 10^{-24}\,cm^2 = 100\,fm^2$

REMEMBER : $\hbar c = 0.4\ GeV^2\,mb$

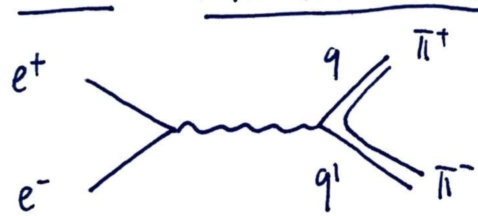
p-p x-sec ~ 0.1 barn

$$\sigma_{e^+e^-\to\mu^+\mu^-} = \frac{4\pi\alpha^2}{3E_{cm}^2} \simeq \frac{\cancel{4}}{\cancel{137^2}\ \cancel{36}\ 9}\ \frac{\overset{0.1}{\cancel{0.4}}\,mb}{E_{cm}^2} \simeq \frac{\overset{10^{-7}b}{0.01^2\,mb}}{E_{cm}^2} \sim \frac{100\,nb}{E_{cm}^2}$$

$$= \frac{87\,nb}{E_{cm}^2}$$. This is valid to about 30 GeV $= E_{cm}$

Above, the off-shell Z contributes

8.5. e^+e^- ANNIHILATION TO HADRONS



π's, K's spin 0

ρ, ω, K* spin 1

$E_{cm} >> GeV$

• q's are spin ½ particles, therefore up to hadronization the x-section $e^+e^- \to q^+q^-$ will behave as above. ($m_q \sim 0$)

However, the number of colors and generations are important and affect the σ. We also have to change the charge from $Q(e^-) = Q(\mu^-) = -1$ to $Q(u,c,t) = \frac{2}{3}$, $Q_{dsb} = -\frac{1}{3}$ & sum over colors $\Rightarrow N_c = 3$.

$$\frac{d\sigma}{dc}\left(e^+e^- \to \text{hadrons}\right) \propto 1+\cos^2\theta = 1+c^2$$

$$\sigma\left(e^+e^- \to \text{hadrons}\right) = \sum_q Q_q^2 N_c \frac{4\pi\alpha^2}{3E_{cm}^2}$$

for u, d, s

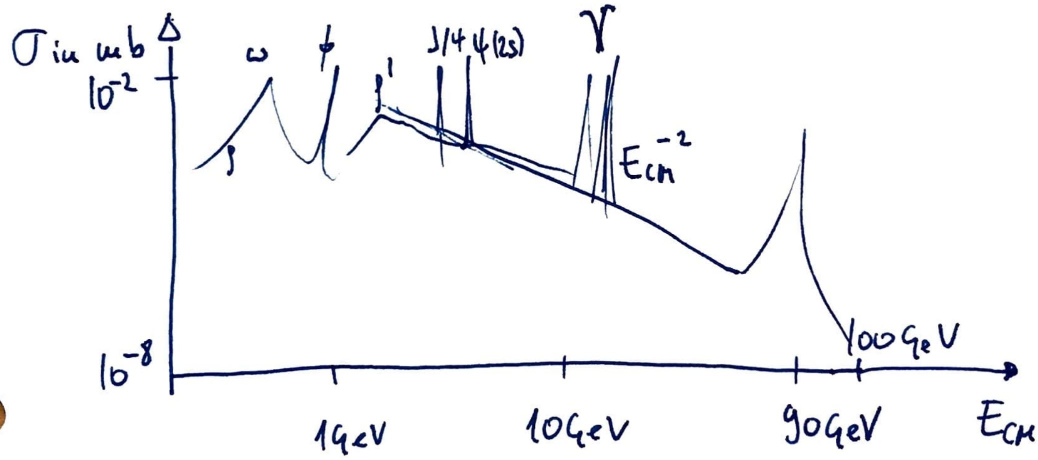
$$R = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum Q_q^2 = \begin{cases} 3\cdot\left(\left(\frac{2}{3}\right)^2\cdot 2+\left(-\frac{1}{3}\right)^2\right) = \frac{4+2}{3} = 2 \\ 3\cdot 2\left(\left(\frac{2}{3}\right)^2+\left(\frac{1}{3}\right)^2\right) = 2\,\frac{5}{3} = 3\frac{1}{3} \\ 3\frac{1}{3} + 3\cdot\frac{1}{9} = 3\frac{2}{3} \end{cases}$$

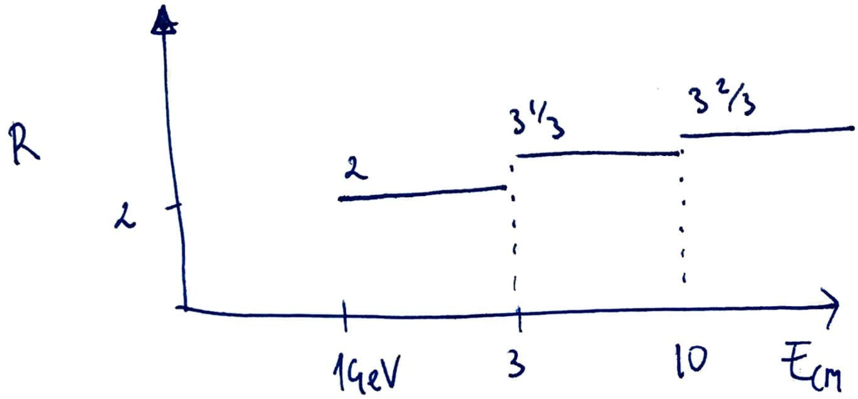
$E_{cm} < 4$…

u, d, s, c

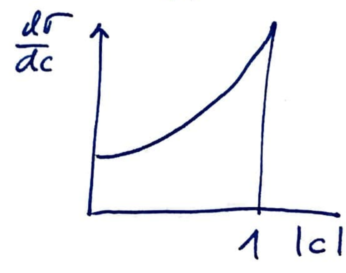
$$R \cong \begin{cases} 2 & E < 2m_c \sim 3\,\text{GeV} \\ 3.\bar{3} & m_c < E < 2m_b \sim 8\,\text{GeV} \\ 3.\bar{6} & E > 8\,\text{GeV} \end{cases}$$

EXPERIMENTAL RESULTS (see also PDG plots)





• angular jet distributions : $\frac{d\sigma}{dc}$



$\simeq 1+c^2$