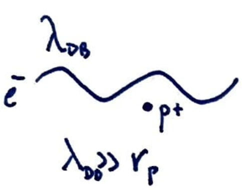
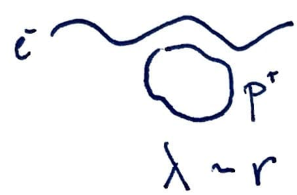
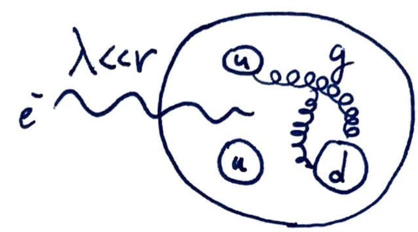
9. DEEP INELASTIC SCATTERING

- Previously we ignored the strong interaction (we only considered the photon, QED interactions).

Protons are composite, can be seen by $ep \to ep$ scattering.







LOW E = LONG WAVELENGTH

ELASTIC

e^- BREAKS the p apart, sees the components

INELASTIC

- This DEEP INELASTIC regime when $\lambda \ll r_p$ happens for $E_e \gtrsim$ GeV and remarkably we can describe this scattering as if e^- bounces off individual constituents.

- We now refer to these parts as quarks and gluons, or collectively partons. This was first measured by the SLAC-MIT experiment in the late 60's.

9.1. THE SLAC-MIT experiment

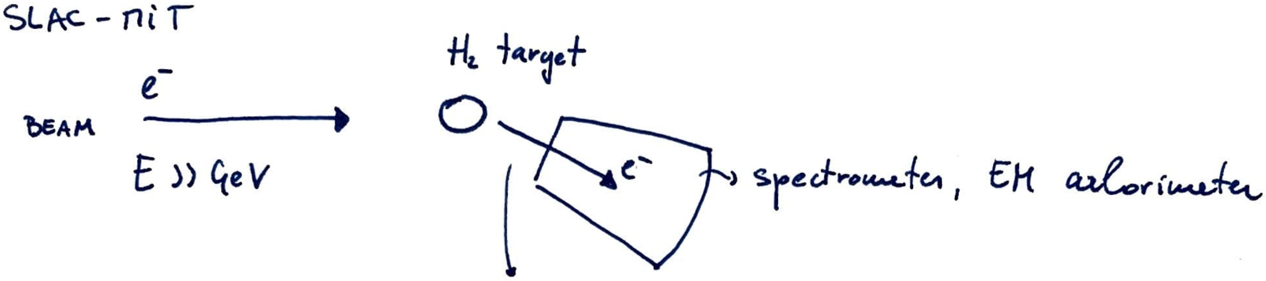
'50s Hofstadter : ep & e-nuclei scattering

$\Downarrow$

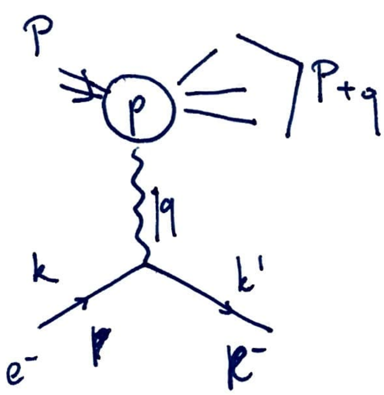
r_p, charge distribution, similar for nuclei

$\lambda_{DB} = \frac{1}{p}$ for $p > m_p \sim \text{GeV} \Rightarrow$ resolve the proton structure ; $\frac{\hbar c}{\text{GeV}} \sim \frac{0.2\ \cancel{\text{GeV}}\ \text{fm}}{\cancel{\text{GeV}}} \sim r_p$

SLAC - MIT



measure p^μ of the outgoing e^-



- we start with an e^- w. k that scatters to e^- w. k'
- the initial proton p w. momentum P disintegrates to partons with the collective momentum $P+q$, where $q = k - k'$ is the momentum transfer in the process.

$$\boxed{\begin{aligned} &E^2 = p^2 + m^2,\ E \sim p + \frac{m^2}{2p} \\ &k = \left(p + \tfrac{1}{2}\left(\tfrac{m_e}{p}\right)^2 p,\ p\right) \\ &P = \left(p + \tfrac{1}{2}\left(\tfrac{m_p}{p}\right)^2 p,\ -p\right) \\ &k - P = q = \Big(\underbrace{\tfrac{m_e^2 - m_p^2}{2p}}_{\simeq 0},\ 2p\Big) \\ &q^2 = -4p^2 < 0 \end{aligned}}$$

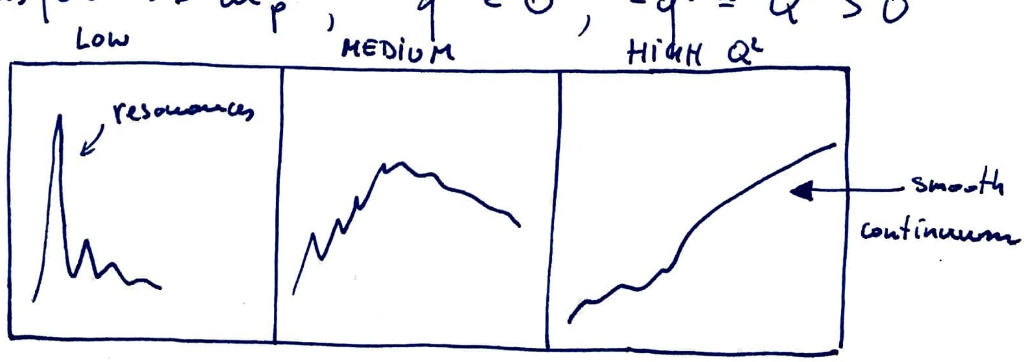
This is the ME : $\langle e^-(k^-) | j^\mu | e^-(k) \rangle$

$$W^2 = (P+q)^2 = P^2 + 2Pq + q^2 = m_p^2 + 2P\cdot q + q^2$$

more specifically

Energy transfer $\gg m_p$, $q^2 < 0$, $-q^2 = Q^2 > 0$

EXPERIMENT :



9.2. The Parton Model

CM frame : $\xrightarrow{e^-} \qquad \Longleftarrow p$

partons move collectively within the proton, carrying fractions of its momentum

- In the DIS regime $p_T \sim m_p$ or less, thus we can ignore it & write

$$\underset{\text{parton momentum}}{p^\mu} = \underset{\text{parton fraction},\ <1}{\xi}\ \underset{\text{proton momentum}}{P^\mu}$$

$\xi \in [0,1]$ & $f_i(\xi)\,d\xi$ is the parton distribution function

- Conservation of probability

$$\int_0^1 d\xi \sum_{\substack{\text{all}\\ \text{partons}}} f_i(\xi)\, d\xi = 1$$

- Similarly, any quantity in this picture is obtained by convoluting over the PDFs. This includes the cross-section

$$\sigma(e^- p \to e^- X) = \int d\xi \sum_{\substack{\text{all}\\ \text{partons } q}} f_q(\xi)\, \sigma(e^- q(\xi P) \to e^- q)$$

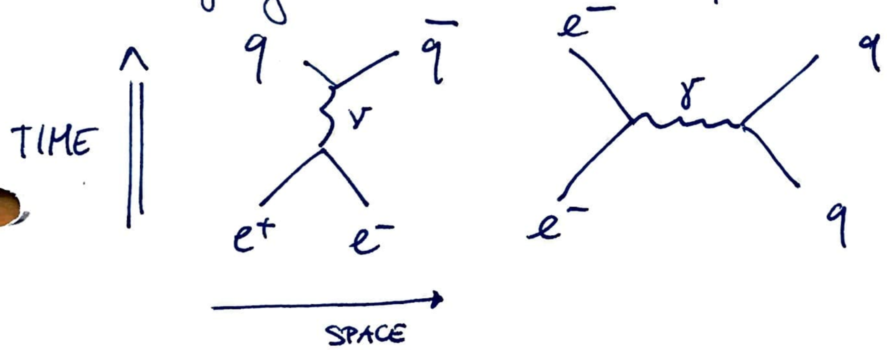
this q then hadronises & forms jets.

- With this picture in mind, it is straightforward to write down the matrix element

$$\mathcal{M}(e^- q_f \to e^- q_f) = -e \langle e^-(k')| j^\mu | e^-(k)\rangle \frac{1}{q^2} (Q_f e) \langle q_f(P+q)| j_\mu | q_f(\xi P)\rangle$$

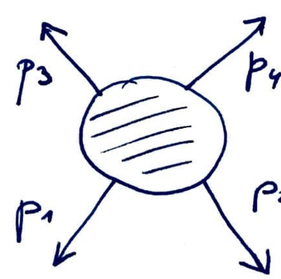
- To evaluate this amplitude, we will employ the <u>crossing</u> symmetry. This symmetry manifestly displays that certain relations exist when incoming and outgoing particles are interchanged.

FOR EXAMPLE: $e^+ e^- \to q \bar{q} \sim e^- q \to e^- q$ by rearranging the initial / final states.



THIS IS USEFUL TO WORK OUT IN SOME GENERALITY

$1 + 2 \to 3 + 4$ OR



all are outgoing, therefore

$p_1 + p_2 + p_3 + p_4 = 0$

$p_i^2 = m_i^2, \; p_1^2 = m_1^2 \ldots$

and introduce the Mandelstam variables

$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t = (p_1+p_3)^2 = (p_2+p_4)^2$$
$$u = (p_1+p_4)^2 = (p_2+p_3)^2$$

. this allows us to simplify the scalar products $p_i \cdot p_j$

↝ e.g. $s = m_1^2 + m_2^2 + 2p_1p_2 = m_3^2 + m_4^2 + 2p_3p_4$ symmetrized s

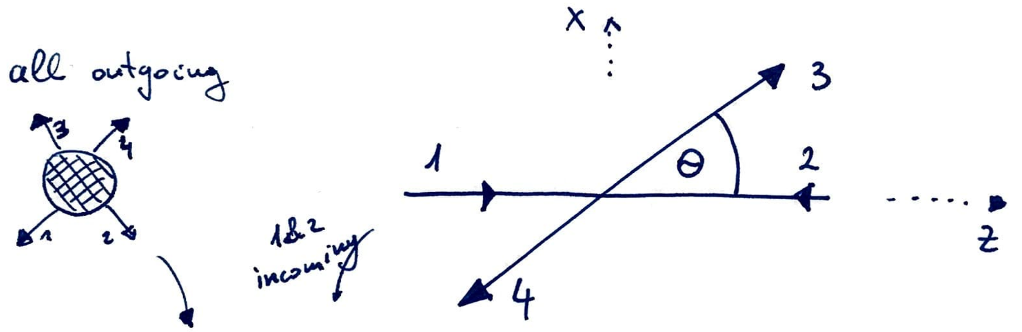
Note that $s+t+u = \frac{1}{2}\big(p_1^2 + 2p_1p_2 + p_2^2 + p_3^2 + 2p_3p_4 + p_4^2 + \dots$

$$= \cancel{\tfrac{1}{2}}\Big(\underbrace{(p_1+p_2+p_3+p_4)^2}_{=0} + \cancel{2}\left(p_1^2+p_2^2+p_3^2+p_4^2\right)\Big)$$

$$= \sum m_i^2 = m_1^2 + m_2^2 + m_3^2 + m_4^2 \text{, so}$$

one scalar product less.

- Thus instead of six p_ip_j we have four less, only 2.
- Let's evaluate these in the CM frame



$$p_1 = -(E, 0, 0, E) \qquad p_3 = (E, Es, 0, Ec)$$

$$p_2 = -(E, 0, 0, -E) \qquad p_4 = (E, -Es, 0, -Ec)$$

$s = (p_1+p_2)^2 = (2E)^2 = (p_3+p_4)^2 = E_{CM}^2$ ↝ true even for $m_i \neq 0$.

- convention $E_{CM} = \sqrt{s}$

$$t = (p_1+p_3)^2 = (\underbrace{E-E}_{0})^2 - (Es)^2 - (1-c)^2E^2$$

$$= -E^2(s^2 + c^2 + 1 - 2c) = -2E^2(1-c) = (p_2+p_4)^2$$

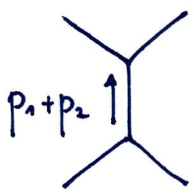
$$u = (p_1 + p_4)^2 = 0^2 - (Es)^2 - E(1+c)^2 = -2E^2(1+c)$$

$$s = 4E^2, \quad t = -2E^2(1-c), \quad u = -2E^2(1+c)$$

$$\Rightarrow \quad s + t + u = \sum m_i^2 = 0 \quad \underline{ok}.$$

How to implement the crossing symmetry?

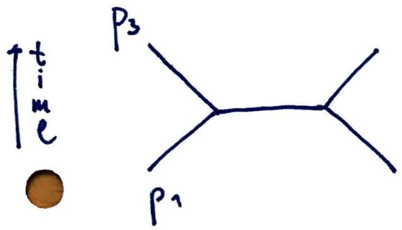
shift / permute s, t & u as the initial & final states change around.



$$\frac{1}{\underbrace{(p_1+p_2)^2}_{s} - m_R^2 + i m_R \Gamma_R}$$

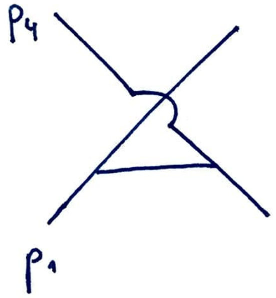
Breit-Wigner resonance

→ this is an <u>s-channel</u> process



$$\frac{1}{(p_1+p_3)^2 - m_R^2 + i m_R \Gamma_R}$$

<u>t-channel</u>



$$\frac{1}{(p_1+p_4)^2 - m_R^2 + i \Gamma_R m_R}$$

<u>u-channel</u>

- for a massless resonance $m_R = 0$:

$$\left| \quad \right|^2 (s\text{-ch}) = \left| \frac{1}{q^2} \right|^2 \sim \frac{1}{s^2} = \frac{1}{E_{CM}^4}$$

* as in e^+e^- annihilation

• note that the θ-dependence in e^+e^- annihilation came from the numerator, the Dirac algebra

t-chan: $\left| \right.$  $\left. \right|^2 \sim \frac{1}{t^2} = \frac{}{4E_{cm}^4(1-\cos\theta)^2} = \frac{1}{16 E_{cm}^4 \sin^4 \theta/2}$

$t = -2E^2(1-\cos\theta)$ $\qquad \sin^2\frac{x}{2} = \frac{1-\cos x}{2}$

diverges when $\theta = 0$

u-chan: $\left| \right.$  $\left. \right|^2 \sim \frac{1}{u^2} = \frac{1}{4E_{cm}^4(1+C_\theta)^2} = \frac{1}{16 E_{cm}^4 \cos^4\frac{\theta}{2}}$

peaks in the backward direction when $\theta = \pi$

9.4. Electron-quark cross-section

Let us apply the crossing "technology" to ~~the~~ convert the $e^+e^- \to \mu^+\mu^-$ calculation to the $e^-q \to e^-q$.

We had: $|\mathcal{M}_{e_R e_L^+ \to q_R \bar{q}_L}|^2 = |\mathcal{M}_{e_L^- e_R^+ \to q_L \bar{q}_R}|^2 = Q_f^2 e^4 (1+\cos\theta)^4 = (1)$

$|\mathcal{M}_{e_R^- e_L \to q_L \bar{q}_R}|^2 = |\mathcal{M}_{e_L^- e_R^+ \to q_R \bar{q}_L}|^2 = Q_f^2 e^4 (1-\cos\theta)^4 = (2)$

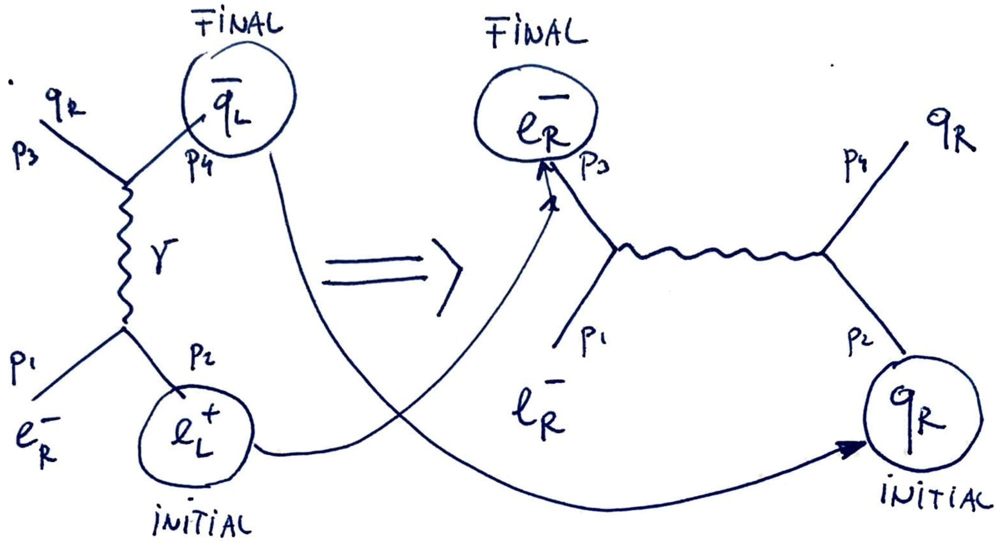
$(1) = 4 Q_f^2 e^4 \frac{u^2}{s^2}$ since $s = E^2$, $t = -2E^2(1-\cos\theta)$, $u = -2E(1+\cos\theta)$

$(2) = 4 Q_f^2 e^4 \frac{t^2}{s^2}$

• The amplitudes are written in $s, t, u \Rightarrow$ LI and apply to any inertial Lorentz frame. Now we simply swap the external legs to implement the crossing, i.e.

annihilation :

$e_L^+ e_R^- \to q_R \bar{q}_R$



• Notice that p_1 ~~and p~~ stays the same, while

$$p_2 \to p_3, \quad p_3 \to p_4, \quad p_4 \to p_2$$

thus $s = (p_1+p_2)^2 \to (p_1+p_3)^2 = t = -2E^2(1-\cos\theta)$

and $t = (p_1+p_3)^2 \to (p_1+p_4)^2 = u = -2E^2(1+\cos\theta)$

$u = (p_1+p_4)^2 \to (p_1+p_2)^2 = s = E^2$

so $(1) = 4Q_f^2 e^4 \dfrac{u^2}{s^2} \xrightarrow{\text{CROSSING}} 4Q_f^2 e^4 \dfrac{s^2}{t^2} \propto \dfrac{1}{(1-\cos\theta)^2}$

and $(2) = 4Q_f^2 e^4 \dfrac{t^2}{s^2} \longrightarrow 4Q_f^2 e^4 \dfrac{u^2}{t^2} \propto \left(\dfrac{1+\cos\theta}{1-\cos\theta}\right)^2$

vanishes for $\cos\theta = -1$ $\quad e_R^- q_L$: $\xrightarrow[\Rightarrow]{e_R^-\ p} \xleftarrow[\Rightarrow]{q_L}$ $\not\to$ $\xleftarrow[\Leftarrow]{e_R^-} \xrightarrow[\Leftarrow]{q_L}$

@ $\theta = \pi$ $\quad s = 1 \quad s = -1$

- We have all the amplitudes that enter in the cross-section

$$\sigma(eq \to eq) = \frac{1}{\underbrace{2E\,2E}_{s}\,2}\,\frac{1}{8\pi}\int_{-1}^{1}\frac{dc_\theta}{2}\,\frac{1}{4}\sum_{\text{spin}}|\mathcal{M}(e^-q \to e^-q)|^2$$

here the color is preserved, γ carries no color index

$e_L q_L, e_R q_R$ $\quad$ $e_L q_R, e_R q_L$

$$\frac{d\sigma}{dc_\theta} = \frac{1}{2s}\,\frac{e^4}{16\pi}\,\frac{2}{4}\left(4Q_f^2\,\frac{s^2+u^2}{t^2}\right)$$

$$\alpha = \frac{e^2}{4\pi}, \quad e^4 = 16\pi^2\alpha^2$$

$$= \frac{1}{\cancel{2}s}\,\pi\alpha^2\,\frac{2}{\cancel{4}}\left(4Q_f^2\,\frac{s^2+u^2}{t^2}\right)$$

$$= \frac{\pi Q_f^2\alpha^2}{s}\,\frac{s^2+u^2}{t^2}.$$

from $s = (2E)^2$, $t = -2E^2(1-\cos\theta)$, $t \in [-4E^2, 0] < 0$
$= [-s, 0]$

$$dt = +2E^2(+d\cos\theta) = \frac{1}{2}s\,dc_\theta$$

$$\frac{d\sigma}{dt} = \frac{d\sigma}{dc_\theta}\frac{dc_\theta}{dt} = \frac{\pi Q_f^2\alpha^2}{s}\,\frac{s^2+u^2}{t^2}\cdot\frac{2}{s} = \frac{2\pi Q_f^2\alpha^2}{s^2}\,\frac{s^2+u^2}{t^2}.$$

We ended up with a Lorentz invariant formulation of the eq cross-section, written solely in terms of the Mandelstam variables s, t & u

$$\frac{d\sigma}{dt} = \frac{2\pi Q_f^2\alpha^2}{s^2}\,\frac{s^2+u^2}{t^2}.$$

9.5. Deep Inelastic Scattering

- promote $s, t, u \longrightarrow \hat{s}, \hat{t}, \hat{u}$ for individual partons
- integrate over the PDFs and over $\hat{t}$

$$\sigma_{ep\to op} = \int_0^1 d\xi \int_{-\hat{s}}^{0} d\hat{t} \sum_f \left(f_f(\xi) + f_{\bar{f}}(\xi)\right) \frac{2\pi Q_f^2 \alpha^2}{\hat{s}^2}\left(\frac{\hat{s}^2+\hat{u}^2}{\hat{t}^2}\right)$$

- sum over all the partons, anti-partons.

- $\hat{t} = q^2 = -Q^2 \Rightarrow$ directly measured in DIS

* remember: $q = k - k' = p_1 + p_3$

- $s = (k+P)^2$

$$= \cancel{m_e^2}^{0} + \cancel{m_p^2}^{0} + 2kP$$

- $\hat{s} = (k+p)^2 = \cancel{m_e^2} + \cancel{m_q^2} + 2kp = \xi 2kP \quad , \quad p = \xi P$

$\Rightarrow \hat{s} = \xi s$

- Introduce the dimensionless DIS variable y

$$y = \frac{2P\cdot q}{2P\cdot k} = \frac{\cancel{2m_p}\, q^0}{\cancel{2m_p}\, k^0} \qquad \text{rest} : P = (m_p, 0, 0, 0)$$

$\ni [0,1]$

$$= \frac{q^0}{k^0}$$

$q^0 \leftarrow$ diminished, transfered E $\qquad q = (q^0, \dots)$, $q = k - k'$

$k^0 \leftarrow$ inital E of e^- $\qquad k = (k^0, 0, 0, k)$

< 1

On the other hand we can relate y to the partonic $p = \xi P$ via

$$y = \frac{2\xi P\cdot q}{2\xi P\cdot k} = \frac{2p(k-k')}{2pk} = \frac{\hat{s} + 2p_1p_4}{\hat{s}} = \frac{\hat{s}+\hat{u}}{\hat{s}}$$

$$= 1 + \frac{\hat{u}}{\hat{s}} \Rightarrow \left(\frac{\hat{u}}{\hat{s}}\right)^2 = (1-y)^2$$

- this is what entered in the partonic amplitude

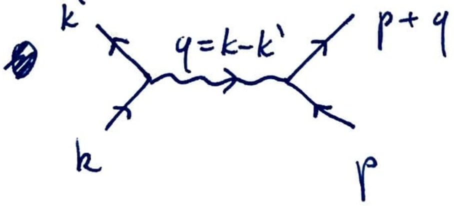
$$\frac{\hat{s}^2+\hat{u}^2}{\hat{s}^2} = 1 + (1-y)^2 \quad \& \quad \hat{t}^2 = (-Q^2)^2$$

$$\hat{t} = -Q^2$$

$$d\hat{t} = -dQ^2 > 0$$

$$\sigma_{ep\to ep} = \int_0^1 d\xi \int_0^{\hat{s}} dQ^2 \sum_{f\bar{f}} f_f(\xi)\, 2\pi Q_f^2 \alpha^2 \frac{1+(1-y)^2}{Q^4}$$

Finally :



$$(p+q)^2 = m_q^2 = 0$$

$$= 2pq + q^2 + \cancelto{0}{m_q^2}$$

$$= 2\xi Pq - Q^2$$

$$\Rightarrow 2\xi Pq = Q^2$$

$$\xi = x = \frac{Q^2}{2Pq}$$

$\downarrow$

the DIS variable

$$x = \frac{Q^2}{2P\cdot q}$$

$$y = \frac{2Pq}{2Pk} = \frac{Q^2}{sx}$$

$$\Rightarrow Q^2 = xys \Rightarrow \quad d\hat{t} = -dQ^2 = -xs\,dy, \text{ for fixed } x$$

We can finally get the distributions over the measurable x & y DIS variables

$$\frac{d\sigma}{dx\,dy}(ep \to ep) = \sum_{f,\bar{f}} \underbrace{Q_f^2\, x f}_{F_2(x)} \overbrace{\frac{2\pi\alpha^2 s}{Q^4}\left(1+(1-y)^2\right)}^{\text{QED } \sigma}$$

$$x = \frac{Q^2}{2P\cdot q} \quad , \quad y = \frac{2Pq}{2Pk} \quad , \quad x, y \in (0,1)$$

$$\frac{d\sigma_{ep\to ex}}{dx\,dy} = \underbrace{F_2(x)}_{} \frac{2\pi\alpha^2 s}{Q^4}\left(1+(1-y)^2\right)$$

form-factor, includes all the partons, independent of Q^2 => Björken scaling