SCALAR DARK MATTER VIA REALIGNMENT

Realignment also known as misalignment:
a mechanism that generates an abundance of cold dark matter for a scalar field.

Start with a periodic potential (not necessary, but will prove useful later)

$$V(\phi) = m^2 f_a^2 \left[1 - \cos\frac{\phi}{f_a}\right] \tag{1}$$

ϕ : real scalar field (1 degree of freedom)

Expand the cosine and consider $m \ll f_a$

$$V(\phi) = \frac{1}{2} m^2 \phi^2 - \frac{1}{4!}\frac{m^2}{f_a^2}\phi^4 + O(\phi^6) = \frac{1}{2} m^2\phi^2 + O(\phi^4) \tag{2}$$

Neglecting interactions we have a free massive scalar field:

$$S = \int d^4x \sqrt{-g}\left[-\frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \frac{1}{2} m^2\phi^2\right] = \int d^4x \, L_\phi \tag{3}$$

FRW metric $\quad ds^2 = -dt^2 + a^2(t)\, d\vec{x}^2$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & a^2(t)\delta_{ij} \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & \frac{1}{a^2(t)}\delta_{ij} \end{pmatrix}$$

From the action we derive the equation of motion

$$\partial_\alpha \frac{\delta S}{\delta(\partial_\alpha\phi)} - \frac{\delta S}{\delta\phi} = 0$$

and the energy-momentum tensor

$$T_{\alpha\beta} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\alpha\beta}}$$

identifying $\quad \underset{\text{ENERGY DENSITY}}{\rho = T_{00}} \qquad \underset{\text{PRESSURE}}{p = \frac{1}{3}\left(g^{\alpha\beta}T_{\alpha\beta} + \rho\right)}$

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The results are

$$\ddot{\phi} - \frac{\nabla^2\phi}{a^2(t)} + 3H\dot{\phi} + m^2\phi = 0 \tag{4}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2a^2(t)}(\vec{\nabla}\phi)^2 + \frac{1}{2}m^2\phi^2 \tag{5}$$

$$p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{6a^2(t)}(\mathrm{grad}\,\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{6}$$

Here $H = \frac{\dot{a}}{a}$, with the dot denoting a derivative with respect to time.

Let's consider ϕ homogeneous over space
(we will see later under which circumstances this is true)

$$\phi(\vec{x},t) \longrightarrow \phi(t)$$

Then we have

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \tag{7}$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}m^2\phi^2 \tag{8}$$

$$p = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2 \tag{9}$$

Consider eq. (7) when the Universe is radiation dominated

$$H = \frac{1}{2t} \qquad H^2 = \frac{\rho_R}{3M_p^2} = g(T)\frac{\pi^2}{90}\frac{T^4}{M_p^2}$$

REDUCED PLANCK MASS $M_p \simeq 2.4 \times 10^{18}$ GeV

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$$\ddot{\phi} + \frac{3}{2t}\dot{\phi} + m^2\phi = 0 \qquad (10)$$

$$\phi(t) = \phi_0\, \Gamma\left(\tfrac{5}{4}\right)\left(\frac{2}{mt}\right)^{\frac{1}{4}} J_{\frac{1}{4}}(mt) \qquad (11)$$

Bessel function

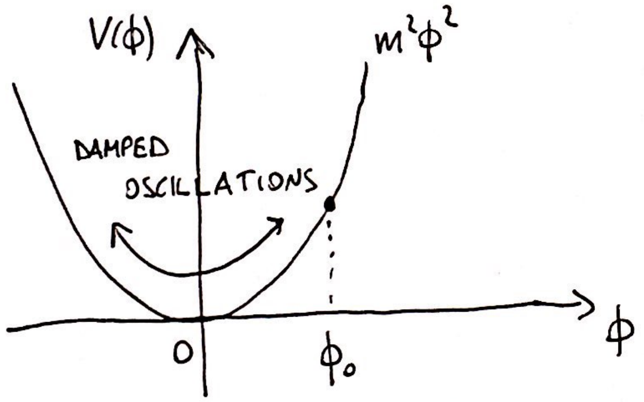
(11) is the solution to (10) with boundary conditions

$$\phi(t=0) = \phi_0 \qquad \dot{\phi}(t=0) = 0$$

At late times

$$\phi(t) \xrightarrow[t \gg \frac{1}{m}]{} \phi_0 \frac{1}{\sqrt{\pi}}\Gamma\left(\tfrac{5}{4}\right)\left(\frac{2}{mt}\right)^{\frac{3}{4}} \sin\left(mt + \frac{\pi}{8}\right) \qquad (12)$$

the damped oscillatory behavior is explicit:



From (12) we can compute

$$\dot{\phi}(t) \xrightarrow[t \gg \frac{1}{m}]{} m\,\phi_0 \frac{1}{\sqrt{\pi}}\Gamma\left(\tfrac{5}{4}\right)\left(\frac{2}{mt}\right)^{\frac{3}{4}} \cos\left(mt + \frac{\pi}{8}\right) + O\left((mt)^{-\frac{7}{4}}\right) \qquad (13)$$

Consider $\langle \phi^2(t) \rangle$, that is the average over one oscillation period

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$$\langle \phi^2(t) \rangle = \left[\phi_0 \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{4}\right) \left(\frac{2}{mt}\right)^{\frac{3}{4}}\right]^2 \underbrace{\frac{m}{2\pi} \int_0^{\frac{2\pi}{m}} dt \ \sin^2\left(mt + \frac{\pi}{8}\right)}_{\langle \sin^2(mt + \frac{\pi}{8}) \rangle = \frac{1}{2}}$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{4}\right)\right]^2 \phi_0^2 \left(\frac{2}{mt}\right)^{\frac{3}{2}} \qquad (14)$$

From (13) it is easy to check that

$$\langle \dot{\phi}^2(t) \rangle = m^2 \langle \phi^2(t) \rangle$$

Then, from (8) and (9) we have

$$\langle \rho \rangle = \frac{1}{2} \langle \dot{\phi}^2 \rangle + \frac{1}{2} m^2 \langle \phi^2 \rangle = m^2 \langle \phi^2(t) \rangle = \frac{1}{2}\left[\frac{1}{\sqrt{\pi}} \Gamma\left(\frac{5}{4}\right)\right]^2 m^2 \phi_0^2 \left(\frac{2}{mt}\right)^{\frac{3}{2}}$$

$$\langle p \rangle = \frac{1}{2} \langle \dot{\phi}^2 \rangle - \frac{1}{2} m^2 \langle \phi^2 \rangle = 0 \qquad (15)$$

Recall that in radiation domination the scale factor $a(t) \sim t^{\frac{1}{2}}$.

Therefore we see that

$$\langle \rho \rangle \sim a^{-3}(t) \qquad \langle p \rangle = 0$$

The oscillating scalar field has an energy density that redshifts as a^{-3}, and zero pressure.

These are the properties of non-relativistic matter in the expanding Universe. So our scalar field gives us a perfectly good COLD DARK MATTER candidate.

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Note we have derived these results starting from the action, and the associated equation of motion, of a CLASSICAL field.

From the point of view of QUANTUM field theory, a spatially homogeneous scalar field describes a collection of free particles of mass m at zero momentum. Clearly they are non relativistic, no matter how small their mass is. Also, particles at rest produce no pressure!

Let's compute the relic density.

In the oscillatory regime, $t \gg \frac{1}{m}$, the number density

$$n = \frac{1}{m} \langle \rho \rangle = m \langle \phi^2(t) \rangle \tag{16}$$

redshifts as a^{-3}. In a Universe with conserved entropy, $aT = \text{constant}$ the ratio $\frac{n}{s}$ remains constant. Here

$$s = \frac{2\pi^2}{45} g_{*s} T^3 \qquad \text{entropy density}$$

So $\frac{n}{s}$ is constant once the field is oscillating. When do the oscillations start?

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Inspect the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0 \qquad H = \frac{1}{2t}$$

At early times $H \gg m$ the ϕ-oscillator is stuck, over damped.

It starts oscillating when $H = m$, roughly:

$$t_* = \frac{1}{2m} \qquad \text{time at which oscillations start} \qquad (17)$$

Even if t_* is not $\gg \frac{1}{m}$ we will use eq (12) to get an estimate of the relic abundance. From (15) and (16) we have

$$n_* = m \langle \phi^2(t_*) \rangle = \frac{1}{2}\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\right]^2 m\,\phi_0^2 \left(\frac{2}{m t_*}\right)^{\frac{3}{2}}$$

$$= 4\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\right]^2 m\,\phi_0^2 \qquad (18)$$

the number density at the onset of oscillations.

At t_* the temperature T_* is

$$H(T_*) = m\,; \qquad \sqrt{\frac{\pi^2}{90} g_*}\,\frac{T_*^2}{M_P} = m\,; \qquad T_* = \left(\frac{\pi^2}{90} g_*\right)^{-\frac{1}{4}} (m M_P)^{\frac{1}{2}}$$

and the entropy density

$$s_* = \frac{2\pi^2}{45} g_* T_*^3 \qquad (\text{here we took } g_{*s} \approx g_*)$$

So we have at t_*

$$\frac{n_*}{s_*} = K \frac{m^{-\frac{1}{2}}\phi_0^2}{g_*^{\frac{1}{4}} M_P^{\frac{3}{2}}} \qquad (19)$$

$$K = 4\left[\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{5}{4}\right)\right]^2 \frac{45}{2\pi^2}\left(\frac{\pi^2}{90}\right)^{\frac{3}{4}}$$

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After t_* the ratio $\frac{n}{s}$ is constant. We can compute the relic abundance today as

$$\Omega_\phi = \frac{\rho_\phi^{today}}{\rho_c} = \frac{m\, n^{today}}{\rho_\gamma/\Omega_\gamma} = \Omega_\gamma \frac{m}{T_\gamma} \frac{86}{33} \frac{n^{today}}{s^{today}}$$

$$= \frac{\Omega_\gamma}{T_\gamma} \frac{86}{33} m \frac{n_*}{s_*} = \frac{\Omega_\gamma}{T_\gamma} \frac{86}{33} K \frac{1}{g_*^{\frac{1}{4}} M_p^{\frac{3}{2}}} m^{\frac{1}{2}} \phi_0^2 \qquad (20)$$

We have used

ρ_c: critical energy density

$\Omega_\gamma = \frac{\rho_\gamma}{\rho_c} \simeq 5 \cdot 10^{-5}$ with

$\rho_\gamma = \frac{2\pi^2}{30} T_\gamma^4$ $\qquad T_\gamma = 2.3 \cdot 10^{-4}\ eV$ (photon temperature today)

$s^{today} = \frac{2\pi^2}{45}\left[2T_\gamma^3 + 3 \cdot \frac{7}{4} T_\nu^3\right] = \frac{2\pi^2}{45} \frac{43}{11} T_\gamma^3$ $\qquad \left(\frac{T_\nu}{T_\gamma}\right)^3 = \frac{4}{11}$

Define $\phi_0 \equiv \theta_0 f_a$ and plug in the numbers

$$\boxed{\Omega_\phi = 0.1\, \theta_0^2 \left(\frac{m}{10^{-22}\, eV}\right)^{\frac{1}{2}} \left(\frac{f_a}{10^{17}\, GeV}\right)^2 \left(\frac{10}{g_*}\right)^{\frac{1}{4}}} \qquad (21)$$

The relic abundance of ϕ is proportional to $\sqrt{m}$ and to ϕ_0^2, with ϕ_0 the initial displacement from the minimum of the potential.

ϕ_0 is an incalculable initial condition (more comments later).

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QUICK SUMMARY

Assuming a free massive spatially homogeneous scalar field, we have its equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

H decreases in the expanding Universe with time. At early times $H \gg m$ the field is stuck at some initial value ϕ_0. When $H \sim m$ the field starts oscillating around the minimum of its potential.
Such an oscillating scalar field behaves like non-relativistic matter, and we computed its relic density at present time. It is proportional to $\sqrt{m}\,\phi_0^2$.

Typically when we discuss this kind of dark matter we have in mind axions. Here by axion I mean either the famous QCD axion or a so-called ALP (axion like particle).
Axions have a periodic potential of the form

$$V(\phi) = m^2 f_a^2 \left[1 - \cos\frac{\phi}{f_a}\right]$$

which is why we started there. Also, in this case $\frac{\phi}{f_a} = \theta$ is clearly an angle, which is why we wrote (21) in terms of f_a and

θ_0 : initial misalignment angle.

As an angle, θ_0 takes values between 0 and 2π, so one expects it to be of order 1 (there is quite a bit literature just on this statement).

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With $\theta_0 \sim 1$, eq. (21) tells us that an axion with

$$m = 10^{-22}\ \text{eV} \qquad f_a = 10^{17}\ \text{GeV}$$

gives us the observed dark matter relic abundance. Such an axion is well motivated by string theory. If you are curious, take a look at 1610.08297 "Ultralight scalars as cosmological dark matter" Hui, Ostriker, Tremaine, Witten

When is the assumption of spatial homogeneity $(\phi(\vec{x},t) \approx \phi(t))$ justified?

Consider a complex scalar field Φ with potential

$$V(\Phi) = \left(|\Phi|^2 - f_a^2 \right)^2$$

We have a global $U(1)$ symmetry

$$\Phi \rightarrow e^{i\alpha}\, \Phi \qquad (22)$$

that is <u>spontaneously broken</u> by the potential, whose minima are at

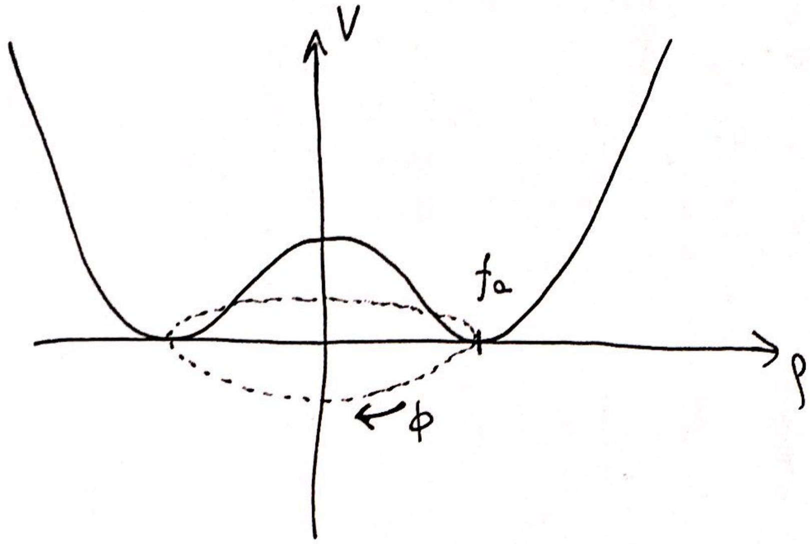
$$\langle |\Phi| \rangle = \pm f_a$$

Parametrize the field as

$$\underset{\substack{\uparrow \\ \text{radial mode}}}{\Phi = \rho}\, e^{i\phi/f_a} \qquad \phi : \text{goldstone boson} = \text{axion}$$

The U(1) symmetry (22), after spontaneous breaking, translates into a continuous shift symmetry of the axion

$$\phi \to \phi + \alpha f_a \qquad (23)$$



f_a is the scale of spontaneous symmetry breaking.

The U(1) can also be <u>explicitly</u> broken at a scale $\sqrt{m f_a}$ with

$$\sqrt{m f_a} \ll f_a$$

in which case the axion develops a potential

$$V(\phi) = m f_a \left[1 - \cos \frac{\phi}{f_a} \right]$$

After explicit breaking the axion has a residual discrete shift symmetry

$$\phi \to \phi + 2\pi n f_a$$

Back to the complex field, at finite temperature its potential is

$$V(\Phi, T) = (|\Phi|^2 - f_a^2)^2 + T^2 |\Phi|^2 + \cdots$$

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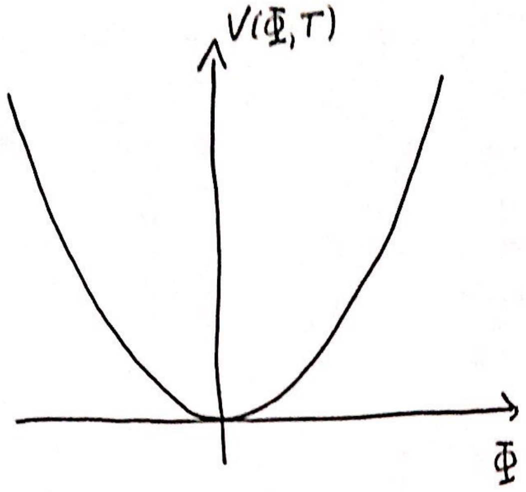
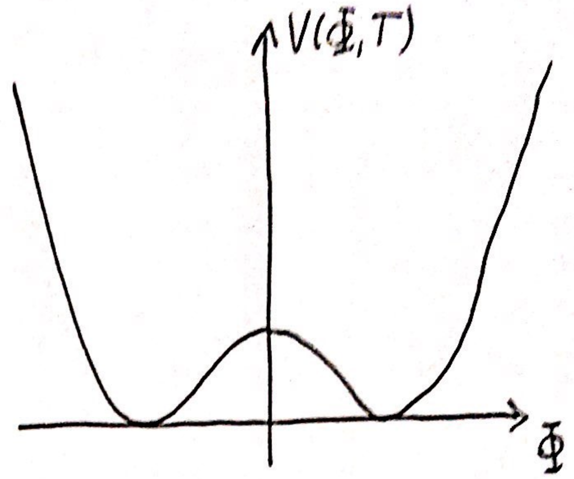


$T > f_a$

UNBROKEN U(1)



$T < f_a$

BROKEN U(1)

$\xrightarrow{\text{PQ phase transition}}$

PQ : Peccei-Quinn (the authors who introduced the sort of U(1) discussed here, usually denoted $U(1)_{PQ}$)

Suppose $\boxed{T_{max} < f_a}$

with T_{max} the highest temperature ever achieved by the Universe.

Consider inflation, the initial epoch of exponentially accelerated expansion. During inflation Hubble is constant, H_I, and we can think of the Universe having a temperature

$$T_{dS} = \frac{H_I}{2\pi} \qquad \text{(de Sitter temperature)}$$

Take $T_{dS} < f_a$, so we are in the broken U(1) phase and there is an axion. Consider

$$\sqrt{m f_a} \ll T_{dS} < f_a \quad \Leftrightarrow \quad 2\pi\sqrt{m f_a} \ll H_I < 2\pi f_a$$

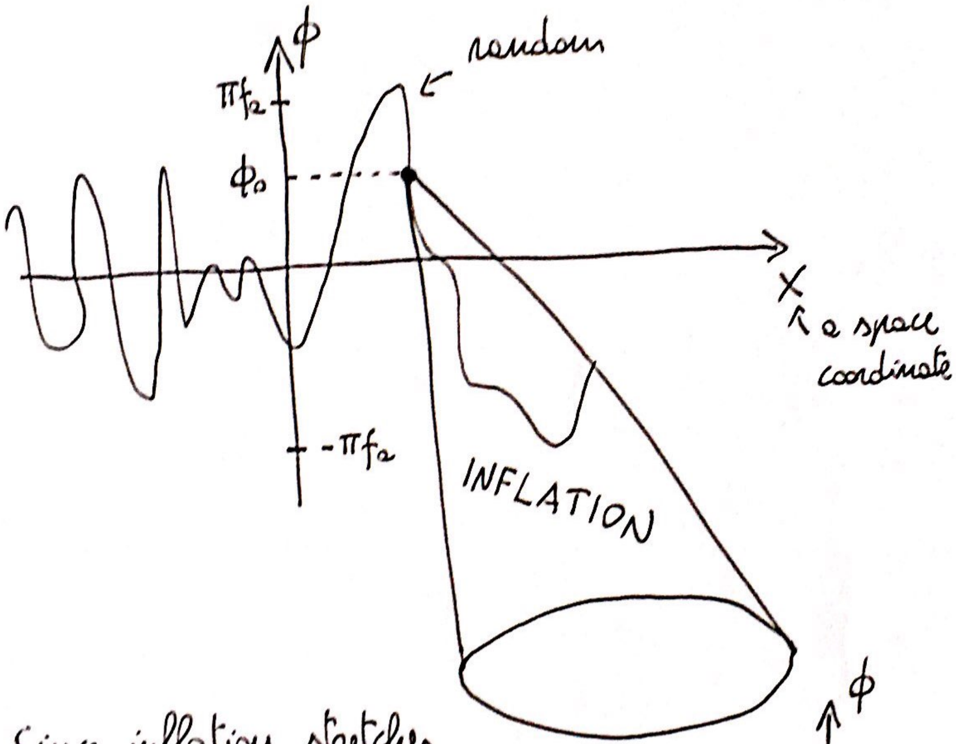
If the inequality $\sqrt{m f_a} \ll H_I$ is satisfied, then we can neglect the axion cosine potential during inflation.

Note for the stringy axion with $m = 10^{-22}$ eV
$f_a = 10^{17}$ GeV

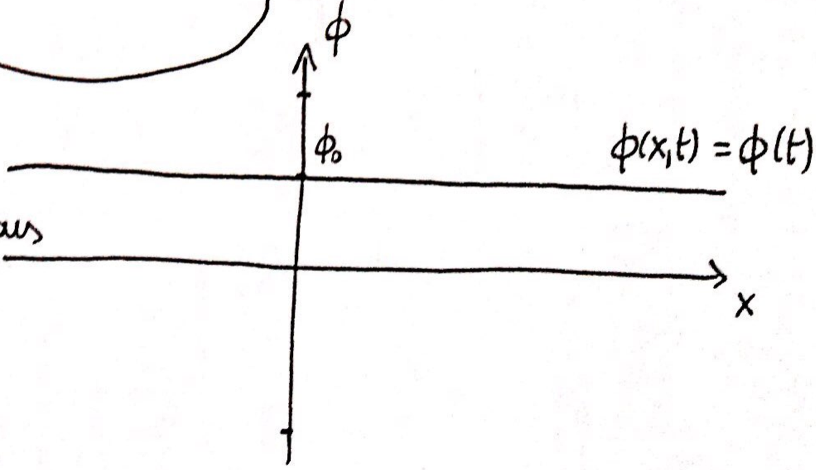
the window is

$$10^{-6} \text{ GeV} \ll H_I < 10^{18} \text{ GeV}$$

which would be satisfied by almost any model of inflation.
In this scenario we can approximate the axion potential as flat during inflation. It follows that any value of the axion field is equally probable at any point in space. The picture roughly looks like this



Since inflation stretches space exponentially, the field ends up being homogeneous with an initial value ϕ_0 selected randomly.



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ϕ_0 is then the initial condition for the equation of motion

$$\ddot{\phi} + 3H\dot{\phi} + m^2\phi = 0$$

which we studied in the radiation dominated Universe, which follows inflation. It is crucial that the reheating temperature, T_{RH}, is also lower than f_a, so that we remain in the broken $U(1)_{PQ}$ phase.

Note for the stringy axion $\quad T_{RH} < f_a = 10^{17}$ GeV

is easy to satisfy.

So the simple treatment that led to eq. (21) for the axion relic abundance is well justified for the stringy axion.

QCD AXION

For this most famous axion, the ball park is $f_a \simeq 10^{11}$ or 10^{12} GeV

Scenario 1 : Max $[T_{ds}, T_{RH}] < f_a$

Always in the broken $U(1)_{PQ}$ phase, axion homogeneous $\phi(\vec{x},t) = \phi(t)$

Scenario 2 : $T_{RH} > f_a$

After reheating we are in the unbroken $U(1)_{PQ}$ phase. Any information about ϕ_0 from inflation is lost. As T drops we go through the PQ phase transition. At $T < f_a$ we are in the broken PQ phase. Now, however, the axion field is no longer homogeneous, it takes random values in different patches:

$$\theta_0^2 \rightarrow \langle \theta_0^2 \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} d\theta_0\, \theta_0^2 = \frac{\pi^2}{3}$$

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The equation of motion now is

$$\ddot{\phi} - \frac{\nabla^2}{a^2}\phi + 3H\dot{\phi} + m^2\phi = 0$$

Not only the zero-momentum, but also the higher-momentum modes enter the calculation.

Also, in the PQ phase transition topological defects form, like axion strings and domain walls. They also give contributions to the final relic abundance.

If you are interested in this scenario a good starting point is the chapter "Axion cosmology" by Sikivie in the book "Axions" edited by Kuster, Raffelt, Beltrán (Springer 2008)

In scenario 2 the final axion relic abundance is calculable in principle. In practice its final value is somewhat uncertain, due to the technical challenges in the computation of the contribution from the topological defects. I won't discuss this scenario any further.

In scenario 1 the abundance is not calculable, due to the fact that the initial condition ϕ_0 is unknown, as we have discussed.

Also, for the QCD axion the final abundance is different from (21).

Let's see how it is different.

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First of all, m and f_a are not independent :

$$m^2 f_a^2 \simeq m_\pi^2 f_\pi^2 \quad \Rightarrow \quad m \simeq \frac{m_\pi f_\pi}{f_a} \approx \frac{\Lambda_{QCD}^2}{f_a}$$

m_π, f_π : pion mass and decay constant

Λ_{QCD} : QCD confinement scale (~ 200 MeV)

In (21) Ω_ϕ is proportional to $m^{\frac{1}{2}} f_a^2$, so for the QCD axion it would be proportional to $f_a^{\frac{3}{2}}$.

However, to derive (21) we considered a mass m constant in time. This is not the case for the QCD axion :

$$m(T) = \begin{cases} \dfrac{\Lambda_{QCD}^2}{f_a} & T \ll \Lambda_{QCD} \\ \dfrac{\Lambda_{QCD}^2}{f_a}\left(\dfrac{\Lambda_{QCD}}{T}\right)^{\frac{\alpha}{2}} & T > \Lambda_{QCD} \end{cases} \tag{23}$$

Here, $\alpha \simeq 8$ can be obtained either via an analytic calculation with the dilute instanton gas, or via numerical calculations on the lattice. Both methods involve several subtleties and the final value of α is still a subject of debate.

What is certain is that the axion mass goes to zero when $T \gg \Lambda_{QCD}$.

Consider the equation of motion for the homogeneous QCD axion

$$\ddot{\phi} + 3H(t)\dot{\phi} + m^2(t)\phi = 0 \tag{24}$$

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When the conditions

OSCILLATORY REGIME

$$H(t) \ll m(t) \qquad \text{and} \qquad \dot{m}(t) \ll m^2(t) \qquad (25)$$

are satisfied, the solution is

$$\phi(t) = \frac{1}{a^{3/2}(t)}\, \phi_0 \sin\left(m(t)\, t + \beta\right)$$

similarly to our previous derivation. Again, averaging over an oscillation period we have

$$\langle \dot{\phi}^2(t) \rangle = \langle m^2(t)\, \phi^2(t) \rangle \qquad (26)$$

still under conditions (25).

Note that

$$\rho = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2} m^2 \phi^2$$

$$\dot{\rho} = \underbrace{\ddot{\phi}\dot{\phi} + m^2 \phi \dot{\phi}}_{\substack{-3H\dot{\phi}^2 \\ \text{using (24)}}} + \dot{m}\, m\, \phi^2 = -3H\dot{\phi}^2 + \frac{\dot{m}}{m}\, m^2 \phi^2$$

Use (26)

$$\langle \dot{\rho} \rangle = \langle m^2 \phi^2 \rangle \left(\frac{\dot{m}}{m} - 3H \right) = \langle \rho \rangle \left(\frac{\dot{m}}{m} - 3H \right)$$

$$\frac{\langle \dot{\rho} \rangle}{\langle \rho \rangle} = \frac{\dot{m}}{m} - 3\frac{\dot{a}}{a} \; ; \quad \frac{d}{dt}\left(\ln \langle \rho \rangle + \ln a^3 - \ln m \right) = 0$$

$$\Rightarrow \frac{\langle \rho \rangle a^3}{m} = \text{constant} \; ; \quad \langle \rho \rangle = \text{constant}\, \frac{m}{a^3} \; ; \quad n = \frac{\langle \rho \rangle}{m} = \frac{\text{constant}}{a^3}$$

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This is another way of seeing that $n \sim \frac{1}{a^3}$ in the oscillatory regime. (17)

Oscillations start when (roughly)

$$m(T_*) = H(T_*)$$

Using (23) with $T > \Lambda_{QCD}$ we get

$$\frac{T_*}{\Lambda_{QCD}} = \left[\frac{M_P}{f_a} \frac{3\sqrt{10}}{\pi \sqrt{g_*}} \right]^{\frac{2}{4+\alpha}} \tag{27}$$

At T_* we have

$$\frac{n_*}{s_*} = \frac{m(T_*)\,\phi_0^2}{\frac{2\pi^2}{45} g_* T_*^3} = \frac{H(T_*)\,\phi_0^2}{\frac{2\pi^2}{45} g_* T_*^3} = \frac{\sqrt{\frac{\pi^2}{90} g_*}\ \frac{T_*^2}{M_P}\,\phi_0^2}{\frac{2\pi^2}{45} g_* T_*^3} = K \frac{\phi_0^2}{\sqrt{g_*}\, M_P T_*} \tag{28}$$

with K a numerical factor of order 1.

The ratio $\frac{n}{s}$ remains constant after T_*. So, repeating the steps of eq (20) we have

$$\Omega_\phi = \Omega_\gamma \frac{m}{T_\gamma} \frac{86}{33} \frac{n_*}{s_*}$$

where $m = \frac{\Lambda_{QCD}^2}{f_a}$ is the zero-temperature mass.

Using $\phi_0 \equiv \theta_0 f_a$, (28) and (27) we get

$$\boxed{\Omega_\phi = 0.1\ K_\alpha \left(\frac{\theta_0}{2.15}\right)^2 \left(\frac{f_a}{10^{12}\,\text{GeV}}\right)^{1+\frac{1}{2+\frac{\alpha}{2}}} \left(\frac{60}{g_*}\right)^{\frac{1}{2}-\frac{1}{4+\alpha}}}$$

Here K_α is a numerical factor ($K_\alpha \in [1,5]$ for $\alpha \in [3,8]$).

For $\alpha = 8$ Ω_ϕ is proportional to $f^{7/6}$.

Without accounting for the temperature dependence of the axion mass Ω_ϕ was proportional to $f_a^{3/2}$.

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