PARTON DISTRIBUTION FUNCTIONS

10. QUANTUM CHROMODYNAMICS (QCD)

PROTON = quarks, anti-quarks and gluons

↓ PDF ... parton distribution function

each with its own pdf $f_{q,\bar{q},g}(x)$

- There are dedicated libraries for fast access to these functions (LHAPDF, MSTW (Mathematica), ...)

- DIS model : $F_2(x) = \sum_f Q_f^2 \, x \left(f_f(x) + f_{\bar{f}}(x) \right)$

10.1. Measurements of parton distribution functions

from the model: $Q_u = \frac{2}{3}$, $Q_d = -\frac{1}{3}$

$$F_2(x) = \frac{4}{9} x f_u + \frac{1}{9} x f_d$$

two u's are counted here, see Q_p

- the rest come from virtual pair-creation $q\bar{q}$, yes, there are parts of s, b and t in the proton, also $\bar{u}$, $\bar{d}$ and $\bar{s}$.

PDFs obey sum rules due to conservation of charge, isospin and strangeness

→

$Q_p = 1 \quad : \quad \int_0^1 dx\, (f_u - f_{\bar{u}}) = 2 \quad \Rightarrow$ two u-quarks

$\sum_{sea} u + \bar{u} = 0$

$I_p = \frac{1}{2} \quad : \quad \int_x (f_d - f_{\bar{d}}) = 1$

$S = 0 \quad : \quad \int_x (f_s - f_{\bar{s}}) = 0$; similarly for b, c, t..

- from DIS $e^- p \to e^- p + X$, one gets PDFs of the proton

- scattering on $\underline{\underline{D}}$ (= deuterium) gives info on the neutron. Due to isospin

$$f^{(neutron)}_{u,\bar{u},d,\bar{d}} = f_{d,\bar{d},u,\bar{u}}$$

unlabeled is for the proton, by default

$$f^{(n)}_{q \neq u,d} = f_{q \neq u,d}$$

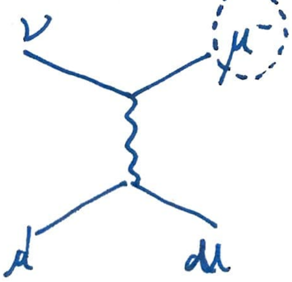
- similarly, $F_2^{(n)}(x) = \frac{4}{9} x f_d + \frac{1}{9} x f_u$.

$\Downarrow$

measuring $F_2, F_2^{(n)}$ gives separate info on $f_{u,d}$

- scattering on $\underline{\nu}$'s also brings in new information

via weak interaction



measure the sign of μ

charge conservation

$: \nu + d \to \mu^- + u$

$\bar{\nu} + \bar{u} \to \mu^+ + d$

on anti-quarks : $\nu + \bar{u} \to \mu^- + \bar{d}$

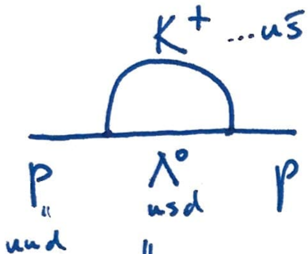
$\bar{\nu} + \bar{d} \to \mu^+ + u$

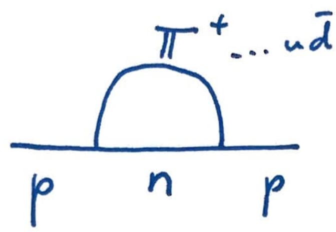
$\Rightarrow f_{u,d}$

- similarly K's may be produced instead of d's

$\Rightarrow f_{s,\bar{s}}(x)$ ... strange PDFs

- virtual contributions add the s & $\bar{s}$ differently in X





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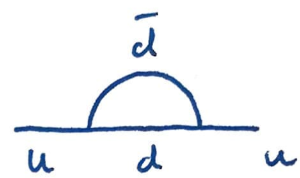
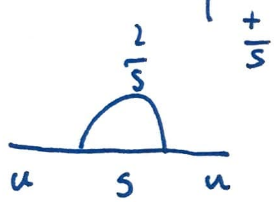


$\Downarrow$ adds s-quark



$\Downarrow$ adds $d+\bar{d}$

$\Downarrow$

we expect more $\bar{d}$ than $\bar{u}$ in the proton

* fig w. pdFs

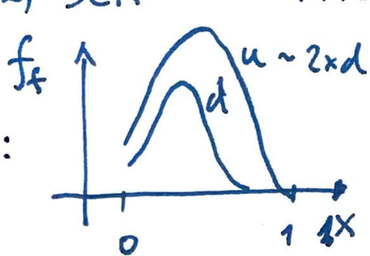
- high momentum proton : p carried by one u-quark
ud pair that remains forms a $I=0$ (from $1/2 \times 1/2$)
& $S=0$ state

QUANTITATIVE PICTURE with

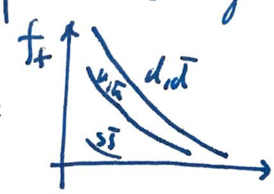
1) VALENCE PDFs (like 2u & 1d for the p)

2) SEA PDFs ($q\bar{q}$ pairs & gluons, virtual states)

1) :



2) :



→ large at small x values, diverges at x=0, renormalized in PDFs

- PDFs are fit to data at different momentum transfer, characterized by Q & x.

- Different collaborations, NNPDF, PDF4LHC, MSTW, CTEQ, ... use various approaches and data to provide global fits, accessible by the LHAPDF wrapper.

- Finally, momenta have to sum up to the total proton momentum $P = x_i p_i$, therefore:

$$\int_0^1 dx \sum_i x f_i(x) = 1$$

* note that divergencies of u & $\bar{u}$ cancel for small x to have the charge well defined, the extra x in the integral makes

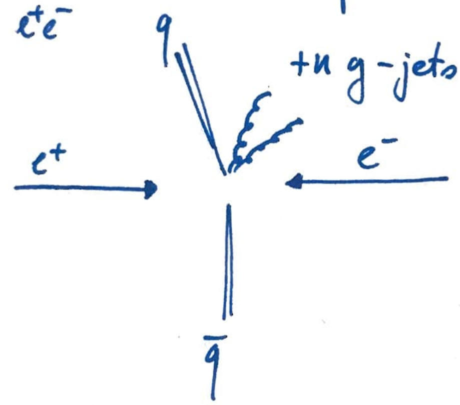
$$\frac{P_{q+\bar{q}}}{P} = \int_0^1 x \sum_i \left(f_i(x) + f_{\bar{i}}(x) \right) \sim 0.5 \quad \underline{\text{finite}},$$

but less than 1. Additional partons are needed!

$\Rightarrow$ $\boxed{\text{GLUON}}$ massless spin 1 mediator of strong interactions via vector current

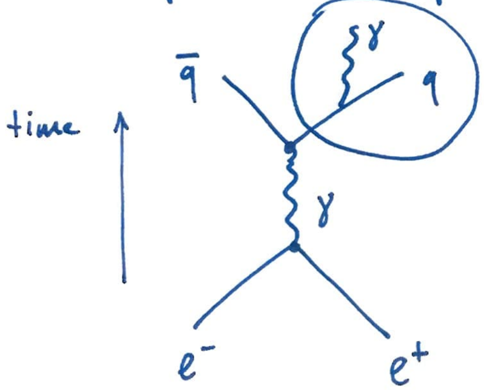
- The gluon should act similarly to a photon and if we have bremsstrahlung of $e^+e^- \to \mu^+\mu^-\gamma$, then gluons may be radiated away in $\underbrace{\underbrace{e^+e^- \to q\bar{q}}_{2\text{jets}} + g}_{3\text{jets}}$

- These events were in fact seen readily in the '70s by PETRA, SLD, TASSO experiments.



10.2. Photon emission in $e^+e^- \to q\bar{q}$

- To warm-up for gluon emission, let's consider the simple QCD process of photon emission by quarks.



... easy for high momentum final state to radiate if the directions doesn't change much

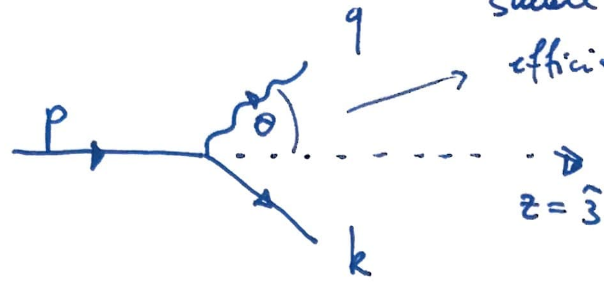
$\Downarrow$

COLLINEAR SPLITTING

$e^+e^- \to \bar{q} q \gamma$

small angle "efficient" radiation

KINEMATICS



$\theta \ll 1$

$p \approx (E, 0, 0, E)$, $m_q \simeq 0$, $p = q + k$

$q = \left(zE \, , \, q_\perp \, , \, 0 \, , \, zE - \frac{q_\perp^2}{2zE} \right)$, $q_\perp < E$ → small transverse momentum transfer

$k = \left((1-z)E \, , \, -q_\perp \, , \, 0 \, , \, (1-z)E - \frac{q_\perp^2}{2(1-z)E} \right)$

- We are producing on-shell photons and quarks $\Rightarrow$ $q^2 = k^2 = 0$.

$\Rightarrow$ the original ($m_q = 0$) quark must be off-shell with

$$p_{\hat{3}} = E - \frac{q_\perp^2}{2zE} + \frac{q_\perp^2}{2(1-z)E} = E - \frac{q_\perp^2}{2z(1-z)E}$$

$$\Rightarrow \quad p^2 = E^2 - E^2 + \frac{q_\perp^2}{z(1-z)} = \frac{q_\perp^2}{z(1-z)} + \mathcal{O}\left(\left(\frac{q_\perp}{E} \right)^4 \right)$$

- In the amplitude, the off-shell quark is $\sim \frac{1}{p^2} \sim \frac{z(1-z)}{q_\perp^2}$, thus the process is enhanced at small $q_\perp$.

$$\mathcal{M}(e^+e^- \to q\bar{q}\gamma) = \mathcal{M}(e^+e^- \to \bar{q}q) \, \frac{1}{p^2} \, \mathcal{A}(q \to q\gamma)$$

· Let us choose the initial q_R : $\langle f | j | i \rangle$

splitting : $\mathcal{M}(q_R \to \gamma \cdot q_R) = Q_f e \langle q_R(k) | j^\mu | q_R(p) \rangle \, \varepsilon^*_\mu(q$

$$j^\mu = \psi_R^\dagger \sigma^\mu \psi_R + \psi_L^\dagger \bar{\sigma}^\mu \psi_L$$

$$\mathcal{M}(q_R \to q_R \gamma) = Q_f e \; u_R^\dagger(k) \sigma^\mu u_R(p) \; \varepsilon^{\mu *}(q)$$

- We need the spinors $u_R^\dagger(k)$ & $u_R(p)$ with E & $p_\perp$

$$u_R(p) = \sqrt{2E} \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_R(k) = \sqrt{2(1-z)E} \begin{pmatrix} 1 \\ -\frac{q_\perp}{2(1-z)E} \end{pmatrix}$$

· We can radiate: $\varepsilon_R = \frac{1}{\sqrt{2}} \left(0, 1, i, \boxed{-\frac{q_\perp}{zE}}\right)$

$$\varepsilon_L = \frac{1}{\sqrt{2}} \left(0, 1, -i, \boxed{-\frac{q_\perp}{zE}}\right)$$

rotated along q : $\underline{\varepsilon_X \cdot q = 0}$.

· For LH γ :

$$u_R^\dagger(k) \sigma \cdot \varepsilon_L^* \, u_R(p) = 2E \sqrt{1-z} \left(1, -\frac{q_\perp}{2(1-z)E}\right)$$

$$\times \frac{1}{\sqrt{2}} \left(\cancel{0 \cdot \sigma^0} + \underbrace{\sigma^1}_{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} + \underbrace{i\sigma^2}_{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}} - \frac{q_\perp}{zE} \underbrace{\sigma^3}_{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \right) \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= \frac{2E\sqrt{1-z}}{\sqrt{2}} \left(-\frac{q_\perp}{\cancel{2(1-z)E}} + \frac{q_\perp}{\cancel{2(1-z)E}} - \frac{q_\perp}{zE} \right)$$

$$= -\sqrt{2}\, q_\perp \frac{\sqrt{1-z}}{z}.$$

- For the right-handed photon, we have

they sum up now

$$u_R^+ \sigma \, \varepsilon_R^* \, u_R = 2E\sqrt{1-z}\left(1-\frac{q_\perp}{2(1-z)E}\right)\frac{1}{\sqrt{2}}\left(\sigma^1 - i\sigma^2 - \frac{q_\perp}{zE}\sigma^3\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$= \frac{2E\sqrt{1-z}}{\sqrt{2}}\left(-\frac{\cancel{2}\,q_\perp}{\cancel{2}(1-z)E} - \frac{q_\perp}{zE}\right)$$

$$= -\sqrt{2}\sqrt{1-z}\; q_\perp \frac{\cancel{z}+1-\cancel{z}}{z\,(1-z)} = -\sqrt{2}\, q_\perp \frac{\sqrt{1-z}}{z(1-z)}\,.$$

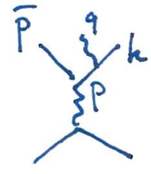
- Summing the two amplitudes together, we have

$$\sum_{LR} |\mathcal{M}(q\to q\gamma)|^2 = (Q_f e)^2 \, 2 q_\perp^2 \,(1-z)\underbrace{\left(\frac{1}{z^2} + \frac{1}{z^2(1-z)^2}\right)}_{\frac{(1-z)^2+1}{z^2(1-z)^2}}$$

- OK, we have the strahlung $|\mathcal{M}|^2$, now we need the production part $e^+e^-\to q\bar{q}$ and integrate over the phase space:

$$\sigma_{e^+e^-\to q\bar{q}\gamma} = \frac{1}{2E_A\, 2E_B\, 2}\int d\Pi_3 \,|\mathcal{M}(e^+e^-\to q\bar{q}\gamma)|^2$$

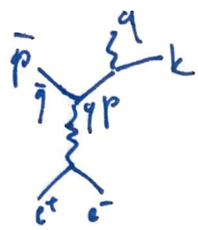
<u>PHASE SPACE</u> $\int d\Pi_3$



$$Q = k + q + \bar{p}$$

$$\int d\Pi_3 = \int \frac{d^3\bar{p}\; d^3k\; d^3q}{(2\pi)^9\; 2\bar{p}\; 2k\; 2q}(2\pi)^4\,\delta^{(4)}(Q-\bar{p}-q-k)$$

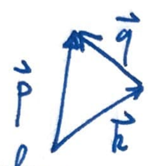
- We can approximate the phase space for small $q_\perp$; divided into longitudinal & perpendicular parts.



$$p = k + q \qquad k \simeq (1-z)\,p \quad \text{when } q_\perp = 0$$

$$d^3p = d^3k \qquad q \simeq z\,p$$

both $\vec{p}$ and $\vec{k}$ cover the entire d^3p volume



- then:

$$d\Pi_3 \simeq \overbrace{\frac{d^3\bar{p}\, d^3p}{(2\pi)^6\, 2\bar{p}\, 2p}\,(2\pi)^{(4)}\delta^{(4)}(Q-p-\bar{p})}^{d\Pi_2} \times \frac{d^3q}{(2\pi)^3\, z\,(1-z)\,2p}$$

$$q = zp \qquad k = (1-z)p$$

$$\text{the } d^3q = \underbrace{dq_z}_{\substack{\text{longitudinal} \\ \parallel \\ p\,dz}}\ \underbrace{d^2q_\perp}_{\substack{\parallel \\ \pi\, dq_\perp^2}}{}^{\to \text{ perpendicular}} = p\,dz\ \pi\, dq_\perp^2$$

- putting it all together, we have:

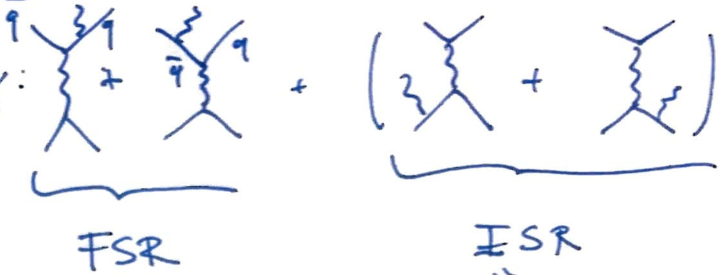
$$\sigma_{e^+e^-\to q\bar{q}\gamma} = \frac{1}{2E_A\, 2E_B\, 2}\int d\Pi_2\, |\mathcal{M}(e^+e^-\to q\bar{q})|^2 \int dz\, dq_\perp^2\, \frac{\pi \cancel{p}}{(2\pi)^3\, 2\cancel{z(1-z)}\cancel{p}} \underbrace{\left(\frac{1}{p^2}\right)^2}_{\substack{\parallel \\ \left(\frac{\cancel{z(1-z)}}{q_\perp^2}\right)^2}}$$

$$\times\ \underbrace{|\mathcal{M}(q\to q\gamma)|^2}_{\substack{\parallel \\ (Q_f e)^2\, 2q_\perp^2\, \frac{(1-z)^2+1}{\cancel{z^2(1-z)}}}}$$

$$= \sigma(e^+e^-\to q\bar{q})\cdot \frac{Q_f^2\,\alpha}{\pi}\int dz \int \frac{dq_\perp}{q_\perp}\, \frac{1+(1-z)^2}{z}.$$

↳ We have the σ for production × emission of a $\gamma \simeq q$.

- The total $e^+e^- \to q\bar{q}\gamma$:



FSR = Final State Radiation

ISR = Initial ...

- Soft divergencies $\propto \underbrace{\frac{1}{q_\perp}}_{\text{collinear}}$ & $\underbrace{\frac{1}{z}}_{\text{soft}}$. $(q \sim zp)$

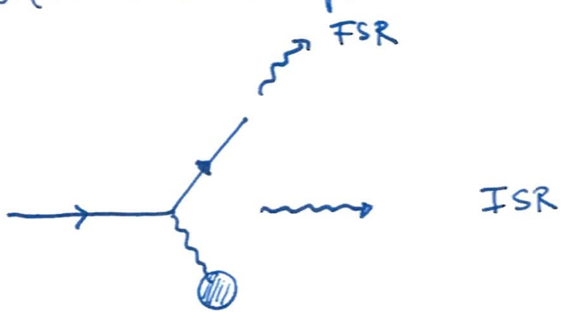
- These expressions (~~soft &~~)(collinear approximation) hold generally as long as $E \gg m_i$

$$\left.\begin{array}{l} \sigma(A \to B + f + \gamma) \sim \sigma(A \to B + f) \\ \sigma(A + f \to B + \gamma) \sim \sigma(A + f \to B) \end{array}\right. \times \left\{ \int dz\, dq_\perp \frac{1}{q_\perp} \frac{Q_f^2 \alpha}{\pi} \times \frac{1 + (1-z)^2}{z} \right. .$$

- Weizsäcker-Williams distribution

for electrons $f = e^-$ & $q_\perp \in (m_e, E_{cm})$, $z \in \left(\frac{m_e}{E_{cm}}, 1\right)$

we have $\sigma(A \to B + e^- + \gamma) \sim \sigma(A \to B + e^-) \frac{2\alpha}{\pi} \ln^2 \frac{E_{cm}}{m_e}$



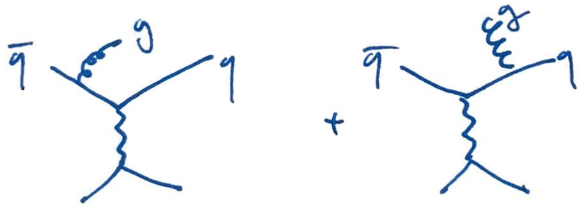
10.3. Three-jet events

We can apply the same reasoning for the emission of gluons by assuming $m_g = 0$, $s_g = 1$, and the vecto[r] current interaction $\bar{\Psi}\gamma^\mu\Psi$, $e \to g_s$ & $\alpha_s = \frac{g_s^2}{4\pi}$

Thus, the $q\bar{q}g \Rightarrow 3$ jet events are given by the same formula

$$\sigma(e^+e^- \to q\bar{q}g) \sim \sigma(e^+e^- \to q\bar{q}) \times \frac{4}{3}\frac{\alpha_s}{\pi}\int dz \int \frac{dq_\perp}{q_\perp}\frac{1+(1-z)^2}{z}.$$

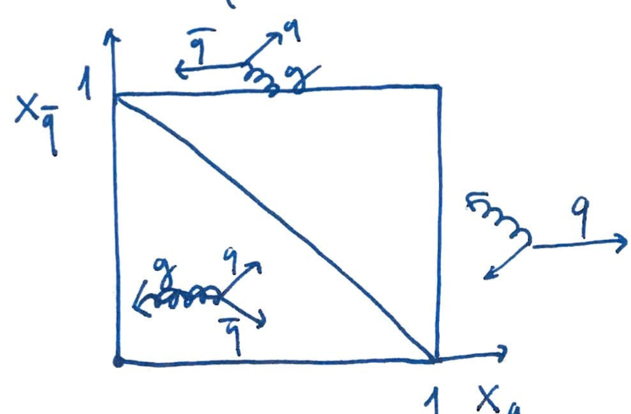
This can be worked out even outside the collinear approx. by interfering the two amplitudes



& approximating the 3 body phase space

CM frame, $E_{CM} = Q = E_q + E_{\bar{q}} + E_g$

& again $x_i = \frac{2E_i}{Q}$, $i = q, \bar{q}, g$, $\sum x_i = 2$



$$\Gamma_{e^+e^-\to \bar{q}qg} = \sigma_{e^+e^-\to q\bar{q}} \times \frac{2\alpha_s}{3\pi}\int dx_q\, dx_{\bar{q}}\, \frac{\overset{(1-z)^2}{x_q^2} + \overset{1}{x_{\bar{q}}^2}}{\underbrace{(1-x_q)}_{z}(1-x_{\bar{q}})}$$

- This agrees with the previous formula in the collinear limit, e.g. $x_{\bar{q}} \to 1$: $\overset{\bar{q}}{\longleftarrow}\ \overset{g}{\leadsto}\ \underset{q}{\longrightarrow}$ g & q are collinear

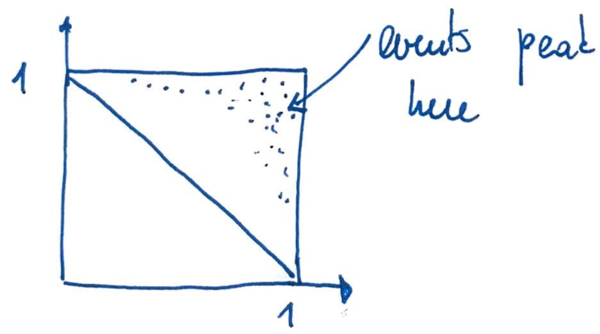
$$x_g \sim z \quad , \quad x_q \sim 1-z \quad , \quad x_{\bar{q}} = \frac{2E_{\bar{q}}}{Q} \simeq \frac{2E_{\bar{q}}Q}{Q^2} = \frac{2\bar{p}Q}{Q^2}$$

$$p^2 = (q+k)^2 = (Q-\bar{p})^2 = Q^2 - 2\bar{p}Q = Q^2(1-x_{\bar{q}}) = \frac{q_\perp^2}{x_g x_q}$$

- then : $$\frac{dx_{\bar{q}}}{1-x_{\bar{q}}} = \frac{dq_\perp^2}{q_\perp^2} = 2\frac{q_\perp dq_\perp}{q_\perp^2} = 2\frac{dq_\perp}{q_\perp}$$

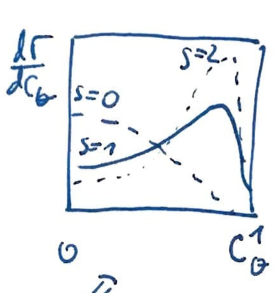
$$\Downarrow$$

$$\frac{4\alpha}{3\pi}\int dz \int \frac{dq_\perp}{q_\perp}\, \frac{1+(1-z)^2}{z}$$ as in the previous collinear case.

- 

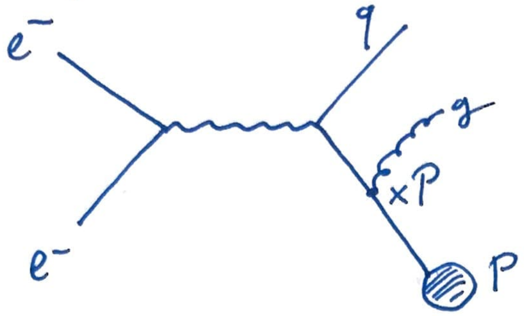
Experiments : TASSO, SLD confirm

- 2-jet events w $s = 1/2$ quarks
- 3-jet events peaking collinearly
  ↳ prediction for x_3 agrees.
- $\frac{d\sigma}{dc_\theta}$ that favors vector $s=1$ over scalar ($s=0$) or tensor ($s=2$)



10.4. Effect of gluon emission on PDFs

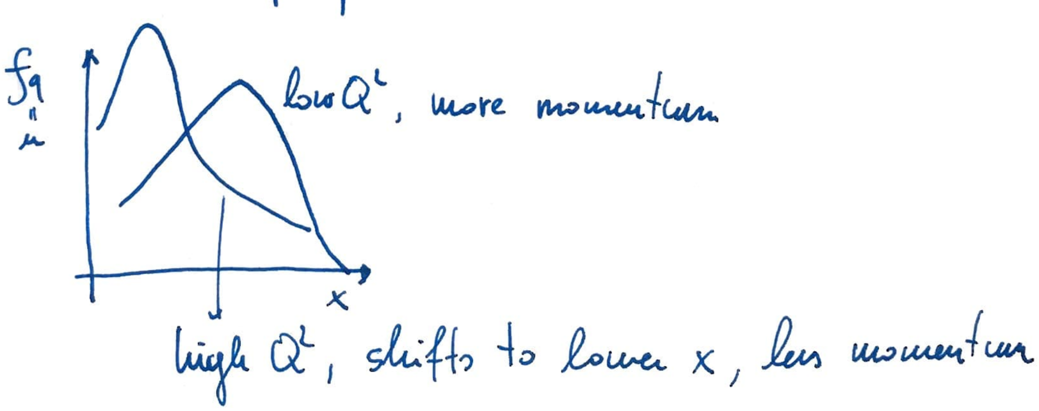
DIS picture :



The gluon can be ISR or FSR in both cases reducing the momentum of the quarks.

$$\propto \frac{4}{3}\frac{\alpha_s}{\pi}\int\frac{dq_\perp}{q_\perp} \sim \frac{4}{3}\frac{\alpha_s}{\pi}\ln\frac{Q}{m_p} \propto \underline{\ln Q}.$$

- So the effect is that the momentum of partons is softened = lower x , in proportion to $\ln Q$



- another effect is the increase of $q\bar{q}$ PDFs coming from $g \to q\bar{q}$ pair creation (like $\gamma \to e^+e^-$)