11. QCD formalism

To construct QCD, we follow QED w. $m_g = 0$, $S_g = 1$

- has to account for $N_c = 3$ colours
- needs to be of vector-like nature

11.1. Lagrangian and gauge invariance

- QED : $\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$ U(1) symmetry

The $\mathcal{L}$ is invariant under the local (or gauge) symmetry $\psi \to e^{i\alpha(x)}\psi$ when $\not{D} = \not{\partial} + ie\not{A}$, $\not{A} = \gamma^\mu A_\mu$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha \Rightarrow F_{\mu\nu} \text{ is invariant}$$

- Lie groups

· from U(1) → non-Abelian $\psi \to e^{i\alpha^a(x) T^a}$

$$[T^a, T^b] = i f^{abc} T^c \quad ; \; a = 1, \dots, \overset{\text{\# of generators}}{\overset{\shortparallel}{d_G}}$$

Lie algebra for the Hermitian generators

structure constants, anti-symm.

specific representations $(t_R^a)_{ij}$, $d_R \times d_R$ matrices,

- For $SU(2)$ & $d_R = 2$ $T^a = \frac{\sigma^a}{2}$ = Pauli matrices

$SU(3)$ & $d_R = 3$ $T^a = \frac{\lambda^a}{2}$ = Gell-Mann matrices

$\underbrace{\hspace{4cm}}$

FUNDAMENTA REPS in $SU(3)$: 3 & $\bar{3}$

in $SU(N)$: N & $\bar{N}$

$\downarrow$

$$\mathrm{tr}(t_N^a t_N^b) = \frac{1}{2}\delta^{ab}$$

$\uparrow$ canonical normalization

- for a general rep : $\mathrm{tr}(t_R^a t_R^b) = \underbrace{C(R)}_{\text{Dynkin index}} \delta^{ab}$

$\frac{1}{2}$ for fundamental

N for adjoint

- Adjoint : $(t_G^a)_{ij} = i f^{abc}$ $\left(SU(2),\ f^{abc} = \varepsilon^{abc} \right)$

Non-Abelian gauge symmetry

$$(D_\mu)_{ij} \Rightarrow \partial_\mu \delta_{ij} - ig A_\mu^a (t_R^a)_{ij}$$ acts on: $\psi_i \to e^{i\alpha^a t^a_{ij}} \psi_j$

$$A_\mu \to A_\mu + \frac{1}{g}\partial_\mu \alpha^a + \underbrace{f^{abc} A_\mu^b \alpha^c}_{\hookrightarrow \text{ extra non-Abelian piece}}$$

- With the covariant derivative

$$(D_\mu \psi) \longrightarrow e^{i\alpha^a t^a_R} (D_\mu \psi) \checkmark$$

- $$[D_\mu, D_\nu] \to e^{i\alpha^a t^a} [D_\mu, D_\nu] e^{-i\alpha^a t^a}$$

$\Downarrow$

transforms as a tensor

- $$\Rightarrow [D_\mu, D_\nu] = -ig\, t^a F^a_{\mu\nu}$$

where $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu$

such that $F^a_{\mu\nu} t^a \Rightarrow U (Ft) U^+$

- $$\& \ \mathrm{tr} \left(F^a_{\mu\nu} t^a\right) \left(F^{\mu\nu a} t^a\right) = \text{invariant}.$$

$\Downarrow$

Non-abelian Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \bar{\psi} (i\not{D} - m) \psi$$

$\Downarrow$ Non-linear Maxwell equations, $\Downarrow$ Dirac equation w. a vector current coupling quarks to gluons.

- Specify QCD w. quark triplets & gluon octets

$$3_i = \psi_i \quad , \quad i = 1,2,3 = \text{color index}$$

Dirac spinor ψ_α , $\alpha = 1,\dots,4$

More generations / flavors ψ_f , $f = u,c,t,d,s,b$

$\psi_{i\alpha f}$ ... Dirac quark triplet field of flavor f

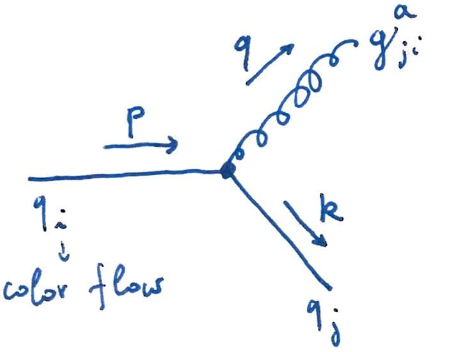
- $\psi \to e^{i\alpha^a t^a}\psi$, $D_\mu = \partial_\mu - i g_s A_\mu^a t^a$

$t^a \to \frac{\lambda^a}{2}$ ... 8 Gell-Mann matrices

$a = 1,\dots,8$ ... gluon fields

- $\Rightarrow$ $$\mathcal{L} = -\frac{1}{4}\sum_a F_{\mu\nu}^a F^{\mu\nu a} + \sum_f \bar{\psi}_f (i\not{D} - m_f)\psi_f$$

- EXAMPLE : the $\frac{4}{3}$ factor in gluon emission



$$\mathcal{M}(q_L(p) \to q_L(k) + g^a(q))$$
$$= g_s \bar{u}_j(k) \gamma^\mu t^a_{ji} u_i(p) \varepsilon_\mu(q)$$

- Now we sum over final colors $\sum_j$ & average over the initial ones : $\frac{1}{3}\sum_i$

• we have : $\frac{1}{3}\sum_{ija} g_s^2 \, |t^a_{ij}|^2 = \frac{g_s^2}{3} \cdot \underbrace{\mathrm{tr}\, t^a t^a}_{\frac{1}{2}\underbrace{\delta^{aa}}_{8}} = \frac{4}{3} g_s^2$

• if we had gluons (8 rep) instead, we would get a different coupling :

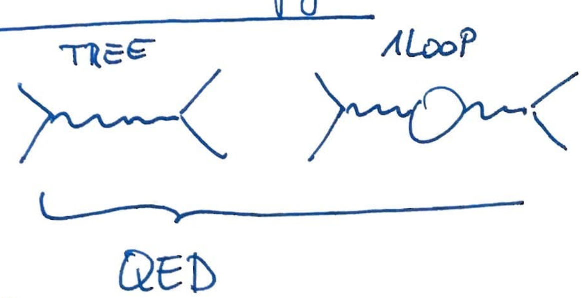
$$\frac{1}{8} \sum_{abc} g_s^2 \, |(t^b_g)_{ac}|^2 = \frac{g_s^2}{8} \underbrace{\mathrm{tr}\, t^a t^a}_{3\delta^{aa} = 24} = \underline{3 g_s^2}$$

average over 8 gluons

• How come things worked out with a perturbative calculation (2 jets, 3 jets) if QCD is supposed to be strong & non-perturbative ?

RGE : running of α_s with energy

VACUUM FLUCTUATIONS :

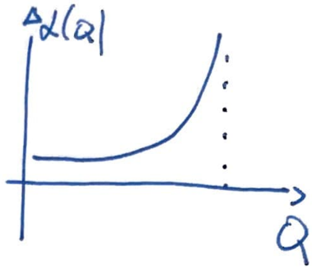


$$\frac{de}{d\ln Q} = \beta(e(Q)) \simeq \frac{e^3}{12\pi^2} \quad @ \; 1\,\mathrm{loop}$$

- The solution for $e(Q)$ is

$$e^2(Q) = \frac{e_0^2}{1 - \frac{e_0^2}{6\pi^2} \ln Q/Q_0}$$

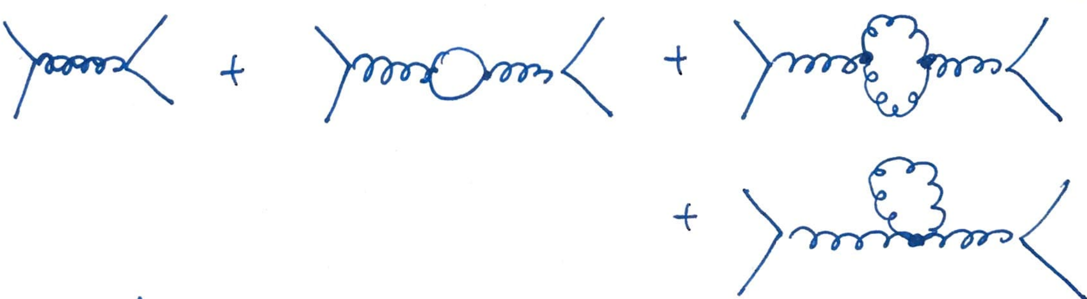
or $$\alpha(Q) = \frac{\alpha_0}{1 - \underbrace{\frac{2\alpha_0}{3\pi} \ln \frac{Q}{Q_0}}_{= 1 \;\Rightarrow\; \text{landau pole}}}$$



$\Rightarrow$ QED becomes <u>stronger</u> at higher energy/momentum transfer

Q	~ 0	30 GeV	91 GeV
α^{-1}	137	130	129

- For QCD the effect is modified by self-interactions



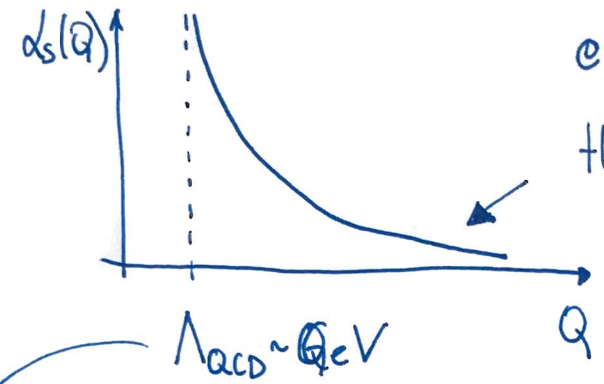
this gives $$\frac{dg_s}{d\ln Q} = \beta(g_s) = -\frac{g_s^3}{16\pi^2}\left(\frac{11}{3}\underbrace{C_2(G)}_{=3} - \frac{4}{3} n_f \underbrace{C(R)}_{=\frac{1}{2} \text{ for fund.}}\right)$$

$$= -\overbrace{\left(11 - \frac{2}{3} n_f\right)}^{b_0} \frac{g_s^3}{(4\pi)^2}$$

· the solution in terms of $\alpha_s = \frac{g_s^2}{4\pi}$ is

$$\alpha_s(Q) = \frac{\alpha_s(Q_0)}{1 - \frac{b_0 \alpha_s(Q_0)}{2\pi} \ln\left(\frac{Q}{Q_0}\right)}$$

$$\Downarrow$$



@ $Q \gg GeV$ (e.g. colliders), the α_s becomes weaker

$\alpha_s(M_Z) \sim 0{,}118$

and perturbative calculations can be done, e.g. $q \to gq, \ldots$

ASYMPTOTIC FREEDOM

$\wr$

TRIVIAL UV FIXED POINT

$\alpha_s \xrightarrow{Q \to \infty} 0.$

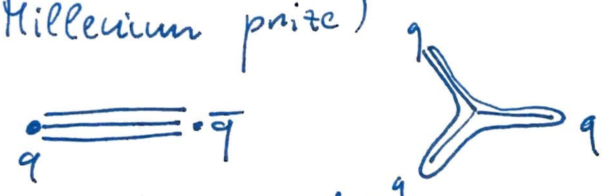
here QCD becomes strongly coupled and non-perturbative – the Λ_{QCD} appears ⇒ DIMENSIONAL TRANSMUTATION

α_s determination (Q)

- 2, 3 jet events in e^+e^-
- PDF scaling with Q^2
- higher order corrections to $R^{NLO} = R^{LO}\left(1 + \frac{\alpha_s}{\pi} + \ldots\right)$

$$R = \frac{\sigma(e^+e^- \to hadrons)}{\sigma(e^+e^- \to \mu^+\mu^-)}$$

- at low Q^2 QCD confines (Millenium prize)
  - flux tubes by Wilson



  - lattice QCD → spectroscopy, very succesful